# Journal for the History of Analytical Philosophy

Volume 9, Number 11



ISSN: 2159-0303

*jhaponline.org*



## Strictures on an Exhibition: Frege on his Primitive Laws

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In *Grundgesetze der Arithmetik*, Frege tried to show that arithmetic is logical by giving gap-free proofs from what he took to be purely logical basic laws. But how do we come to judge these laws as true, and to recognize them as logical? The answer must involve giving an account of the apparent arguments Frege provides for his axioms. Following Sanford Shieh, I take these apparent arguments to instead be exhibitions: the exercise of a logical capacity in order to bring us into a state of judgement. I provide an account of what sort of inferential capacities are at play in such exhibitions, and explain how they lead us to judge that Frege's primitive laws are general and undeniable. I will also situate my account with respect to other rival interpretations, particularly the elucidatory interpretations of Joan Weiner and Thomas Ricketts.

*Special Issue:* Frege in Philosophical-Historical Context
Edited by Sanford Shieh and Juliet Floyd

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## 1. Introduction

In his seminal *Grundgesetze der Arithmetik*, Frege tried to demonstrate that arithmetic is purely logical, and can be justified without using either intuition or sense perception. He thought this demonstration necessary precisely because it isn't immediately clear that arithmetic is purely logical—it concerns a specific subject matter (numbers) and seems to have forms of inference peculiar to it (such as mathematical induction). Part of Frege's strategy is to demonstrate arithmetical truths on the basis of primitive laws, since the nature of these primitive laws is clear in a way that the nature of derived arithmetical truths is not. That Frege's strategy went so far is beyond exegetical dispute—what is far more contentious is just how Frege treated his primitive truths. Is their nature obvious to us as soon as we understand the formalism in which they are expressed? Or do we need to reason about their sense to appreciate their nature?

To shed light on this puzzle, I will provide a reconstruction, consistent with Fregean commitments, of the role of Frege's apparent arguments for his primitive logical laws in Volume I of *Grundgesetze*.[1] In the account I offer here, these arguments play the role of exercising his reader's inferential capacities in such a way as to warrant them in judging his basic laws to be true and purely logical. Following Sanford Shieh, I shall call such exercises *exhibitions* (2019, 230). Frege's arguments take us through the sense of, say, Law I, so that we come to acknowledge it as true; once we reflect on the fact that we can acknowledge Law I as true without using intuition or perception, we are warranted in the further judgement "Law I is logical". This reading crucially relies on a distinction Shieh draws between inference and inferring—the former is when derived judgements are shown to rest on more basic ones, and the latter encompasses the exercise of inferential capacities more broadly. In his arguments, Frege can't be deriving his primitive laws from more basic ones, or giving them some deeper grounding—accordingly, what we have here is not inference, but inferring in the context of an exhibition.

I begin in Section 2 with a brief overview of Frege's foundationalism, and some comments about Shieh's inference/inferring distinction. In Section 3, I present Frege's arguments for his basic laws. In Section 4, I list several accounts of what role these arguments play, and argue that we cannot take them as attempted justifications of his primitive logical laws. In Section 5, I discuss the sense in which Frege took logic to be general and undeniable, and connect this with the way in which he contrasts logical truths with truths which rely on intuition or sense/perception. I consider the elucidatory interpretation of Frege's arguments in Section 6, give the details of my own reconstruction in Section 7, and provide some final remarks and clarifications in Section 8.

## 2. Frege's Foundationalism

Frege's way of determining whether or not a given truth, or collection of truths, is purely logical ties in closely with his foundationalist epistemology. We start by proving everything that is provable, which allows us to discover the laws upon which the truths in question rest (1879 [1967], §13; 1884 [1953], §2; 1893 [2013], vi; 1914 [1979], 204–5), and then check whether these primitive laws are purely logical. His foundationalism was not

[1] To save space, I shall drop the "apparent", and simply call them "arguments", where this is to be construed broadly enough to avoid prejudging whether it is a genuine case of inference in the strong sense tied up with justification.

specific to logic—Frege was a foundationalist about geometry as well (1903 [1984], 319/CP 273).[2]

Frege was well aware that there are different ways of axiomatizing logic—he says of his *Begriffsschrift* axiomatization that "the way followed here is not the only one in which the reduction can be done" and that "there is perhaps another set of judgements from which, when those contained in the rules are added, all laws of thought could likewise be deduced" (1879 [1967], §13). However, he was equally clear elsewhere that we are not at liberty to take any truth whatsoever as an axiom. He criticized theories which took arithmetical identities as axioms, and also insisted that axioms must be true and certain (1903 [1984], 319/CP 273),[3] independently recognizable as true (1899-1906 [1979], 168), and unprovable (1903 [1984], 319/CP 273). In a word, axioms must be *primitive*. The leeway in choice of axioms, together with the constraints on this choice, jointly imply that primitive (certain, unprovable, independently recognizable as true) logical laws are not all mutually independent. While all laws taken as axioms in a formalization must be primitive laws, not all such primitive laws need be taken as axioms—in setting up an axiomatic system for logic, one chooses, from among the primitive laws, a set of mutually independent axioms sufficient to generate the rest of the logical truths.

When Frege says that axioms are unprovable, he can't mean simply that they're underivable, since axioms trivially count as underived. Instead, it suggests a view of truths as resting in an objective justificatory ordering bottoming out in primitive truths. These truths can be said to be objectively unprovable since, in this objective ordering, they are grounded in no more fundamental truths. Whether a given primitive truth is an axiom or a theorem will depend on the axiomatization in question—in some axiomatizations we may derive that primitive truth from others as a preliminary step in proving some non-primitive truth. In order for such derivations to be consistent with the objective unprovability of the primitive law, Frege must have held that when we derive one primitive truth from another, we do not *prove* it in the sense of justifying it on the basis of something more basic.[4]

Before moving on, it's necessary to say a bit about Frege's conception of judgement and inference. Late in his career, Frege says that inference "is the pronouncement of a judgement made in accordance with logical laws on the basis of previously passed judgements", where judgements are determinate thoughts recognized as true (1906 [1984], 387/CP 318). It's clear that at this point he takes such recognition to be factive, since if it weren't, we could have cases of inference from false premises, a possibility he denies explicitly (1906 [1984], 424/CP 335; Shieh 2019, 127). While this conception of judgement fits well with his foundationalism, there are other places where Frege speaks of judging in a less restrictive sense, as a non-factive holding something to be true (Ricketts 1996, 131; Kremer 2000; Shieh 2019, 124–26). Also, Frege's early view of content implies a less restrictive notion of inference and judgement. In *Begriffsschrift*, he says that the content of two judgements is same when "the consequences derivable from the first, when it is combined with certain other judgements, always follow also from the second, when it is combined

[2]I give the original page numbers of Frege's published writings translated and collected in Frege (1984), followed by "CP" and the page numbers in Frege (1984).

[3]Although Frege is concerned with geometrical axioms here, he is explicit that he intends what he says to hold for logical axioms as well.

[4]The view outlined thus far is close to that of Tyler Burge in "Frege on Knowing the Foundation" (1998)—all axioms are objectively primitive, but not all objectively primitive laws need be selected as axioms. The difference between our views is the role of pragmatic considerations such as simplicity, precision, and flexibility in Frege's work (1893 [2013], ix–x). Burge maintains that pragmatic considerations can help one realize that Frege's basic logical laws are self-evident (*einleuchten/selbstverständlich*), even if they aren't self-evident to oneself (1998, 354), while I maintain that pragmatic considerations only help us identify which truths are objectively unprovable, and hence basic, but play no role in getting us to see that his laws are self-evident.

with these same judgements" (Frege 1879 [1967], §3). Clearly, this condition only makes sense if we can infer from false judgements.

Of course, Frege's view of content shifts and splits after the introduction of his distinction between sense and reference. However, he continues to speak from time to time of judging as holding something to be true (1892 [1984], 34/CP 163). What's more, *Grundgesetze* explicitly allows derivations which don't count as inferences in the restrictive sense. An example would be a derivation of

$\vdash a \supset (a \supset a)$

from

$\vdash (a \supset (a \supset a)) \supset ((b \supset (b \supset b)) \supset (a \supset (a \supset a)))$

by using modus ponens twice (§14), together with the appropriate minor premises, which are in turn inferred from Law I (§18) via Roman letter replacement (§48). This circular derivation clearly can't be inference in the restrictive, justificatory sense, as the conclusion isn't derived from something more basic. And if, as argued above, primitive laws are not all independent, there can be derivations of primitive laws from other primitive laws. Many of Frege's axioms in *Begriffsschrift* (1879 [1967], §§13–22) are as plausibly primitive as his axioms in *Grundgesetze*, but in the latter work, axioms of the former (or, to be precise, analogues thereof) must be derived.

Shieh's proposal, which I will here follow, is that we should make a distinction between *judgement*, which is a factive recognition as true, and non-factive *judging*, which has judgement as its aim (2019, 127). He points out that this regimented use of English coincides with a tendency in Frege's writings to use *das Urteil* and *das Anerkennung* to refer to cases of successful holding as true, while using the verb *urteilen* and the verbal noun *das Urteilen* when discussing acts by particular judging subjects (2019, 128). Shieh further suggests that we should make a similar distinction between inference, in the strict sense stated explicitly in 1906, and the activity of inferring (2019, 200). I suggest that the moves made in Frege's §18, §20, and §25 arguments for his basic laws count as inferring in Shieh's sense—they cannot be cases of inference, since the "conclusions" are, as primitive truths, objectively unprovable. More broadly, it seems like *some* inferential capacities must be implicit in our understanding of anything whatsoever, even if these capacities don't pass muster as "inference" as officially characterized in (1906 [1984]). For instance, if I know it's true that A and that it's false that B, I know it's false that $A \supset B$.

Whether Frege would be happy in calling the sort of moves made in §18 inferential is beside the point. Rather than quibble over terminology, what is essential to my reading is that the moves made in Frege's arguments for his basic laws are evaluable in light of logical laws, and thus separate from mere rhetoric. To put it one way: If, incredibly, Frege's brief argument for Law I contained some misstep, something would be lost, even if by some strange error all readers glossed over the misstep and came to believe Law I anyway. Or, put another way: Frege would be doing wrong by his readers if, instead of an argument for his basic laws, §18 instead triggered belief in Law I by non-rational means such as hypnosis, an incantation, or instructions to strike oneself on the head with *Grundgesetze* at precisely the right angle. Something rational is taking place, although the inferential capacities are not playing the usual role of justifying laws by inferring them from something more fundamental.

## 3. Frege's Arguments

In this section, I'll make some observations about the apparent purpose and structure of Frege's arguments for his logical laws. The question of whether the apparent purpose of the arguments is the real purpose will be addressed in Section 4.

In the introductory sections of *Grundgesetze*, Frege explains his logical terminology. To choose one example, Frege introduces his conditional stroke as follows:

> I introduce the function with two arguments $\zeta \supset \xi$ by means of the specification that its value shall be the False if the True is taken as the $\zeta$-argument, while any object that is not the True is taken as the $\xi$-argument; that in all other cases the value of the function shall be the True (Frege 1893 [2013], §12).

This specification (*Bestimmung*) concerns a function, not a symbol. Nevertheless, it is clearly doing double-duty, since Frege later uses it to argue that "$\zeta \supset \xi$" refers to something (1893 [2013], §31).

A few sections later, Frege gives an argument for Law I:

> Law I: According to §12, $\Gamma \supset (\Delta \supset \Gamma)$ would be the False only if $\Gamma$ and $\Delta$ were the True while $\Gamma$ was not the True. This is impossible; accordingly $\vdash a \supset (b \supset a)$ (Frege 1893 [2013], §18).

This certainly appears to be a straightforward reductio argument for the truth of Law I; his argument for Law IV is also a reductio (1893 [2013], §18). Not all of his arguments for his basic laws take this form, however. In his argument for Law IIa, he assumes that $\forall \mathfrak{a} \Phi(\mathfrak{a})$, and then explains how this implies the truth of $\Phi(\Gamma)$ for any arbitrary $\Gamma$. He then discharges the assumption, and generalizes (with Roman letters) over the $\Gamma$-place and $\Phi$-place, giving us the law $\vdash \forall \mathfrak{a} f(\mathfrak{a}) \supset f(a)$ (1893 [2013], §20).[5] Finally, for Laws V and VI, he simply asserts the laws after citing the relevant specification (1893 [2013], §18; §20).

Two observations. Firstly, Frege's arguments appear to be just that—arguments, ones which begin with statements about what values Frege's functions take for various inputs, and which culminate in a judgement, the acknowledgement of the truth of a thought expressed in Frege's concept-script. Secondly, these arguments are not themselves expressed in his concept script, but rather in a mix of natural language and concept-script augmented with Greek meta-variables.

Here's an expanded version of the argument for Law I, with some liberties taken in filling in some implicit steps:

Expanded Argument for Law I:

1. $\zeta \supset \xi$ is the False iff the $\zeta$-argument is the True and the $\xi$-argument is not the True [Premise]
2. $\zeta \supset \xi$ is the True iff it's not the case that the $\zeta$-argument is the True and the $\xi$-argument is not the True [Premise]
3. Assume $\Gamma \supset (\Delta \supset \Gamma)$ is the False [Assumption]
4. $\Gamma \supset (\Delta \supset \Gamma)$ is the False iff $\Gamma$ is the True and $\Delta \supset \Gamma$ is not the True [1]
5. $\Gamma$ is the True and $\Delta \supset \Gamma$ is not the True [3,4]
6. $\Delta \supset \Gamma$ is not the True [5]
7. $\Delta \supset \Gamma$ is the True iff it's not the case that $\Delta$ is the True and $\Gamma$ is not the True [2]
8. $\Delta$ is the True and $\Gamma$ is not the True [6,7]
9. $\Gamma$ is not the True [8]
10. $\Gamma$ is the True [5]
11. $\Gamma \supset (\Delta \supset \Gamma)$ is not the False [reductio, discharging 3]
12. Either $\Gamma \supset (\Delta \supset \Gamma)$ is the True or $\Gamma \supset (\Delta \supset \Gamma)$ is the False [premise]
13. $\Gamma \supset (\Delta \supset \Gamma)$ is the True [11,12]
14. $\vdash a \supset (b \supset a)$ [13]

The premises are drawn from his exposition of his function signs, where he tells us which values the corresponding functions take on when given certain arguments. In particular, he specifies what happens when functions do or don't take the True as an argument; accordingly, it's fair to say that he's taking us through truth conditions. And indeed, he later says that the sense of the

[5] For reasons of space, I shall for the most part use contemporary notation, using "$\forall$" instead of Frege's concavity stroke, and "$\supset$" instead of his conditional stroke, etc. However, I will adhere quite strictly to Frege's use of variables, since the distinction between German letter and Roman letter variables is important. In brief, the former are used to delineate scope by filling places which are to be generalized over via Frege's concavity stroke, whereas the latter always takes widest possible scope. I shall also adhere closely to Frege's use of Greek letters for meta-variables.

name of a truth-value is that the conditions under which it refers to the True are fulfilled (1893 [2013], §32); thus, it's clear that Frege's arguments, whatever else they may be doing, are taking us through the sense of his primitive laws.

A few notes on Frege's conception of truth: some have taken his view, particularly as articulated in the article "Thought" (1918 [1984]), to preclude the use of a truth predicate (Ricketts 1996; Kemp 1995). But others offer convincing arguments that Frege could, and did, make use of a truth predicate (Stanley 1996; Heck and May 2018, 208–12,; Shieh 2019, 89–96). This paper rests in part on the view that Frege was not opposed to truth-predicates in principle, but that he was merely opposed to the view that truth is fundamentally a property of thoughts (Heck and May 2018, 211) and that to judge a thought is to make a predication of truth. I do not wish to claim more than this, however. In particular, I do not claim that Frege was aiming to give his logical laws any sort of semantic grounding.

Note that Frege's argument reconstructed above, though couched in natural language, seems to proceed via logically valid rules. We get (5.) via modus ponens from (3.) and (4.), and we get (13.) from (11.) and (12.) via disjunctive syllogism. Since I make no claims for the unique appropriateness of my reconstruction, I make no claim that any particular one of these rules is essentially involved. However, it should be clear that Frege's arguments for his basic laws employ logical rules of a broadly familiar sort, albeit in the informal setting of natural language. Given this, it's natural to ask whether these arguments are circular, and, if so, just where the circularity lies, and how problematic it is. To answer this question, we need to look more closely at what Frege hoped to accomplish with his arguments, and in particular whether the passages in question contain an implicit argument for Frege's primitive laws being logical.

## 4. Four Readings of Frege's Arguments

Frege's arguments for his basic laws terminate with an acknowledgement of the truth of the laws in question. They do not explicitly result in judgements such as "Law I is purely logical". Yet there is every reason to think it's essential, for his purposes, that we come to see his primitive laws as logical. After all, the self-avowed point of *Grundgesetze* is to see what arithmetical principles rest on so we "gain a basis for an assessment of the epistemological nature of the proven law" (1893 [2013], vii). If we were left quite in the dark about the nature of his primitive laws, clearly reducing arithmetic to his logical principles would be of little help. This much is uncontroversial: what is far more controversial is whether Frege thought it was necessary, or even possible, to say anything substantive in favor of the logical nature of some particular law.

It will be illustrative to contrast four readings of how Frege's apparent arguments for his basic laws relate to their epistemological nature:

- Semantic Interpretation: Frege gives arguments which aim to justify the truth of his primitive laws, qua interpreted formulae (Heck 2012, 27–50),
- Logicality Interpretation: Frege gives arguments which aim to justify the claim that his primitive laws, qua thoughts, are logical.
- Elucidatory Interpretation: Frege's apparent arguments are, like the rest of his commentary on his symbolism, elucidations which familiarize us with his formal system, which among other things leads us to recognize that his laws are general and undeniable (Weiner 1990, 2010).
- Warrant Interpretation: Frege is engaged in an exhibition in which his reader's logical capacities are exercised in such

> a way that they are warranted in acknowledging that his primitive laws are true, and judging that they are logical (Shieh 2019, 229–30).

One point against the semantic interpretation is that if this was Frege's primary goal, one would have expected him to use uniform semantic ascent—to argue explicitly, on the basis of semantic specifications, that certain formulae *refer* to the true,[6] rather than that such-and-such *is* the true. This point is hardly decisive, however—again, Frege's specifications must be semantic in some sense, since they're used in an argument that every name refers (1893 [2013], §31). In any case, the semantic interpretation is, strictly speaking, consistent with the warrant interpretation, since they concern the relevance of Frege's arguments for formulae and for thoughts respectively.

The logicality interpretation suffers from circularity, and it's instructive to pinpoint just where the circularity lies. Since Frege does not explicitly conclude "Law I is purely logical", the logicality interpretation is committed to there being an implicit argument with premises concerning the §18 argument for the truth of Law I. The implicit argument is reconstructed below. In expressing it, I use "I/SP" as short-hand for "intuition or sense-perception"

1. The premises of the §18 argument for Law I are unreliant on I/SP. [Premise]
2. The rules used in the §18 argument are free of I/SP. [Premise]
3. The argument for Law I makes no use of I/SP. [1,2]
4. Anyone who grasps the thought that $\vdash a \supset (b \supset a)$ must be able to follow the argument given in §18 and acknowledge this law as true. [Premise]
5. Anyone who grasps Law I can, without any use of I/SP, acknowledge Law I as true. [3,4]
6. If anyone who grasps a law can, without any use of I/SP, acknowledge this law as true, then it is purely logical. [Premise]
7. Law I is purely logical.

The idea is this: in §18, Frege provides an argument for the truth of Law I, and a modicum of reflection upon this argument yields all the premises one needs to formulate the second argument, outlined above, which concludes that Law I is purely logical. But consider premise 2, the claim that Frege's rules are free of I/SP. Frege's treatment of his inference rules is of a piece with his treatment of his primitive laws. Take, for example, his treatment of modus ponens:

> From the propositions '$\vdash \Delta \supset \Gamma$' and '$\vdash \Delta$' one can infer: '$\vdash \Gamma$'; for if $\Gamma$ were not the True, then since $\Delta$ is the True, $\Delta \supset \Gamma$ would be the False (Frege 1893 [2013], §14).

As one would expect, Frege expresses the inference rule syntactically. His argument for the validity of this rule, however, consists of precisely the sort of reasoning about truth conditions which he gives for his primitive laws. Thus, if §18 can be read as an implicit argument for the logical nature of Law I, §14 can similarly be read as an implicit argument that modus ponens makes no use of I/SP.

We have a dual regress, one epistemological and one ontological (see Shieh 2019, 226–29 for a detailed discussion of the types of regress at play). On the one hand, the argument for the logicality of Law I gives us no epistemic mileage, since the status of Frege's inference rules is in as much or as little doubt as the logicality of his primitive laws—after all, he argues for modus ponens in much the same way as he does for Law I. This is the epistemological regress. There's also an ontological regress. Suppose the logicality of Law I follows from other truths via inference. Frege sees inference as preserving the objective justificatory ordering in which thoughts stand. This implies that the logicality of Law I is grounded in objectively more basic facts, such as that certain

[6] Or, to be more accurate, *indicate* the True, since his primitive laws contain Roman letters, and are thus Roman object-markers, rather than names of truth-values

rules are free of I/SP. If §14 can be read as an implicit argument that modus ponens is free from I/SP, then this fact will follow, via inference, from objectively more basic premises, and these premises will likely also say that certain *other* rules are free of I/SP. The logicality of Law I would be grounded in facts about rules, which would be grounded in yet more facts about other rules, and so on forever. This second regress is ontological, because it concerns the objective justificatory ordering in which thoughts stand to one another.

I take the dual regress as a conclusive case against the logicality interpretation in the strictest sense. The warrant view, however, gives space for an alternative role for Frege's arguments: they are not justifications at all, but rather a way of exercising the reader's inferential capacity to confer a special sort of warrant on their assertion of Frege's primitive laws.

## 5. Heuristic Indicators of Logicality

In the previous section, I argued, contra the logicality interpretation, that Frege is not, in §18, §20, and §25, trying to justify the epistemic status of his primitive laws on some deeper basis. In this section, I examine the different dimensions of the generality and undeniability of logic. This is necessary because Frege's exhibitions lead us to recognize the logical nature of his primitive laws precisely by making manifest that they have the sort of generality and undeniability which is distinctive of the purely logical.

In §14 of *Grundlagen*, Frege uses generality and undeniability as *heuristic indicators* of logicality. In other words, he uses them to make tentative determinations concerning whether geometrical or arithmetical truths are logical.[7] For instance, he takes as fact that we can assume the contrary of a geometrical principle without involving ourselves in contradictions to show that such principles are independent of logical truths, whereas for arithmetic "we have only to try denying any one of them, and complete confusion ensues. Even to think at all seems no longer possible" (1884 [1953], §14). Similarly, he suggests that arithmetic is "connected very intimately with the laws of thought" since it governs all that is numerable, which is the "widest domain of all", including "everything thinkable" (1884 [1953], §14).

This falls short of a conclusive case for the logical nature of arithmetic—if it was conclusive, he could have ended the book then and there. It is clear, however, that he is using undeniability and generality here as evidence of logicality. The question is then how his treatment of his primitive laws in his later work leads to a more conclusive case for logicality. In particular, we must ask whether Frege's strategy involves *more* than just reducing a more contentious case (logicality of arithmetic) to what he hoped was a less contentious case (logicality of his basic laws), and if so what this additional element consists of.

### 5.1. Generality

There are several senses, some very close together, of what one might mean by logic being general (see Proops 2007). A logical truth is *maximally general* if it contains only logical terminology and variables. Clearly not all logical truths are maximally general—"Hesperus=Hesperus" contains non-logical terminology, but Frege would have thought it logical nonetheless, as does modern classical logic (and indeed most logical systems). However, Frege's *primitive* logical laws (both those found in *Begriffsschrift* and those found in *Grundgesetze*) are maximally general. But even though maximal generality is a feature of expressions of primitive logical truths, it is useless as an *indicator* of logicality, as Frege says even less about what makes a constant logical than he does about what makes a truth logical.

[7] §§12–17 fall under the section "Are the laws of arithmetic synthetic a priori or analytic?" But of course to be analytic, as Frege uses the term, just is to be justified on the basis of purely logical truths (1884 [1953], §3).

Frege also took logic to be general in the sense of being *universally applicable*. He says that logical inference is "a way that, disregarding the particular characteristics of objects, depends solely on those laws upon which all knowledge rests" (1879 [1967], 5), and that it is "to be independent of all particular properties of things" (1885 [1984], 98/CP 114). Why did Frege take logic to be universally applicable? One answer is that this isn't an uncommon view, and he may have thought such universality is part of what one means when one calls a truth logical. But there is also a more illuminating answer, suggested by passages where Frege emphasizes the contrast between the logical and the more restricted domains of the intuitable and the perceptible. In a letter to Marty, Frege says:

> The field of geometry is the field of possible spatial intuition; arithmetic recognizes no limitation. . . the area of the enumerable is as wide as that of conceptual thought, and a source of knowledge more restricted in scope, like spatial intuition or sense perception, would not suffice to guarantee the general validity of arithmetical propositions (Frege 1980, 100).

This is another example of the heuristic indicator strategy—arithmetic is likely logical, because it has widest possible scope. And this issue of scope isn't merely a restatement of universal applicability, but a suggestion that logic is general in the deep sense of being *all-encompassing*. It's helpful here to think in terms of the "logical source of knowledge", a term used in Frege's correspondence (1899 [1980], 37) and which is implicit in his discussion elsewhere of tracking down sources of knowledge (1897 [1984], 363/CP 235; Shieh 2019, 201–2, 230). Frege's claim that logic is universally applicable can be traced to the all-encompassing nature of the logical source of knowledge, and it is this nature which figures most essentially in Frege's use of generality as a heuristic indicator of the logical nature of arithmetic.

### 5.2. Undeniability

At least at the time *Grundlagen* was published, Frege took primitive logical laws to be, in some important sense, undeniable—this much follows from his use, in §14, of undeniability as initial evidence for the logical nature of arithmetic. There's some controversy over whether he still thought laws were undeniable by the time he wrote *Grundgesetze*. The oft-quoted and much debated relevant passage is as follows:

> Stepping outside logic, one can say: our nature and external circumstances force us to judge, and when we judge we cannot discard this law—of identity, for example—but have to acknowledge it if we do not want to lead our thinking into confusion and in the end abandon judgement altogether. I neither want to dispute nor endorse this opinion, but merely note that what we have here is not a logical conclusion. What is offered here is not a ground of being true but of our taking to be true. And further: this impossibility, to which we are subject, of rejecting the law does not prevent us from supposing beings who do so; but it does prevent us from supposing that such beings do so rightly; and it prevents us, moreover, from doubting whether it is we or they who are right (Frege 1893 [2013], xvii).

I read this passage as affirming undeniability, but (rightly) refraining from endorsing the psychologistic claim that our nature leads us to endorse certain laws. A point in favor of my reading is that this passage succeeds a protracted criticism of psychologism. Some have read it instead as an abandonment of the claim that logical laws are undeniable (Baker and Hacker 1984, 44; Linnebo 2003, §4; Mezzadri 2015, 596–97; Pedriali 2019, 68–69), whereas others think, as I do, that Frege maintained undeniability throughout his career (MacFarlane 2002, 39; Taschek 2008, 385–86; Steinberger 2017, 151). Clearly, the matter cannot be decided on the basis of this passage alone—it is my hope that the account in this paper is compelling enough, and fits well enough with Frege's other commitments, that it indirectly lends support to my interpretation of this passage.

What does undeniability amount to? Distinguish between the following two claims:

- Rational undeniability: We mustn't deny logical laws if we want assured passage from truth to truth, since they concern how one should reason insofar as one cares about truth.
- Sense-analyticity: A sustained denial of a logical law must involve some defect in our grasp thereof.

Mere rational undeniability is weak—it just says that if we want to pass securely from truth to truth, we're at risk of failing to do so if we violate those principles governing the secure passage from truth to truth. If rational undeniability was *all* that was claimed for primitive logical laws, then Frege's use of undeniability as a heuristic indicator in *Grundlagen* reduces to so much foot-stomping—arithmetical principles are likely logical because they're rationally undeniable, but by saying that they're rationally undeniable, we're saying that they're laws of being true, which is just to say that they're logical.

### 5.3. Doubts about undeniability

Even if one accepts that primitive logical laws are undeniable in the stronger sense of being sense-analytic, this will be of less help as a heuristic indicator if one takes geometric axioms to be undeniable in the same way. Consider the following passage from Frege's 1914 "Logic in Mathematics":

> . . . so long as I understand the words 'straight line', 'parallel', and 'intersect' as I do, I cannot but accept the axiom of parallels. If someone else does not accept it, I can only assume he understands these words differently. Their sense is indissolubly bound up in the axiom of parallels (Frege 1914 [1979], 247).

This seems like a straight-forward claim that geometry is sense-analytic—anyone who understands geometrical terms in the same sense as we do must accept the axiom of parallels. Of course, "Logic in Mathematics" is an unpublished work, and where it conflicts with his published writings, we should, all else being equal, give precedence to the latter.[8] But Frege says something along the same lines in his "Foundations of Geometry II":

> If we take the words 'point' and 'straight line' in Mr. Hilbert's so-called Axiom II.1 in the proper Euclidean sense, and similarly the worlds 'lie' and 'between', then we obtain a proposition that has a sense, and we can acknowledge the thought expressed therein as a real axiom. . . Now if one has acknowledged [II.1] as true, one has grasped the sense of the words 'point', 'straight line', 'lie', 'between'; and from this the truth of [II.2] immediately follows, so that one will be unable to avoid acknowledging the latter as well (Frege 1906 [1984], 423/CP 333–34).

Frege uses [II.1] and [II.2] to mean Hilbert's "geometrical" axioms II.1 and II.2, but with "point", ''line", and ''between" taken in their ''proper Euclidean sense", rather than being taken as schematic and multiply interpretable. By "immediately follows", Frege does not mean that we get [II.2] from [II.1] via inference—we couldn't classify such a transition as inference in any case, since both [II.1] and [II.2] are primitive geometrical truths. His point is that if we grasp the Euclidean sense of "point", "straight line", etc, we are unable to avoid acknowledging these two truths: they are sense-analytic.

These two passages imply that both geometrical axioms and basic logical laws are sense-analytic—anyone who "denied" them would simply be assigning deviant senses to signs which

[8] What Frege says in 1914 about sense and analysis is also in tension with Frege's published writings. He says that we perform logical analysis by means of definitions with the consequence that the analysans must have the same sense as the analysandum, to which it must be self-evidently equivalent (1914 [1979], 209–10), which doesn't fit well with the examples of analysis he actually gives us elsewhere—see Blanchette (2012, 79–89) for a careful treatment of the issue.

figure in propositions meant to express them. The would-be deniers would fail to understand; in so doing they fail to judge, and consequently they fail to genuinely deny. Accordingly, the undeniability of logic must be stronger than *mere* sense-analyticity. So what must be added? We're told that "assuming the contrary" of any given geometrical axiom results in no contradictions, while "trying to deny" any given arithmetical principle will result in "complete confusion" (1884 [1953], §14), a point Frege reiterates a year later (1885 [1984], 94-5/CP 112).[9] Another crucial contrast with the geometrical case is that when we assume the contrary of a geometrical axiom, we suspend our faculty of intuition:

> Empirical propositions hold good of what is physically or psychologically actual, the truths of geometry govern all that is spatially intuitable, whether actual or product of our fancy. The wildest visions of delirium, the boldest inventions of legend and poetry...all these remain, so long as they remain intuitable, still subject to the axioms of geometry. Conceptual thought alone can after a fashion shake off this yoke, when it assumes, say, a space of four dimensions or positive curvature. To study such conceptions is not useless by any means; but it is to leave the ground of intuition entirely behind (Frege 1884 [1953], §14).

Again, it's helpful to put matters in terms of sources of knowledge. The geometrical source of knowledge is something which *can* be shaken off "after a fashion", while we can never shake off the yoke of logic, precisely because it is all-encompassing, and figures essentially in all our reasoning.

The upshot is that for generality and undeniability to be useful as heuristic indicators of logicality, they must be jointly present. Glossing generality in terms of logic's all-encompassing nature, and undeniability as sense-analyticity, we can combine them into a single indicator for ease of exposition:

[9]It will also presumably result in contradictions, the source of the confusion in question. But Frege thought this was something that needed demonstrating—showing that a denial of "1 + 1 = 2" results in a contradiction requires nothing less than a demonstration of the logical nature of this identity.

> Strong Sense-analyticity: Logical truths are recognizable as true by anyone who grasps them, in a way that doesn't involve intuition or sense perception.

This is merely a heuristic indicator of logicality, not some sort of deep analysis of what it is to be logical. The question is now whether it's just *obvious* that primitive logical laws are strongly sense-analytic, or whether some more conceptual legwork is required.

## 6. The Warrant and Elucidatory Interpretations: A Comparison

### 6.1. Commonalities

Many aspects of the elucidatory account of Frege's exposition on his symbolism are compatible with the present reconstruction. In a memorable turn of phrase, Thomas Ricketts says that Frege

> . . . lacks any general conception of logical consequence, any overarching conception of logic. Frege has only a retail conception of logic, not a wholesale one. He tells us what logic is by identifying specific laws and inferences as logical. The universalist conception of logic specifies the sort of content logical laws have; it does not state a defining criterion of the logical (Ricketts 1996, 124).

The success of my account does not require logic to have a defining criterion—as I will discuss in Section 7.1, Frege probably took logicality to be peculiarly fundamental. Indeed, although Frege says there are certain concepts proper to logic (1906 [1984], 428/CP 338) he never, as Ricketts himself points out (1997, 174), states any precise criteria for what counts as logical vocabulary. Richard Kimberly Heck gives some tantalizing suggestions about how Frege might have proceeded (2012, 38), and Aldo Antonelli and Robert May show how one could characterize logical notions within the framework Frege sketches in "Foundations of Geometry II" (2000, 258–62). But even if Frege could have provided sharp criteria, he didn't. And what's more, aside from his well-founded uneasiness surrounding Law V (1893 [2013], vii), there's every reason to believe Frege thought *Grundgesetze* successfully made the case for the logicality of arithmetic. For this to be so, he must have thought his work contained all that is needed for us to see his primitive laws as logical, even though he provides no analysis of what being logical amounts to, or indeed any formally precise criteria, beyond the necessary condition of being derivable within his system, for identifying logical laws.

Weiner's version of the elucidatory interpretation is particularly close to my own in several respects. She says:

> ... consider an example: the claim that every object is identical to itself. Since its truth is self-evident, it satisfies the first eligibility requirement for primitive laws. Supposing this to be a primitive law, is it analytic? In Grundlagen, Frege mentions two features of analytic truths. One is maximal generality... Another is that we cannot deny them in conceptual thought... The law that every object is identical to itself exemplifies both of these features. First, this law surely tells us, not just about every actual (spatio-temporal) object or every intuitable object, but about every object. Second, it seems we cannot deny it without involving ourselves in contradictions. Given these criteria, the law in question is analytic (Weiner 2010, 34–35).

Both my own account and Weiner's maintain that the epistemic status of primitive truths is settled once we know whether they are general and undeniable. The heuristic indicators are crucially involved in making Frege's foundationalist strategy more plausible—the claim is not that when we grasp a primitive law, we feel an inexplicable urge to label it as "logical" or "non-logical". Rather, in grasping such a law we are thereby in a position to see whether it has the sort of generality and undeniability required of logical laws (or, in my way of speaking, whether the thought satisfies the conjunctive condition of strong sense-analyticity discussed in Section 5.3). Frege's "arguments" for his primitive logical laws are not justifications, but rather careful articulations of content, which aim to bring us to a recognition of the generality and undeniability of his axioms. The difference between the warrant and elucidatory accounts lies in how such articulation brings about the recognition in question.

### 6.2. Differences

These similarities aside, the warrant and elucidatory interpretations differ in subtle but important ways. Firstly, Weiner uses the term "elucidation" in a much broader way than I use "exhibition". Exhibitions exercise our inferential capacities in order to warrant us in judging a thought as true or logical, where judging and inferring are taken in the less restrictive sense discussed in Section 2. Frege uses "elucidation" (*Erläuterung*) to mean informal explanations which help the reader to understanding certain aspects of his formal system. In his most extensive comments on the matter, Frege says elucidations help others understand logically primitive, indefinable elements (1906 [1984], 301/CP 300–1), and this fits with earlier comments where he says one must "make do with gesturing" at what indefinable primitive terminology means (1893 [2013], 4). However, as Weiner recognizes (2010, 60–61), Fregean elucidation is used for more than just logically simple terms—defined terms can have elucidations (1893 [2013], §35) as can complex terms in which they figure (1893 [2013], §34). Weiner argues that elucidation is more prevalent still—she says that much of Frege's discussion of his basic laws, including his §31 argument that every name in his language has a reference, is elucidatory (1990, 229, 239–43), as is his discussion of the concept/object distinction (1990, 246).

The substantive issue is not whether exhibitions should be labeled "elucidation", either as that term is understood by Frege or interpreted by Weiner. The real question is whether Frege's apparent arguments for his basic laws serve the same function as other parts of his exposition. Much of this exposition plays just the role Weiner supposes, familiarizing us with his formal system and thus sharpening our understanding of thoughts expressed within it. I maintain that Frege's arguments for his basic laws serve a different goal, and should be treated differently. If getting us thoroughly familiarized with his formalism was all that was necessary to convince us that his primitive laws were logical, he could have met this goal while omitting his apparent arguments for his basic laws, and arguments of the sort found in §10 and §§29–31.

This brings us to the second difference, which concerns just what is needed for us to judge that Frege's primitive laws are strongly sense-analytic. Consider the following view:

> *UL*: Frege believed that primitive logical laws are *obviously* general and undeniable to anyone who fully understands them.

Weiner is committed to UL, explicitly maintaining that the proper epistemic categorization of primitive truths is obvious once their content is fully understood (1990, 46, 53–54; 2005, 346; 2010, 58). I reject UL—anyone who fully understands a primitive law recognizes it as true, but such full understanding may still fall short of guaranteeing a recognition of logicality. What more there is to be done, over and above bringing readers to full understanding, will depend on the law in question. Sometimes, what is needed is very little. Consider Law VI: after giving specifications in §11 governing his "$\backslash\xi$" operator, Frege simply states the law in which it figures, and claims that it follows from the reference of the function-name (1893 [2013], §18). We can reconstruct this very brief argument as follows:

1. If $\xi = \dot{\varepsilon}(\Delta = \varepsilon)$ then $\Delta = \backslash\xi$ (§11 specification)
2. $\dot{\varepsilon}(\Delta = \varepsilon) = \dot{\varepsilon}(\Delta = \varepsilon)$ (Identity)
3. $\Delta = \backslash\dot{\varepsilon}(\Delta = \varepsilon)$ (From 1,2 Modus ponens)
4. $\vdash a = \backslash\dot{\varepsilon}(a = \varepsilon)$ (From 3)

This argument is not, of course, inference in the restrictive justificatory sense, for the reasons I outlined in Section 4—it is an exhibition which exercises our inferential capacities in order to warrant our acknowledgement that Law VI is true. To recognize that VI is logical, what must be added is the additional step of reflecting upon the argument above. The sort of reflection in question will essentially involve forming judgements about the reasoning itself—judgements such as "Frege's argument for VI involves no use of I/SP". We could read the above argument, fully understand each step, and nevertheless fail to recognize VI as logical if we did not go through the additional stage of reflecting upon it. The gap, however small, between understanding a basic law and recognizing it as logical must be bridged by *judgements* about the sort of inferential activity which leads us to recognize the truth of the law in question.

Such an account essentially requires a meta-perspective. While everyone agrees Frege had a meta-perspective in at least the weak sense of using natural language to introduce and discuss his formal system (see for instance Weiner 2010, 58), the present account makes the bolder claim that Frege's way of warranting us in judging that his primitive laws are logical essentially involves us being led through a particular sort of non-justificatory reasoning about such laws. Of course, the passages of *Grundgesetze* I have called exhibitions are expressed in natural language, rather than the concept-script. While exhibitions needn't be formalized to be effective, this needn't render them unformalizable. Indeed, in his later writings, Frege makes it clear he's open to there being a science, compete with its own laws, which treats of thoughts, and could be used in independence proofs (1906 [1984], 423–30/CP 333–40; Antonelli and May 2000).

The key motivation of the warrant interpretation is to explain how we are rationally warranted in judging that Frege's primitive laws are true and logical. In an oft quoted remark, Frege says that it is precisely at the unprovable foundation of chains of inferences that "epistemology comes in" (1879-91 [1979], 3). Whatever we take "epistemology" to consist in here, it had better not be rhetoric. To reiterate a point from Section 2, part of the task is to explain what would be missing if Frege's arguments for his basic laws were flawed, or replaced with some non-rational means of jolting us into an acknowledgement of truth. Let us grant that primitive laws are basic, and permit no justification on the basis of other truths; let us even grant that logical laws are so deeply bound up with understanding that they figure implicitly in our grasp of any thought whatsoever. What's crucial is that the way in which Frege brings us to judge his axioms involves inferential capacities, capacities which are recognizably logical, even if he isn't engaged in inference in the strong justificatory sense.

## 7. Warrant and Inferential Capacities

### 7.1. A Two-Stage Exhibition

The view I'm putting forward[10] is that Frege's arguments for his primitive laws have precisely the structure they appear to have, complete with "premises" and a "conclusion",[11] but not the role which they appear to have. They really are arguments which proceed from specifications (*Bestimmungen*) to primitive laws via logically valid rules, but they are not examples of inference in the strict justificatory sense.

For any given primitive law, the way in which Frege gets us to recognize its logicality will have two distinguishable stages. Stage 1 is an exhibition which consists of an argument which leads to a recognition of the truth of that primitive law—these are what Frege provides for laws I, IV, and VI in §18, for laws IIa, III, and V in §20, and for law IIb in §25. Stage 2 consists both of the argument which figures in stage 1, and an implicit reflective component. This implicit component is schematized below: again, "I/SP" is short-hand for "intuition or sense perception".

Implicit Component of Stage 2

1. The "premises" of the argument for $\Theta$ are unreliant on I/SP. ["Premise"]
2. The rules used in the argument for $\Theta$ are free of I/SP. ["Premise"]
3. The argument for $\Theta$ makes no use of I/SP. [1,2]
4. Anyone who grasps the thought that $\Theta$ must be able to follow the argument for $\Theta$, and acknowledge this law as true. ["Premise"]
5. Anyone who grasps $\Theta$ can, without any use of I/SP, acknowledge $\Theta$ as true. [3,4]
6. If grasping a law is sufficient, without any use of I/SP, for acknowledging it as true, then that law is logical. ["Premise"]
7. $\Theta$ is purely logical. ["Conclusion"]

Note that Frege's arguments for his primitive laws will figure in *both* stage 1 and stage 2, playing a different role in each stage. In the first stage, it warrants us in acknowledging that $\Theta$ is true, while in the latter case, the addition of the implicit component above lets the original exhibition lead us to a recognition of the logical nature of $\Theta$.

This looks just like an attempt at justifying the claim that $\Theta$ is logical—indeed, it's a schematized version of the argument for the logicality of Law I which I discussed in Section 4. This yields a puzzle. Stage 2 had better not be a justification, since we already rejected the logicality interpretation on grounds of circularity. But denying that this is a justification seems odd, more so than denying the same of his arguments for primitive laws. It was never an option to see Frege's arguments for primitive laws as justifications of those laws qua thoughts, for the obvious reason that if they could be so justified, they wouldn't be primitive. In

[10] This view, and some of the terms with which it is expressed, is inspired by that of Sanford Shieh (2019, 229–30), and also owes something to Barry Stroud's "Inference, Belief, and Understanding" (1979).

[11] The use of scare quotes here and below is just to emphasize that while the apparent arguments are exercises in inferring, they are not cases of conclusions being justified on the basis of premises.

the case of arguments for the logicality of a law, however, one might, aside from worries about circularity, think it plausible that the conclusion really does rest on statements about the absence of intuition and perception.

However, denying that stage 2 is a justification isn't really so odd, for there are independent reasons to avoid taking it as justificatory. Firstly, if stage 2 is a justification, then the logicality of Θ isn't something internal to it, but rather due to the status of some other thoughts or rules. Secondly, if the conclusion is objectively justified on the basis of the premises, then logicality is a derivative feature, rather than something *sui generis*. This is surprising, because while Frege nowhere says that "Law I is logical" is a primitive truth, comments in "Foundations of Geometry II" suggests that the distinction between the logical and the non-logical is peculiarly basic. There, Frege considers a science in which one could demonstrate the independence of thoughts, and provides some tentative ideas about what basic truths could serve as axioms for this "new realm" (1906 [1984], 426/CP 336). One would need a law which serves as "an emanation of the formal nature of logical laws" (1906 [1984], 426/CP 337), and he affirms that this law would be basic (1906 [1984], 429/CP 339). In stating this law, we would have to determine what counts as logical inferences and what is "proper to logic", i.e., what counts as logical terminology (1906 [1984], 429/CP 339). Frege stops short of saying that these determinations *must* be taken as basic, but he does go as far as to say that it will "probably have to be taken as axiomatic" that the concept *point*, and relation of a point's lying in a plane, do not belong to logic (1906 [1984], 429/CP 339). Although he's tentative here, it certainly suggests that he took the distinction between the logical and non-logical to be basic. All the more reason to see stage 2 as an exhibition, rather than a justification.

### 7.2. Grasp of thoughts and willingness to infer

The implicit component of stage 2 is not a justification. However, it is still no good if its "premises" are false. (1.), (2.), and (5.) are plausible, although more must be said later about how stage 2 avoids the reappearance of the epistemic regress. (4.)—that anyone who grasps Θ must be able to follow Frege's argument for Θ—is crucial, but perhaps more controversial. Thinking through just what (4.) involves provides one of the linchpins of the present reconstruction—the reason exhibitions effectively lead us to an acknowledgement of truth and logicality is precisely because there is a sense in which logical laws are implicit in judgement. I'll now focus on this connection, in order to shed light on the way in which Frege's exhibitions let us recognize his primitive laws as strongly sense-analytic, and hence logical.

Let's proceed by considering a particularly enigmatic comment of Frege's. In "Logic", a fragment of an unfinished textbook written sometime between 1879 and 1891, Frege says that the logical laws of logic are "an unfolding of the content of the word 'true"' (1879-91 [1979], 3), a sentiment he repeats a few decades later (1918 [1984], 59/CP 352). Since logical laws neither define truth (1897 [1997], 228; 1906 [1979], 174; 1918 [1984], 60/CP 353), nor contain truth predicates, the only sense in which they could unfold the content of "true" is if they bring to the surface what is implicit in our recognition of truth, in a sense which must be teased out. Frege thought that truth-values are implicitly recognized by anyone who judges (1892 [1984], 34/CP 163), so logical laws, if they are to reveal what is implicit in our recognition of truth, must be implicit in judgement. Logical laws should also be importantly connected to our grasp of thoughts, since judgement is the acknowledgement of the truth of a thought (1918 [1984], 62/CP 356), and one must grasp a thought before it can figure in our judgement. Thus, in order to properly understand the way in which logical laws connect to truth, we must appreciate how they figure implicitly in our judgement and our grasp of thoughts.

Reading Frege as committed to logical laws being implicit in judgement and understanding is not unique to the warrant interpretation.[12] In particular, Joan Weiner reads Frege's view of conceptual content as suggesting that "the primitive laws of logic must be inextricably bound up in our understanding of the content of our judgements" (1990, 77). She provides a convincing account of how this "common understanding of correct inference" underlies the efficacy of elucidations of logically simple elements (1990, 237), and elsewhere suggests that this understanding/inference connection lets us understand Frege's treatment of primitive laws (2010, 57–58). Though similar in many respects, the warrant and elucidatory accounts still differ in the ways discussed in Section 6.2. One of the purposes of the present reconstruction is to carefully separate two strains of thought which are commonly run together. On the one hand, we have the view that Frege takes understanding and communication to be deeply bound up with logical laws, and that somehow he is exercising our inferential capacities in order to get us to recognize something. On the other hand, there's the view that Frege's conception of logic allows for no meta-theoretical perspective. The warrant account incorporates the former view, while rejecting the latter.[13]

I claim that explaining the sense in which Frege thought logical laws were implicit in judgement involves the following:

> *INF*: If one is not willing and able to infer with the content of $\Psi$ and its analysed constituents, then one fails to grasp the content of $\Psi$.

In the sense intended here, ability to infer with $\Psi$ or its subexpressions is consistent with performative failures on particular occasions. I wish to say for ability what Chomsky says for his competence/performance distinction—he says we abstract away from "irrelevant conditions as memory limitations, distractions, shifts of attention and interest, and errors" (Chomsky 1965, 3). We should understand inferential ability in the same way.

To properly understand INF, it's important to recognize the interpretive background commitments upon which it relies, commitments which are broadly Dummettian. As suggested by the use of "analysis" in the statement of INF, I'm committing myself here to Dummett's familiar distinction (1973, 27–33; 1981, 261–89) between the *constituents* into which a thought can be *analysed*, and the *components* into which it may be *decomposed*. Some, however, have taken issue with the distinction between analysis and decomposition—see Sluga (1980, 90–95) and Bell (1981, 220) for criticisms of the view that thoughts come with a set, articulated structure, Levine (2002) for a critical discussion of the analysis/decomposition distinction, and Sullivan (2010, 97–104) for a defense.

The importance of analysis is this: there is a sense in which the arguments which Frege provides for his primitive laws are keyed to the structure of the formulae expressing these laws.[14] To choose one example, Law I, $\vdash a \supset (b \supset a)$, consists of two conditionals, one nested in the consequent of the other. Implicit in Frege's reductio argument for its truth is an acknowledgement of what the semantic values of analysed constituents of the law

[12] Daniele Mezzadri provides a helpful overview of the issues at play, and argues that such an understanding/inference connection is consistent with Frege's anti-psychologism and his conception of the normative role of logic (2018, 738–41).

[13] The understanding/inference connection ties in naturally with accounts emphasizing sense-analyticity or the constitutivity of logical laws for thought. Walter Pedriali raises a concern that such accounts fail to leave enough of a gap between a grasp of a thought and a recognition of the validity (*Gültigkeit*) of that thought, and maintains that this gap is needed to make sense of someone taking a critical stance concerning the extent of the validity of a primitive law (that is, whether it holds generally, or is restricted to the intuitable or perceptible) (2019, 67, 72). The present account sidesteps this concern, since, as mentioned in Section 6, I deny that the extent of the validity of a primitive law will be obvious to anyone who fully understands it.

[14] I should also note that it is not essential for the warrant interpretation that analyses be unique—it may be that there are other ways of analysing Frege's basic laws and, accordingly, different ways of formulating exhibitions, keyed to the structure of these analyses, which also lead to an acknowledgement of truth. But for the present purposes, all that is needed is, for each primitive law, one such analysis.

imply for the semantic values of analysed sub-constituents. If $\Gamma \supset (\Delta \supset \Gamma)$ were the False, $\Gamma$ would be the True and $\Delta \supset \Gamma$ would not be the True; the later implies that $\Delta$ is the True and $\Gamma$ is the not the True. And moreover, anyone who couldn't reason from the falsity of a Fregean conditional to the truth of its antecedent wouldn't understand the expression in which the conditional figures.[15]

If this is so, we have a straightforward sense in which inferential capacities are implicit in understanding—they figure in our implicit recognition of what it is that the values of simpler constituents imply about the values of more complex analysed parts of the truth in question. I shall refer to any inferring with which we must be willing and able to carry out in order to grasp a given $\Psi$ as *requisite inferring* of $\Psi$. The requisite inferring of $\Psi$ may include certain transitions to and from $\Psi$ itself—being unwilling/unable to use modus ponens, for instance, would disqualify one from having understood a conditional claim which constitutes the major premise. But requisite inferring should also be construed broadly enough to encompass the sorts of transitions that figure in the paragraph above, which contain truth operators "is the True" and "is the False" and schematic variables such as "$\Gamma$" and "$\Delta$".

As the present account is already a reconstruction, we may leave it a bit open ended just which inferring will be requisite; it certainly need not be limited to something very specific such as introduction and elimination rules. For the argument of this paper, it suffices that the sorts of moves Frege makes in his arguments for his primitive laws are all plausible candidates for requisite inferring with respect to the laws in question. Nevertheless, the notion could be precisified in various ways. If we accept the compositionality of sense,[16] then grasping $\Psi$ requires that we also grasp those sub-expressions which we arrive at in the course of analysing $\Psi$. This gives us the following constraint:

> *Restriction 1*: Requisite inferring for sub-expressions is also requisite inferring for expressions in which they figure as analysed parts.

For example, $\exists \mathfrak{a} G\mathfrak{a}$ is an analysed constituent of $\forall \mathfrak{b} F\mathfrak{b} \wedge \exists \mathfrak{a} G\mathfrak{a}$. Since we must understand the former to understand the latter, requisite inferring for former must be be requisite inferring for the latter.

Another possible restriction would be to follow even more closely the analysis/decomposition distinction as understood by Dummett:

> *Restriction 2*: Requisite inferring for $\Psi$ consist only of transitions between expressions in which analysed constituents of $\Psi$ or their negations figure in the appropriate way.

This restriction needs, of course, to be stated more precisely so as not to disallow, among other things, inferring with:

- expressions in which Roman letter variables occurring in analysed constituents are replaced with Greek meta-variables
- expressions involving the operators "is the True" and "is the False"

[15] Frege's argument is also keyed to Law I's generality. Heck argues that the meta-variables $\Gamma$ and $\Delta$ which figure in Frege's arguments are "a formal device, a new name, added to the language, subject only to the condition that it should refer to some object in the domain" (2012, 57–58). In effect, this allows Frege the use of objectual quantification when talking about his concept-script—it is the meta-theoretic analogue of Roman letter generality.

[16] While there is general agreement that Frege took reference to be compositional, it's more contentious whether he took sense to be so as well. Sceptics point out that Frege did not explicitly state the principle (Pelletier 2001, 91–92), and some emphasize that thought comes without intrinsic articulation and has structure read into it (Sluga 1977, 239; Bell 1981, 220; Haaparanta 1985, 75). See Klement (2002, 85–88) for a defense of the compositionality of sense. Pelletier (2001) also contains an illuminating discussion, as well as additional references.

- expressions which, like Frege's specifications in §§5–13, speak of argument-places of functions which figure in Ψ as an analysed constituent

- negations of analysed constituents

But after some sharpening, restriction 2 follows from (a.) the fact that those decomposed components of Ψ which are not also analysed constituents need not figure in our grasp of Ψ and (b.) the additional claim that one cannot grasp Ψ without also grasping ¬Ψ. (b.) follows from Frege's view of content—we first grasp the content expressed by a formula, and, prior to making a judgement, can question whether it is true or false (1879 [1967], §2; 1918 [1984], 60/CP 353; 1919 [1979], 253). Since, for Frege, the falsehood of Ψ just amounts to the truth of ¬Ψ (1893 [2013], §6), we could not question whether Ψ is true or false if we did not also grasp ¬Ψ.[17]

If we accept INF, then we have a concrete sense in which logical rules are closely tied to an implicit appreciation of what the referents of subsentential expressions in a sentence Ψ contribute to the truth or falsity of Ψ, since, at least in the case of primitive laws, each step in tracing this contribution will plausibly involve requisite inferring. Since grasping the thought is, for Frege, a necessary part of acknowledging it as true, logical laws are tied to judgement—if we can't or won't infer rightly, we do not grasp, and if we do not grasp, we do not judge. This gives us a sense in which logical laws unfold the content of "true": it is because when we acknowledge a thought as true, i.e. judge, we must grasp the thought judged as true, and thus be willing and able to infer.

[17]Frege did not seem to anticipate the difficulties that arise when his treatment of falsity is coupled with his claim that sentences with bearerless names lack truth-values (1892 [1984], 33/CP 162–63) and his claim that "*P*" and "It is true that *P*" have the same sense (1918 [1984], 61/CP 354). It's not difficult to show, under these assumptions, that truth gaps are also truth gluts. A detailed discussion of such problems can be found in Milne (2010).

### 7.3. Why Exhibitions Succeed

We're now in a position to say just how it is that Frege's exhibitions succeed in their task. Let's begin with stage 1. When we follow Frege's arguments for his primitive laws, we are led to reflect on the analysis of these laws (or, hedging, *an* analysis). This reflection consists in requisite inferring. We must be in a position to do requisite inferring since, according to INF, anyone not willing and able to infer in the required way wouldn't understand the law in question. The requisite inferring which Frege leads us through is keyed to the analysed structure of the primitive law under consideration, and can be said to manifest the way in which the values of simpler constituents contribute to the values of more complex constituents. At the end of stage 1, we are warranted in judging Frege's primitive laws as true, not because we have given them some deeper grounding, but because our inferential capacities have been appropriately exercised. What is essential for the success of this exhibition is the reasoning through which we are led; our warrant in judging a primitive law as true is not to be found in our arrival at understanding, but in the very manner in which the exhibition articulates our understanding via inferring.

Now for stage 2, which consists of Frege's arguments for his basic laws together with the implicit reflective component reconstructed in Section 7.1. Frege gets us to judge that his primitive laws are logical by getting us to judge that they are undeniable and general, conditions which, following the discussion in Section 5 and for ease of exposition, can be sharpened and combined in the condition of strong sense-analyticity. Surveying a Fregean argument for some primitive law Θ, we come to appreciate how this argument led, in stage 1, to an acknowledgement of its truth. But the rules used in this argument are free of I/SP. Since these rules are being used in requisite inferring, and his arguments articulate an understanding of the law in question, his argument exhibits how an understanding of Θ leads to a recognition of its truth, without any admixture of I/SP. The law is strongly sense-analytic—once we have reflected on the exhibition for $\Theta$ and judged that this is the case, we are warranted in judging that $\Theta$ is logical.

Understanding how stage 2 avoids regress tells us something crucial about how such exhibitions work. In Section 4, I said that the logicality interpretation suffered from both an ontological and an epistemological regress. The ontological regress drew essentially on Frege's foundationalist structure of inference. Since neither Frege's arguments for his primitive laws nor the implicit component of stage 2 should be understood as "inference" in the strict sense subject to Frege's foundationalism, the ontological regress cannot arise. The epistemological regress isn't a problem for stage 1—it is sufficient that the premises and rules in Frege's arguments *be* free of I/SP, with no requirement, at that stage, that we *know* them to be such. But what of stage 2? Let us grant that the claim some $\Theta$ is logical isn't grounded in facts about some rule which figures in a Fregean argument for $\Theta$. Still, aren't we on a regress if the judgement that $\Theta$ is logical must succeed a judgement that some rule, such as modus ponens, is free of I/SP? After all, Frege gives an argument for the validity of modus ponens in §14, which itself makes informal use of logical rules—if judgements that these rules are free of I/SP must precede judgements that modus ponens is free of I/SP, then we're on a regress.

However, the regress can end after one step. Although the logical nature of $\Theta$ is not *grounded* in facts about rules which figure in Fregean arguments for $\Theta$, it is indeed true that the implicit component of stage 2 requires us to judge that certain rules are free of I/SP before judging that $\Theta$ is logical. My suggestion is that we can begin with the judgement that certain rules are free from I/SP, because this is, in the context of exhibitions, transparent. When these rules occur in the context of a Fregean argument for primitive laws, they are instances of inferring, and it is only when in use as an instance of inferring that their epistemic status is transparent.

There are different senses in which one can speak of an inference rule. There is a syntactic sense—given contentful formulae with a certain structure, we can infer another contentful formula with a certain structure. But this is not the only sense. In *Begriffsschrift*, Frege says that

> the truth contained in some other kind of inference can be stated in one judgement, of the form: if $M$ holds and if $N$ holds, then $\Lambda$ holds also, or, in signs, $\vdash N \supset (M \supset \Lambda)$ (Frege 1879 [1967], §6).

When he speaks of "truth contained in an inference rule", he's referring to a corresponding conditional statement, and not to some truth about syntax, such as "From '$M$' and '$N$" we may infer '$\Lambda$' ". Elsewhere he says that he seeks

> as far as possible to translate into formulae everything that could also be expressed verbally as a rule of inference, so as not to make use of the same thing in different forms (Frege 1880-81 [1979], 37).

Peter Sullivan argues that these passages imply that there is, for Frege, no deep extra-systemic significance between primitive laws and inference rules (2004, 673–74), though Frege himself points out that the distinction must be present within a particular axiomatization of logic (1879 [1967], §13). This is worth taking quite seriously, and ties in with my previous suggestion—arguments such as those in §14 can help us see the epistemic status of modus ponens when it is expressed in the form of a judgement about a conditional, but when modus ponens occurs in an act of inferring, its epistemic status is transparent.[18]

[18]In most cases it's quite clear which inference rule would correspond to a given primitive law. In *Grundgesetze*, Roman letters figure in every such law. Laws I-IV have conditionals as their main connective, and correspond to rules which would let you infer a thought with the same form as the consequent from a thought with the same form as the antecedent (we can easily characterize form by replacing the Roman letters by the upper-case Greek letters Frege uses when discussing inference rules). Law V corresponds to a rule that lets

If this is so, we can now say something more precise about UL, which we discussed in Section 6.2. We can understand a primitive law such as $\vdash \forall \mathfrak{a} f(\mathfrak{a}) \supset f(a)$ without appreciating that it is logical precisely because it is expressed in the form of a judgement. But if the primitive law/inference rule distinction has no extra-systemic significance, then in an alternative axiomatization of logic, we could take the rule expressed by Law IIa instead as an inference rule. If this inference rule then figured informally in an Fregean exhibition/argument for some other axiom, its freedom from I/SP would be a transparent matter. Thus, there is a qualified sense in which UL is correct—any primitive logical law would be transparently logical if it were, in some axiomatization, used as an inference rule. However, once one turns one's attention to the law itself, one is not seeing it in action, as it were, but apprehending it as a judgement. The question "Is Law Θ logical?" arises precisely in contexts outside of inferring, contexts where its logical nature is not transparent. It is then that we follow Frege's exhibitions, which, precisely by inferring, lead us to judge Law Θ as logical.

As discussed in Section 5.1, Frege believes that "logical source of knowledge" is of widest possible scope—"as wide as conceptual thought", unlike the more restricted spatial intuition or sense perception (1980, 100). Shieh encourages the following analogy: just as spatial intuition figures in warranting our acknowledgement of the truth of geometrical axioms, so the logical source of knowledge figures in warranting our acknowledgement of primitive logical laws. To warrant us in acknowledging the truth and logicality of a primitive logical law,

> what are required are exercises of the capacity that is, or is associated with, the logical source of knowledge (Shieh 2019, 230).

According to my variant of the warrant interpretation, the capacity in question is our capacity to infer, and Fregean exhibitions draw on the logical source of knowledge precisely within contexts in which we are inferring. It is because of the logical source of knowledge that the epistemic status of Frege's rules is transparent. And this does justice to Frege's conception of logic as having widest possible scope—the logical source of knowledge is there whenever there is inferring, and, on the account I've given, articulate understanding is inseparable from such inferring. Since one judges as true only content which one grasps, the logical source of knowledge will always be in the background: it is all-encompassing in a deep sense.

### 7.4. Why some exhibitions fail

Of course, Frege's arguments for his basic laws do not always succeed as exhibitions leading to an acknowledgement of their truth and a judgement that they are logical. This is, notoriously, the case with Law V:

$$\vdash (\dot{\varepsilon} f(\varepsilon) = \dot{\alpha} g(\alpha)) = (\forall a (f(a) = g(a)))$$

Frege had already had doubts about this law (1893 [2013], vii), which were well-founded, as he came to realize (1903 [2013], 253). Not only does this law fail to be logical—it fails to be true as well. So where did Frege go wrong? Thankfully, what is required here is not some deep diagnosis of what kind of misunderstanding V embodies, but just an explanation of why Frege's error doesn't threaten the general strategy for warranting his readers in certain judgements.

Whereas for laws I-IV, Frege takes us through their content, for V and VI, Frege simply cites the passages in which the relevant logical vocabulary (§3 and §§9–10 for "$\dot{\varepsilon}\Phi(\varepsilon)$", and §11 for "$\backslash\xi$") is specified and discussed. His specification of the meaning of "$\dot{\varepsilon}(\Phi(\varepsilon))$" begins in §3 with an informal statement of Law V:

> I use the words "The function $\Phi(\xi)$ has the same *value-range* as the function $\Psi(\xi)$" always as co-referential with the words "the functions $\Phi(\xi)$ and $\Psi(\xi)$ always have the same value for the same argument" (Frege 1893 [2013], §3).

you move to and from value-range identities. Law VI would be a principle of substitution which allows you to add and remove terms using the "\" operator.

This is false: Frege did not use these as co-referential, because they cannot, on pain of contradiction, be so used. In §9, he begins by formalizing one side of the false equivalence:

> If $\forall\mathfrak{a}\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$ is the True, we can, according to our previous specification §3, also say that the function $\Phi(\xi)$ has the same value range as the function $\Psi(\xi)$ (Frege 1893 [2013], §9).

Frege then spends the rest of §9 explaining, as he did when introducing his concavity stroke in §8, how to determine, given an application of the second-level function "$\dot{\varepsilon}\Phi(\varepsilon)$", which first-level function $\Phi$ it is being applied to. If the specifications given in §9 properly secured a reference for the second-level function "$\dot{\varepsilon}\phi(\varepsilon)$" function, then Frege's argument for V, though trivial, would have been adequate as an exhibition. There simply isn't that much to do to get us to see that V is strongly sense-analytic, since it is no more than a restatement, within the concept script, of the specification in question. The same is true for VI—his argument for it, though trivial, is perfectly adequate provided Frege has properly secured a reference for the "$\backslash\xi$" operator which occurs within it.

However, Frege was simply mistaken in thinking that he had secured a reference for the function-sign "$\dot{\varepsilon}\Phi(\varepsilon)$". According to §29, to know whether "$\dot{\varepsilon}\Phi(\varepsilon)$" refers, we need to know whether "$\dot{\varepsilon}\phi(\varepsilon)$" refers for every referring first-level function-name "$\phi(\xi)$". But to know whether "$\dot{\varepsilon}\phi(\varepsilon)$" refers, we must know that "$\psi(\dot{\varepsilon}\phi(\varepsilon))$" refers for any referring "$\psi(\xi)$". And while this last condition is met when "$\psi(\xi)$" is a primitive first-level function-name such as Frege's horizontal stroke, negation stroke, or backslash operator, proving that it holds for *any* "$\psi(\xi)$" can't be done—inductive arguments to this effect are inescapably circular (Heck 2012, 79–81; Dummett 1991b, 218–19). Lack of proof of reference is not proof of lack of reference, but we should bear in mind that what Frege says in his §3 and §9 specifications is *false*—we cannot, on pain of contradiction, use identities between value-range terms as co-referential with formulae like "$\forall\mathfrak{a}\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$". Unless Frege believed that successful specifications could be false, he must, after discovering the contradiction, have recognized that he failed to secure a reference for "$\dot{\varepsilon}\Phi(\varepsilon)$".

Since "$\dot{\varepsilon}\Phi(\varepsilon)$" fails to refer, so does any expression in which it occurs—including Law V. So, does "$\dot{\varepsilon}\Phi(\varepsilon)$" have a sense? Frege does allow for referentless terms to have sense—he claims that "Odysseus" has a sense, but no referent (1892 [1984], 32–33/CP 162). However, "Odysseus" is a term in natural language, and "$\dot{\varepsilon}\Phi(\varepsilon)$" is a term in a formal language. This matters, because Frege is quite explicit that "scientific rigor" requires we make sure we are never reasoning with referentless signs (1891 [1984], 19/CP 148) and that reference failure is a defect of unrigorous symbolic systems (1892 [1984], 40/ CP 168–69). Secondly, Frege was broadly descriptivist about terms like "Aristotle", 1892 [1984], 27 n4/CP 158, whereas "$\dot{\varepsilon}\Phi(\varepsilon)$" has a totally different character—it's a primitive name. Nor can the structure of "$\dot{\varepsilon}\Phi(\varepsilon)$" be part of its sense—as a primitive second-level function-name, it's syntactically simple. Thus, unlike "Odysseus", where we can cash out the sense in terms of descriptive content, it's unclear what the sense of "$\dot{\varepsilon}\Phi(\varepsilon)$" could be, other than the way it picks out its referent—if that's the only option, then where the referent of "$\dot{\varepsilon}\Phi(\varepsilon)$" is lacking, sense is lacking as well. Under the plausible assumption that a sentence cannot express a thought if one of its constituent terms is senseless, Law V does not, despite initial appearances, express a thought. This is where Frege's exhibition goes awry: he couldn't have led us through its sense, because there is simply no sense through which to be led.

## 8. Conclusion

The central point of *Grundgesetze* was to demonstrate that "nothing but logic forms the basis" of arithmetic (1893 [2013], vii). Frege gives gapless chains of inference in order to catalog every presupposition on which arithmetical laws rest so that we "gain a basis for an assessment of the epistemological nature of the proven law" (1893 [2013], vii). Doubts about Law V aside, he believed that he'd succeeded in his task. But Frege couldn't have thought this unless he thought we were warranted in acknowledging his primitive laws as true, and judging them as logical. Since he does not explicitly argue that these primitive laws are logical, what warrants us in judging them as such must be implicit. Though necessarily conjectural at points, this paper attempts to reconstruct what this implicit case could amount to by showing how the tools which Frege provides, and the way in which he sets up his arguments for his primitive laws, give his readers all the resources they need to warrant them in judging that his primitive laws are general and undeniable in a way which is inseparable from logicality.

The picture that emerged was as follows. Our grasp of the sense of formulae is deeply tied up with our willingness and ability to infer via logically valid rules. This gives deep substance to the claim that logic is all-encompassing. Not only does logic apply to all subject matter and govern inferring between any judgements whatsoever, but inferential capacities lurk in the background of each individual judgement because of the role they play in our grasp of the thoughts judged. This gives a clear sense in which logical laws unfold the content of the word "true"—requisite inferring, by leading us through the truth-conditions of formulae, makes explicit what is implicit when we acknowledge the truth of a thought.

I've contrasted my view with the elucidatory interpretation, which shares a great deal with the current account. Frege provides no sharp formal criteria for which primitive laws are logical, and no deep analysis of what logicality amounts to—his conception of logic is retail, not wholesale. And Frege mustn't be seen as undertaking the impossible task of giving the truth or the logicality of his primitive laws some deeper grounding in language or any other discipline. The differences between the elucidatory and warrant interpretations are threefold. Firstly, it is part of my account, unlike the elucidatory account, that one can have an external standpoint on particular systematizations of logic. Secondly, on the present account, the logical nature of a primitive law will not be obvious to everyone who understands it. Finally, exhibitions, though a propaedeutic in the literal sense of bringing us into the right relation to Frege's logical foundations, are not mere clarifications, but an integral part of Frege's overall project to show that arithmetic is logical. They are integral in the following sense—if Frege's arguments for his primitive laws were flawed, or if they were replaced with some non-rational means of convincing us to believe that these laws are true and logical, his project would have failed. And fail it did, precisely because his specifications concerning value-range terms contain false statements.

The general background on which this reconstruction rests draws on themes found in Sullivan (2004). There is no external standpoint from which one can evaluate or justify logic, and there is nothing Frege says which would convince a modern-day proponent of a rival logic to adopt his particular brand of classical logic, even once Law V has been removed or modified. However, we *can* have an external standpoint on particular languages and particular axiomatizations of logic, and can speak in ways which articulate the semantic structure of formulae expressing his primitive laws. The point of doing so is not to provide something like a soundness proof. Rather, the point of a perspective external to his systematization of logic is that it gives Frege room to articulate, via the use of requisite inferring, the truth-conditions of formulae in a way that also manifests what it is that our grasp of these laws in part consists. It also gives him the space to reason about the nature of the rules used in his exhibitions, and such reflection is what warrants us in judging that his laws are logical.

## Acknowledgements

I'm grateful for audience feedback from talks given at the University of Stirling, University of St Andrews, and especially the Masterclass *Frege in Kantian Context* in Boston. Thanks to Peter Milne, Peter Sullivan, and Philip Ebert for comments on earlier drafts. Additional thanks to Sanford Shieh and two anonymous reviewers for this journal for extended comments which helped improve this paper.



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