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Frege on the Fruitfulness of Definitions

Rachel Boddy

What, in Frege's view, makes definitions fruitful? In *Grundlagen* §70, Frege offers an answer: Unfruitful definitions are definitions that "could just as well be omitted and leave no link missing in the chain of our proofs". The §70 passage, however, poses an interpretive puzzle as its characterization of fruitfulness appears to conflict with other conditions that Frege imposes on definitions, namely, eliminability and conservativeness. It appears that the only way to resolve this conflict is to attribute to Frege a notion of fruitfulness that is trivially satisfied and, hence, poorly motivated. I argue that this worry is misplaced. This is because Frege distinguishes between two roles of definitions, namely, between definitions *qua* explanations of concepts (analytic definitions), and definitions *qua* resources of a proof system (logical definitions). I use this distinction to argue that a fruitful definition, for Frege, is a definition that plays both roles, and that to play both roles, the definition has to be used in the proof of sentences containing the term so defined. Starting from §70, I develop and defend this reading of Frege's notion of fruitful definition.

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1. Introduction

"Even I agree that definitions must show their worth by their fruitfulness: it must be possible to use them for constructing proofs." This passage, taken from the introduction of *Grundlagen*, signals Frege's agreement with what he took to be a common view among mathematicians (Frege 1884 [1953]). But what precisely does this claim amount to? Frege's answer in §70 of *Grundlagen* is that *un*fruitful definitions are definitions that "could just as well be omitted and leave no link missing in the chain of our proofs". Fruitful definitions, Frege seems to tell us, are at least indispensable for the construction of some gap-free proof. In line with this characterization of fruitfulness, he goes on to show, in the subsequent paragraphs, how his definition of Number can be used to derive some of the well-known properties of the numbers, and that it is, therefore, fruitful.

But if, for Frege, fruitful definitions somehow play an essential role in proofs, then precisely what do definitions contribute that accounts for their fruitfulness? It is this question that is at the base of an interpretive puzzle about the *Grundlagen* §70 passage; namely, it is unclear how to reconcile this notion of fruitfulness with Frege's (often stated) view that definitions must be non-creative and abbreviatory. These latter two conditions correspond to the familiar criteria of *conservativeness* and *eliminability*.[1] As several commentators have noted, it appears that definitions that are fruitful in the sense of §70 are creative and that eliminable definitions cannot be fruitful.[2] Frege thus appears to express conflicting views about definitions. It is clear what it means for a definition to be eliminable and non-creative. The remaining question, then, is what it precisely means for a definition to be fruitful.

The goal of this paper is to examine Frege's answer; it is to examine what, on his view, fruitful definitions are. My view is that Frege's answer relies on a distinction between two roles of definitions, namely, between definitions *qua* explanations of concepts and definitions *qua* resources of a proof system. I shall call these "analytic definitions" and "logical definitions", respectively.[3] This terminology is not, as such, directly found in Frege's writing, but rather it is meant to help bring into focus a distinction between two ways that Frege makes use of definitions.[4]

In particular, in *Grundlagen*, I believe, Frege employs definitions as explanations of concepts. These definitions can be viewed as answers to what-questions. Frege begins *Grundlagen* with a defining question of arithmetic: *What is a number*?[5] Problematically, he tells us, this question has not yet been given a satisfactory answer (1884 [1953], ii).

[1]This was first discussed in Belnap (1993). In what follows, I shall use "conservative" and "non-creative" interchangeably.

[2]This includes Benacerraf (1981); Horty (2007); Shieh (2008); Tappenden (1995); Weiner (1990).

[3]For lack of a better term, I call definitions *qua* explanations of concepts "analytic definitions". We must clearly distinguish this from Frege's use of "analytic definition" (*zerlegende Definition*) in Frege (1914 [1979]). Frege characterizes the latter type of definition as follows: "We have a simple sign with a long established use. We believe that we can give a logical analysis of its sense, obtaining a complex expression which in our opinion has the same sense" (1914 [1979], 210). Frege rightfully rejects this notion of definition for the reason that these statements are really to be regarded as axioms. But again, my view is *not* that in *Grundlagen* Frege's concern is with analytic definitions *in this sense*.

[4]Another note on terminology: What I call "logical definition" does correspond to Frege's use of "constructive definition" (*aufbauende Definition*) in Frege (1914 [1979]).

[5]For Frege, this question becomes: *What is the concept of a number*?

The main task of *Grundlagen* is to say what a number is, and for this, Frege thinks, it is sufficient to provide a definition of Number. In §68, he lands on the following answer: "the Number which belongs to the concept F is the extension of the concept 'equal to the concept F''. This is an *analytic* definition in the sense that it offers an analysis of a fundamental notion of arithmetic. It tells us what, according to Frege's analysis, the number of a concept is.

It is against this background that Frege appeals to fruitfulness in *Grundlagen* §70. A fruitful definition, for Frege, is a definition that plays both roles, i.e., as an analytic definition and as a logical definition. His view is that there is only one way in which definitions can be used (and justified) in proofs and, hence, any analytic definition is worthless if it cannot play its appropriate role in proofs. Specifically, the only way to bring in an analytic definition is in the proof of a sentence containing the term so defined. Such a proof is not gratuitous: It is in virtue of Frege's analytic definition of Number that the *Grundgesetze* theorems are recognizable as theorems *about cardinal numbers*.[6] This also means that Frege distinguishes between two ways of viewing a proof, namely, *qua* a proof of a sentence ("*de dicto* proof" for short) and *qua* a proof of a thought ("*de re* proof" for short), and that §70 is about *de dicto* proof. In what follows, I further develop and defend this interpretation.[7]

[6]All references to *Grundgesetze der Arithmetik* are to Ebert and Rossberg's translation, cited as Frege (1893 [2013], 1903 [2013]).

[7]This issue has been discussed in the literature. According to Weiner (1990), Frege's notion of fruitfulness is trivial. Starting from Frege's view of logical definitions (i.e., definitions appropriate for constructing proofs), she argues that the fruitfulness condition from §70 is simply a pragmatic requirement, such that "whatever mathematicians find useful for constructing proofs is fruitful" (1990, 92). On her interpretation, fruitfulness is just a matter of being usable in proofs. In contrast, Tappenden (1995) argues that Frege's conception of fruitfulness is more substantive: Fruitful definitions contain quantified expressions (1995, 438). Following Tappenden, my view is that the question of fruitfulness, for Frege, is not trivial. But, where I agree with Weiner is that when this question is posed about logical definitions, Frege has no answer. See Sections 4–6 for further discussion.

2. Fruitfulness in *Grundlagen* §70

When Frege requires that definitions be fruitful in *Grundlagen* §70, it is unclear what he means by this in part because the requirement appears to conflict with other properties that he requires in definitions, namely, eliminability and non-creativity. Here, a definition is eliminable when its *definiendum* expresses exactly the same content as its *definiens* and can, therefore, be replaced by its *definiens* in any sentence of the language because any sentence that contains the defined term expresses the same content as the sentences obtained by replacing its occurrences by the *definiens*. In Frege's words: "if the *definiens* occurs in a sentence and we replace it by the *definiendum*, this does not affect the thought at all" (1914 [1979], 208). A definition is non-creative when it is *only* used to introduce a new term, not new content; i.e., the definition is semantically conservative over its ground language. The most important definition to be vindicated in the context of Frege's logicist project is the definition of Number:

(#D) The number of $F =_{def}$ the extension of the concept *being equinumerous with F*.

Having just presented this definition in §68 of *Grundlagen*, Frege states the fruitfulness requirement as follows:

> *Our definition completed and its worth proved.*
>
> Definitions show their worth by proving fruitful. Those that could just as well be omitted and leave no link missing in the chain of our proofs should be rejected as completely worthless.
>
> Let us try, therefore, whether we can derive from our definition of the Number which belongs to the concept F any of the well-known properties of the numbers (1884 [1953], §70).[8]

[8]Here is the original passage:

Ergänzung und Bewärung unserer Definition

§70. Definitionen bewähren sich durch ihre Fruchtbarkeit. Solche, die eben-

Fruitful definitions, Frege tells us, cannot be omitted without leaving gaps in proofs, and are, in this regard, necessary for these proofs. Specifically, his view is that (#D) is shown to be fruitful by being used to derive the well-known properties of the numbers, and among these is, foremost to Frege's mind, the property that equinumerous concepts have the same number.[9] Accordingly, in the paragraphs following §70, Frege shows how the definition can be used to this end.

As a first approximation, we can thus render Frege's notion of fruitful definition as follows:

> A definition is *fruitful* only if it is indispensable for the gap-free derivation of some proposition.

A fruitful definition is not just usable in proofs, but rather it is indispensable for certain proofs. This characterization, however, immediately raises the question: What is meant by "proposition" here? Specifically, are we to take a proposition to be a *thought* or a *sentence*? Here "thought" is Frege's technical term for (roughly) the propositional content expressed by a sentence, and sentences are the linguistic vehicles that express these contents. Frege's remark about fruitfulness in §70 leaves room for two interpretations of "proposition", corresponding to two ways to view proofs: On (what I shall call) the *de re* view of proof, a proof is a proof *of a thought*, whereas on the *de dicto* view, a proof is a proof *of a sentence*.[10] Accordingly, on the *de re* interpretation of §70, a fruitful definition is indispensable for the proof of some thought, whereas on the *de dicto* interpretation, a fruitful definition is indispensable for the proof of some sentence.[11]

The *de re* interpretation is in tension with Frege's (often stated) view that definitions must be eliminable and non-creative.[12] In what follows, I shall argue for the *de dicto* interpretation. My view is that §70 is best read as follows:

> A definition of Number is fruitful only if it is used for gap-free proofs of the sentences that express the well-known properties of the numbers.

Since these sentences contain the *definiendum* of the definition, it follows that the definition is indispensable for the proof of these sentences. Thus formulated, Frege's appeal to fruitfulness is not in conflict with the requirements of eliminability and non-creativity. I shall use this observation to argue that Frege supports at least the following condition on definitions:

> **Logical adequacy condition.** A definition is *fruitful* only if it is used for a gap-free derivation of a sentence containing the term so defined.

Although this is clearly a weaker characterization, it is at least part of what it means for a definition to be fruitful. Being logically adequate is at least a necessary condition for fruitfulness. Any

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sogut wegbleiben könnten, ohne eine Lücke in der Beweisführung zu öffnen, sind als völlig werthlos zu verwerfen.

Versuchen wir also, ob sich bekannte Eigenschaften der Zahlen aus unserer Erklärung der Anzahl, welche dem Begriffe F zukommt, ableiten lassen!

[9] That is, any two numbers are the same if they belong to concepts that can be placed in a one-to-one correspondence. This is the right-to-left direction of Hume's Principle ($HP^{\leftarrow}$). This, *as it is stated*, cannot be proved without a definition of number. This is also the first main theorem that Frege derives in Part II of *Grundgesetze*, now labeled theorem 32. Part II of *Grundgesetze* is entitled "Proofs of the Basic Laws of Cardinal Number" and its first section, called "A", is devoted to the proof of theorem 32. Whereas $HP^{\leftarrow}$ is listed as the first basic law of cardinal number, the left-to-right direction, $HP^{\rightarrow}$, is not listed as a law of cardinal number at all (May and Wehmeier 2019). Frege derives $HP^{\rightarrow}$ as theorem 49 in the proof of the proposition that each cardinal number has a unique successor, which he lists as a Law of Cardinal Number. Theorem 49 is marked as a "notable theorem".

[10] Frege does not use this terminology.

[11] Henceforth, I shall refer to these interpretations of §70 as the "*de re* interpretation" and the "*de dicto* interpretation", respectively.

[12] More on this below. Interpreters have used this observation to argue that Frege's view of the fruitfulness of definitions, and the corresponding fruitfulness requirement, is *not* the notion from *Grundlagen* §70, and have, as part of this move, tried to downplay the relevance of §70. However, in my view, doing so is unjustified. I defend this claim in Boddy (2019b).

definition, to Frege's mind, has to be *usable* in proof, otherwise it is not justified logically. What is required for fruitfulness is that the definition is used in proofs.

In what follows, I shall further extend this interpretation, and defend it against objections from the literature. In particular, I shall discuss two influential objections against the *de dicto* interpretation. First, it is usually taken to be characteristic of Frege's view of logical proof that what is proven are always thoughts, rather than sentences.[13] What this means is that Frege only accepts the *de re* view of proof. If so, then the fruitfulness of a definition cannot be a matter of its role in the "proof" of a new *sentence* because what is proven are thoughts, not sentences. Second, the *de dicto* interpretation appears to trivialize the fruitfulness requirement. For if it (the requirement) were just about sentences, then it would be a trivial matter to show of any stipulative definition that it is fruitful. What this would mean, the objection goes, is that Frege ultimately has nothing to say about the contribution of definitions and, hence, no answer to our opening question.[14] Before addressing these points, I shall briefly review why the *de re* interpretation of §70 is problematic.

3. Stipulative Definitions

Frege's view of definitions with respect to logical proof is constant throughout his works: Definitions are stipulations that serve to introduce novel terms into a language as replacements for complex terms, whose meaning has already been established. *Begriffsschrift* contains an early statement: "Our sole purpose in introducing such definitions is to bring about an extrinsic simplification by stipulating an abbreviation" (Frege 1879 [1967], §24).[15] In *Grundgesetze*, Frege re-affirms his position stated in *Begriffsschrift* §24, that for the purposes of logic, definitions are abbreviatory, and that as a consequence of their explicit form, they are eliminable and non-creative.[16] Frege's view is that definitions are sentences of the form *definiendum* $=_{def}$ *definiens*, such that the *definiendum* and *definiens* express the same sense, and thus have the same reference (1893 [2013], §27).

From Frege's perspective, any definition, to play its appropriate role in proof, has to function as a *stipulative* definition, a *logical* definition.[17] And again, logical definitions, Frege insists, are semantically conservative over their ground language. If definitions are strictly abbreviatory, then they clearly are not necessary for the construction of the proof of any thought.[18] The problem with the *de re* interpretation is clear: Logical definitions, as conceived by Frege, are never indispensable for the proofs of *new thoughts* precisely because they are conservative. Interpreters have taken this to show that Frege must have held that fruitfulness, at least for logical definitions, is *not* a matter of being *indispensable* for proof.[19] There is an alternative path, however: On the *de dicto* interpretation of §70, a fruitful definition is indispensable for the proof of a sentence containing the

[13]See Blanchette (2012) for a defense of this interpretation of Frege.

[14]This is essentially the position of Weiner (1990). See Section 3 for further discussion.

[15]I quote the English translation by Bauer-Mengelberg in van Heijenoort (1967). The original German version is Frege (1879).

[16]While there is an important distinction between the logic of the *Begriffsschrift* and the logic of *Grundgesetze*, in that the former has no semantics and is formal in the Kantian sense of carrying no ontological assumptions, Frege's view of definitions with respect to logical proof is constant between these works.

[17]In what follows, I shall use "stipulative definition" and "logical definition" interchangeably. I use "logical definition" to emphasize that, considered from the point of view of logic, definitions are purely abbreviatory.

[18]This observation is at the basis of Weiner's argument against the *de re* interpretation. Her conclusion is that "Frege cannot be requiring that a definition's worth is dependent on its being *necessary* for some proof" (1990, 90–91).

[19]For example, Horty (2007); Tappenden (1995); Benacerraf (1981); Weiner (1990) have argued for this in their work. Taking this route involves downplaying the relevance of §70.

term so defined. But, again, if this was Frege's path, then we are owed an account of what fruitful definitions, unlike unfruitful definitions, contribute.

3.1. Can sentences be proven?

It is often assumed that since Frege held that logical laws govern relationships between thoughts, proofs are proofs *of thoughts*, rather than of the sentences that express those thoughts. What this means is that it is wrong to assume that Frege would consider the *de dicto* view of proof. Frege, however, explicitly considers this view in "Logic in Mathematics" (1914 [1979]),[20] where he writes:

> Of course it may look as if a definition makes it possible to give a new proof. But here we have to distinguish between a sentence and the thought it expresses. If the *definiens* occurs in a sentence and we replace it by the *definiendum*, this does not affect the thought at all. We then obtain a different sentence, but not a different thought. Of course we need the definition if, in the proof of this thought, we want it to assume the form of the second sentence. But if the thought can be proved at all, it can also be proved in such a way that it assumes the form of the first sentence, and in that case we have no need of the definition. *So if we take the sentence as that which is proved, a definition may be essential, but not if we regard the thought as that which is to be proved* (Frege 1914 [1979], 208; my emphasis).

This textual evidence shows that Frege, at least the Frege of (1914 [1979]), is open to the idea that sentences can also be proved (i.e., the *de dicto* view of proof). This observation, however, immediately raises the question: Why think that Frege was interested in constructing proofs of thoughts *expressed with certain sentences*? Before considering this question, it is worth noting that this interpretation is at least consistent with Frege's conception of logical proof.

On Frege's conception of language, a language is a system of *symbols*, such that a symbol is a sign that expresses a particular sense.[21] To change the symbols is to change the language.[22] In Frege's *Grundgesetze* logic, then, the sharp distinction between the proof of sentences and the proof of thoughts cannot be drawn. Each sentence, *qua* a sentence of a language, is fully interpreted and expresses a thought, and it is only by deriving sentences that we can prove thoughts.[23] What this means is that any proof, given that it is represented (in a language), can be viewed both *de re* and *de dicto*. While Frege held that logical relations hold between thoughts, since any concept-script sentence expresses a fixed thought, proofs can be conducted completely in the formal mode, i.e., purely syntactically. But it does not follow that such proofs are *about* sentences: "Rather, the [inference] rules here follow necessarily from the reference of the signs, and this reference is to the proper objects of arithmetic" (Frege 1903 [2013], 156).

[20] This is a set of lecture notes from 1914.

[21] I shall use "sign" as a translation of the German "Zeichen". This is the German word that Frege typically uses for uninterpreted signs. Thus, a sign is a syntactic unit without an associated meaning and a symbol is an interpreted sign, i.e., a syntactic unit plus a sense. Although Frege does not systematically use a pair of words that could be translated to "symbol" and "sign", he does make this distinction between symbol and sign.

[22] There are two ways of changing a symbol: By keeping the sign and changing the sense associated with it, or by changing the sign and keeping the sense.

[23] It is worth noting that I do not use the technical distinction between derivation and proof introduced in Blanchette (2012). Blanchette distinguishes between the *derivation* of sentences and the *proof* of thoughts, such that a derivation expresses a proof. In contrast, on my reading, a derivation just is a proof (for the reasons just given), so I do not distinguish between a derivation and a proof, as Blanchette does.

4. Two Roles of Definitions

In general, it is a trivial matter to establish that a logical definition is logically adequate.[24] Frege's *Grundgesetze* definitions, just like any other logical definitions, are trivially shown to satisfy this requirement. The logical perspective, however, is not the only perspective from which to consider definitions: Definitions, according to Frege, also play an important *theoretical* role in science. From the point of view of scientific theorizing, definitions are used to explain and identify the fundamental notions that the theory is about. In this role, definitions are answers to defining questions of the science, i.e., to *what*-questions about the fundamental notions of the science. For arithmetic, the foremost question, in Frege's mind, is "What is a number?". Once we take this broader perspective on definitions into consideration, things change.

4.1. Analytic definitions

Frege's discussion of definitions, primarily in *Grundlagen*, but also in the critical sections of *Grundgesetze*, shows that he distinguishes two roles of definitions, namely, between definitions *qua* explanations of concepts (analytic definitions) and definitions *qua* resources of a proof system (logical definitions). In their analytic role, definitions isolate the concepts that the theory is about.[25] These concepts are not just used in proofs; the proofs are about them.[26] Frege has this latter role in mind when in *Grundgesetze* he explains that by means of a definition we "make something prominent by demarcation and designate it with a name" (1893 [2013], XIII).[27] Bear in mind that, for Frege, what a theory is *about*, and the way in which definitions *isolate* the fundamental concepts of the theory are informal notions, not technical. Frege presuppose an intuitive understanding of these notions.[28]

Grundlagen as a whole is about definitions whose role is to introduce and explain concepts, i.e., analytic definitions.[29] Frege begins his discussion with the question "What is a number?" This question, he tells us, has not yet been given a satisfactory answer. To Frege, the resulting situation is clearly problematic, as it undermines the scientific standing of arithmetic:

> If a concept fundamental to a mighty science gives rise to difficulties, then it is surely an imperative task to investigate it more closely until those difficulties are overcome; especially as we shall hardly succeed in finally clearing up negative numbers, or fractional or complex numbers, so long as our insight into the foundation of the whole structure of arithmetic is still defective (Frege 1884 [1953], ii).

[24] It is sufficient to use the definition in the proof of a sentence containing the term so defined, regardless of whether or not the sentence is marked as a notable theorem.

[25] See, e.g., the sections placed by Frege in the Table of Contents under the heading "Critique of theories concerning irrational numbers" in Frege (1903 [2013], VIII), where this is especially clear. Frege (1903 [2013], §§57-67) discusses principles "that should be observed when defining" (1903 [2013], §67), i.e., when constructing analytic definitions. Frege appeals to these principles to criticize the definitions proposed by others. A criticism Frege uses against competing analytic definitions is that these definitions are unsuitable as logical definitions. Another important reference is Frege (1880-81 [1979]).

[26] As was mentioned in footnote 3, we must clearly distinguish this notion of an analytic definition—i.e., an explication of a concept—from the Frege (1914 [1979]) notion of an analytic definition ("zerlegende Definition"). Frege (1914 [1979]) uses the notion to make a different rhetorical point; namely, a negative point against the idea that definitions can be used to assert sense-identity of defined and defining expression, where the defined expression already has a sense.

[27] Similarly, Frege (1884 [1953], §88) tells us that definitions are used for "demarcating an area" and for "drawing boundary lines".

[28] Thus, with these remarks, I do not intend to put forth a precise characterization of these notions, as Frege's use is non-technical. This also applies to the notion of explanation (*Erklärung*).

[29] See, e.g., the discussion at Frege (1884 [1953], xx-i, and §§9, 12, 19, 28, 72, 88 and 103). Another example is §99, where in criticizing Hankel's definitions of addition and multiplication, Frege writes: "But we have not the slightest right to suppose that we can use it as a method for introducing addition and multiplication. It does not give their actual definitions, but only lays down the lines for them" (in the sense of being a guide towards a definition).

The main task of *Grundlagen* is to offer an answer in the shape of a definition of Number.

The answer Frege lands on, in §68, is the definition of Number, (#D). That is, his answer is that numbers are extensions of concepts.[30] The definition is *analytic* in the sense that it tells us what, according to Frege's analysis, the number of a concept is; it is part of his account of *what numbers are*. From this perspective, it is not a stipulation about the meaning of term newly introduced as an abbreviation for another expression of the language (even if, from the logical perspective, it is just this).

A hallmark of an analytic definition is that it is subject to philosophical criticism. This is because the answer to a defining question and, correspondingly, the adequacy of an analytic definition, is open to discussion and investigation. Frege's definition of Number is the result of an extended argument spanning *Grundlagen* §§5–68.[31] Its correctness, *qua* analytic definition, can be questioned: "That this definition is correct will perhaps be hardly evident at first. For do we not think of the extensions of concept as something quite different from the numbers?" (1884 [1953], §69). Regardless, the definition, in Frege's mind, is well-supported: It is explicative of the concept Number and it can play its appropriate role in proofs of arithmetical truths.[32] Its adequacy is to be assessed in the context of the overall system in which it is employed. Any serious criticism, Frege tells us in *Grundgesetze*, must start from an alternative (analytic) definition and then "try to develop a sound and usable symbolic exposition on that basis".[33] In contrast, a logical definition is not open to discussion in this way; it is just an abbreviation.

The construction of analytic definitions is a proper part of mathematical work: "While the mathematician defines objects, concepts and relations, the psychological logician is listening in on the coming and going of ideas, and in the end the mathematician's defining can only appear foolish to him, since it does not convey the nature of ideas" (Frege 1893 [2013], XXV).[34] Frege's logicist definitions of arithmetical notions are important examples of analytic definitions. Included among these are the definitions of Number, and of the numbers zero and one. These definitions characterize fundamental notions of arithmetic and allow for their identification within the *Grundgesetze* development of arithmetic.[35]

The details of this interpretation are worked out in Boddy (2019a,b).[36] What is important for our discussion is that the analytic/stipulative definition distinction, as Frege understands it, is not based on the logical form of definitions. For Frege, there is only one way in which definitions can be used (and justified) in proofs and, hence, *from the perspective of logical proof*, there really is only one notion of definition at play. But from the perspective of the overall scientific theory, these definitions can still play an analytic role. An analytic definition is identifiable on the basis of its place within a theory, rather than on the basis of its logical form. What this means is that in the context of a scientific theory,

[30] In *Grundlagen* numbers are extensions of concepts whereas in *Grundgesetze* numbers are value ranges of functions. Note, however, that concepts are functions from objects to truth values. As such, extensions of concepts are just the value ranges of a particular type of function.

[31] Though *Grundlagen* as a whole plausibly can be read as a defense of Frege's definition of Number.

[32] In support of (#D), Frege has argued that numbers are objects, that arithmetical truths are analytic (in Frege's sense) and that the definition can be used to derive the well-known properties of numbers in a system of logic.

[33] Here is the full passage:

By this act I am to confirm the conception of cardinal number which I set forth in [*Grundlagen*]. The basis for my results is articulated there in §46, namely that a statement of number contains a predication about a concept; and the exposition here rests upon it. If someone takes a different view, he should try to develop a sound and usable symbolic exposition on that basis; he will find that it will not work (Frege 1893 [2013], IX).

[34] See also, e.g., Frege (1884 [1953], v; 1914 [1979], 203).

[35] I shall return to this point in the next section.

[36] Horty (2007) makes a similar point.

definitions are not to be assessed only in terms of their logical justification.[37]

4.2. Frege's logicist definitions

Logicism, for Frege, is a scientific project: To carry through this project is to construct a scientific theory.[38] Importantly, in *Grundgesetze*, Frege does not just present logical derivations, but rather what is on offer is a fully rigorous development of arithmetic (see Frege 1893 [2013], VI). Analytic definitions play an essential role in this project; the analytic definitions must *specify a content* in a way that allows for the *justification of arithmetic* and are not, as such, arbitrary.[39] Frege introduces these definitions under the heading "*special definitions*", where he refers back directly to the relevant sections from *Grundlagen*.[40] The *Grundgesetze* theorems are recognizable as theorems *about cardinal numbers* in virtue of the analytic definitions of the theory. These definitions turn the *Grundgesetze* logic into a scientific theory about numbers.

Weiner (1990) and Tappenden (1995) approach the question of the fruitfulness of definitions as a question about logical definitions. If, along this line, we only consider this viewpoint, then it seems that fruitfulness must be characterized logically. The idea would then be that the notion of fruitfulness, as conceived by Frege, must be generally applicable to logical definitions. Starting from this idea, Weiner concludes that fruitfulness, for Frege, is a trivial notion. This is because from the logical perspective, if a definition meets Frege's strictures on definitions, then there is nothing further to say about its justification. Tappenden's view is that fruitfulness *can* be characterized logically and, hence, that Frege drew a distinction between fruitful and unfruitful logical definitions. I shall return to this point in Section 6. However, my view is that we cannot properly understand Frege's view of fruitfulness if we only focus on logical definitions. For Frege, the question of fruitfulness only arises for definitions that play an analytic role within their theory. *Grundlagen* §70 has to be read in this broader, scientific context.

5. Back to Fruitfulness

In *Grundlagen*, Frege's attention is on analytic definitions. His view is that the *Grundlagen* definitions of arithmetical notions have (logicist) worth only to the extent that they allow for the gap-free proofs of propositions that are recognizable as the laws of cardinal number. It is against this background that Frege appeals to fruitfulness in §70; it is meant to assure his readers that his definition of Number can serve its intended purpose. If it cannot be used to this end, he says, then it "should be rejected as completely worthless" (1884 [1953], §70). With this remark, Frege transitions from his theoretical discussion of the definition of Number, *qua* an explanation of the concept of Number, to a discussion of that definition *qua* an abbreviation, i.e., from an analytic to a logical definition. A fruitful definition, for Frege, is a definition that plays both roles, i.e., as an analytic definition and as a logical definition.

Frege's view is that a definition of Number is fruitful only if it is used for gap-free proofs of the sentences that express the well-known properties of the numbers. At issue is not the logical justification of the definition, but rather what we could call its "scientific vindication" as it involves the *worth* of Frege's analysis of Number in delivering a definition that can play its intended role in proofs. Thus, from the analytic perspective, the definition introduces the notion of number, but considered purely logically,

[37] To repeat, the logical justification of a definition is just matter of it being eliminable and non-creative with respect to its ground language.

[38] Frege's conception of science is discussed in May (2018).

[39] See Boddy (2019a) for discussion of this point.

[40] The *Grundgesetze* definitions of arithmetical notions are essentially just the *Grundlagen* definitions (Heck 1993, 269). See *Grundgesetze* I §38–40 where Frege introduces his definition of number, namely, definition Z. These passages explicitly refer back to *Grundlagen*'s definition (1884 [1953], §68).

it merely introduces an abbreviation for a complex term of the language.

Taken in this way, the fruitful definitions of *Grundgesetze* are exactly those logical definitions that have an analytic counterpart and that are, therefore, justified in the context of the overall scientific theory. To repeat, from the perspective of logical proof, these definitions are purely abbreviatory and dispensable; but from the perspective of the *Grundgesetze* theory, *qua* scientific theory, these definitions isolate the fundamental concepts of the theory and turn the *Grundgesetze* logic into a scientific theory about cardinal numbers. Bear in mind that it does *not* follow that each of the *Grundgesetze* definitions functions both as an analytic definition and as a logical definition. Frege does not require that all *Grundgesetze* definitions are analytically justified; only the definitions that characterize the concepts that the theory is about require this type of justification. The question of fruitfulness only arises for analytic definitions. In *Grundgesetze*, Frege introduces several definitions that play an important role in the logic—e.g., by facilitating proofs.[41] These definitions, while very useful, are not fruitful.[42] In line with Frege's remarks about fruitfulness, I reserve the term "fruitful" for the scientific notion of fruitfulness.

At this point in our discussion, we can see that Frege's view of fruitfulness relies on the *de dicto* view of proof. This is because an analytic definition can contribute to a scientific theory only by being called upon in proof and, specifically, only by being used in the proof of a sentence containing the term so defined. What this means is that *de dicto* proof is the only way to employ a definition, *qua* explanation of a concept. The value of a *de dicto* proof is identifiability: The *Grundgesetze* theorems about numbers are recognizable as such in virtue of the analytic definitions of the theory.[43] Along this line, the *Grundgesetze* theorems that are labeled by Frege as "the basic laws of cardinal number" are stated in terms of the relevant defined terms.

From Frege's perspective, the extent to which a definition is fruitful is relative to the scientific goals it helps establish. One function of definitions is unification: Definitions can help reveal connections between "matters apparently remote from another, this leading to an advance in order and regularity" (Frege 1884 [1953], ix). The resulting "simplification is in itself a goal worth pursuing" (1884 [1953], §2). Along this line, Frege intended to use his definition of Number to show that arithmetic is a branch of logic. According to Frege, definitions that support proofs that establish connections between "matters apparently remote from another" are fruitful. This last quotation is part of a passage in which Frege remarks on what he takes to be the usual practice of mathematicians. Here is the full passage:

> If a definition shows itself tractable when used in proofs, if no contradictions are anywhere encountered, and if connexions are revealed between matters apparently remote from another, this leading to an advance in order and regularity, it is usual to regard the definition as sufficiently established, and few questions are asked to its logical justification. This procedure has at least the advantage that it makes it difficult to miss the goal altogether. I also think that definitions must prove themselves by their fruitfulness; by the possibility of constructing proofs with them (Frege 1884 [1953], ix; translation altered).[44]

[41] Two important examples of definitions that are crucial to the *Grundgesetze* proof structure but that are not analytically justified are the definition of membership (A) and the definition of the converse of a relation (E). Problematically, definition (A) is not analytically justified. This is problematic because the assumption that membership is a logical notion is not justified. Thanks are due to Robert May for pointing this out to me.

[42] For discussion of the importance of these definitions in the *Grundgesetze* logic, see Heck (2012).

[43] Not all *Grundgesetze* theorems are about numbers: "Propositions also occur which are not about cardinal numbers but which are needed in proofs. They treat, for example, of following in a series, of single-valuedness of relations, of composite and coupled relations, of mapping by means of relations, and such like. These propositions could perhaps be allocated to an extended theory of combinations" (Frege 1893 [2013], V).

[44] Austin translates the last sentence differently: "Even I agree that definitions must show their worth by their fruitfulness: it must be possible to use them for constructing proofs."

Frege here discusses analytic definitions (namely, definitions of mathematical concepts). In agreement with "most mathematicians", he notes that the adequacy of a definition is tied to its fruitfulness, where this is commonly taken to be a matter of its use in proofs. But this, he says, is itself not sufficient to establish a definition; in addition, it has to be logically justified (i.e., non-creative). Similarly, at *Grundlagen* §70 he appeals to fruitfulness as a justificatory notion for analytic definitions: Any characterization of the concept Number has worth only insofar as it can be used as a stipulative definition in proof. Specifically, any definition of Number is *required* to be fruitful because its task is to construct proofs of the laws of cardinal number. In turn, these proofs establish the scientific unification of logic and arithmetic. This focus on analytic definitions might suggest that fruitfulness is a constraint on analytic definitions, rather than on stipulative definitions. However, this suggestion would be misleading, I think, because a definition is fruitful in virtue of playing both roles.

As pointed out by Tappenden (1995, 455), Frege no longer uses the "fruitful definitions" terminology in his post-1884 writings.[45] The fruitfulness condition is also not among (or implied by) the principles of definition listed in *Grundgesetze*. This absence of the fruitfulness terminology is unsurprising, however, since in *Grundgesetze*, Frege's focus has moved away from arguing against competing definitions of (cardinal) Number. His view is that the definition of Number should be used to derive the laws of cardinal number, and the fruitfulness requirement separates definitions that meet this standard from those that do not.

[45]Frege appeals to the notion of fruitfulness in his post-*Grundlagen* writings, even though he does not mention the fruitfulness condition of definitions. He typically uses it to mean "leading to extensions of knowledge" or "leading to many interesting results". For example, in *Grundlagen* §17 and Frege (1885 [1984]) he calls logic fruitful. The interesting results that are provable from principles of logic are, of course, the laws of cardinal number. Thanks to Jamie Tappenden for pointing this out to me.

5.1. Unfruitful definitions

Being introduced as an analytic definition is clearly not sufficient for fruitfulness. Frege's view is that unused definitions have no legitimacy within a proof system, and hence, none as analytic definitions either. Consider the following passage from Frege (1914 [1979]):

> If no use is ever made of a definition, there might as well not be one. However wide it may be of the actual target, no-one will notice (Frege 1914 [1979], 217).[46]

Although Frege does not use the term "fruitful" (*fruchtbar*) in (1914 [1979]), he criticizes competing theories for relying on definitions of mathematical concepts that are not used in any proofs. These definitions can be recognized "by the fact that no use is made of them, that no proof ever draws upon them" (1914 [1979], 212). In this context, he compares Weierstrass's definition of Number to stucco-embellishments on buildings, which only *appear* to support their building. Similarly, he says, Weierstrass's definition appears to support his proofs, but, in the terminology of *Grundlagen*, its omission would not leave a link missing in any proof. The definition is only "ornamental" and is "only included because it is in fact usual to do so" (1914 [1979], 212). Frege takes this to show that the theorems that Weierstrass proves are not supported by his definition, and consequently, that the definition does not fix the reference of occurrences of "its" *definiendum* in Weierstrass's theorems. The claim that these theorems express truths about Weierstrass's numbers (per his definition) is, therefore, ungrounded.[47]

[46]In the (unpublished) text "Logical Defects in Mathematics", Frege makes the same point: "What use to us are explanations when they have no intrinsic connection with a piece of work, but are only stuck on to the outside like a useless ornament?" (1979, 166).

[47]Frege further notes that Weierstrass does appeal, implicitly, to some notion of what he thinks numbers are, but that this notion does not match his definition.

Frege assumes that to explain what numbers are is to specify the referents of number terms as they occur in the sentences of arithmetic. His view is that in sentences of arithmetic (or any other science), each term must be introduced either as a primitive term or its reference must be specified by definition prior to its use in derivation. If a sentence contains a defined term whose definition is (currently) unused, then its proof must contain implicit steps. Given these implicit steps, it is not clear whether the proof relies on unstated assumptions about the referent of the defined term. Thus, his objection is that Weierstrass's definition of Number does not offer (or contribute to) an explanation of arithmetic because it is not used in any arithmetical proof.

6. Is "Fruitful" a Technical Term?

Tappenden (1995) argues that Frege has an intuitive notion and a sharp notion of fruitfulness: Intuitively, fruitful definitions support proofs involving an advance in order and regularity. The sharp notion is that fruitful definitions are given by quantified expressions, such that more fruitful definitions contribute quantifier structure to proofs, unlike less fruitful definitions. Specifically, he argues that Frege sharpens the intuitive notion "as part of a (fragmentary and incomplete) *substantive account* of the intuitive notion" (1995, 436). In contrast, I have argued that for Frege, fruitfulness is not a matter of the logical form of a definition (specifically, of its *definiens*).

Some of Frege's remarks in *Grundlagen* suggest that fruitful definitions are definitions that are used to construct fruitful concepts, which in turn, play a role in accounting for the informativeness of (Fregean) analytic truths. Tappenden uses this observation to argue that Frege's notion of fruitful definition is directly tied to the view that logical proofs can be *informative* or *ampliative*. Specifically, it is part of Frege's account of "the structure of fruitful concepts as intuitively understood" that is intended to help explain how deductive reasoning can be ampliative (1995, 438).[48] Fruitful definitions are definitions of fruitful concepts, where fruitful concepts are concepts that "draw new boundary lines" and that support quantifier inferences, and where more fruitful definitions contribute quantifier structure to proofs, unlike less fruitful definitions. Tappenden attributes the following view to Frege:

> To extend knowledge a logical proof has to involve quantifier inferences. Definitions contributing such quantifier structure to proofs that extend knowledge are "more fruitful" in the sense of the focal passage (Tappenden 1995, 432).

The "focal passage" is *Grundlagen* §88, which is another main place where Frege appeals to fruitfulness:

> [Kant] seems to think of concepts as defined by giving a simple list of characteristics in no special order; but of all ways of forming concepts, that is one of the least fruitful. If we look through the definitions given in the course of this book, we shall scarcely find one that is of this description. The same is true of the really fruitful definitions in mathematics, such as that of the continuity of a function. What we find in these is not a simple list of characteristics; every element in the definition is intimately, I might almost say organically, connected with the others.. . . But the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. . . The truth is that [the conclusions] are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house. Often we need several definitions for the proof of some proposition, which consequently is not contained in any of them alone, yet does follow purely logically from all of them together (Frege 1884 [1953], §88).

As Tappenden observes, a difference between fruitful definitions from mathematics and Kant's definitions, Frege tells us, lies in

[48] Tappenden uses this interpretation to argue, against Weiner (1990), that Frege does not appeal to considerations of fruitfulness merely to distinguish definitions from illustrative examples (per Weiner's interpretation), and relatedly, that the idea that definitions are just conventions of abbreviation is consistent with some non-trivial notion of fruitfulness.

the use of quantification. New boundary lines can be drawn by Frege's definitions because these definitions are constructed in a language that is capable of expressing complex names for certain concepts, and those concepts cannot be expressed in Kant's syllogistic logic.

Frege's 1880 paper "On Boole's Logical Calculus and the Concept-Script" (1880-81 [1979]) contains passages that closely correspond to §88 of *Grundlagen*. This supports Tappenden's claim that Frege's use of "fruitful" in §88 indicates that he assumes an account of fruitfulness—specifically, an account that ties the fruitfulness of concepts/definitions to the use of quantified logic, which in turn helps explain why definitions can lead to advances in knowledge. Thus, the fruitfulness of a definition is tied to the logical techniques involved in its formulation: Fruitful definitions are constructed with the use of quantifiers. It is the logical complexity of certain concepts (stemming from these techniques) that makes their analysis informative and, hence, that makes them fruitful.[49]

This reading, I think, is too strong. Frege's use of the term "fruitful" in *Grundlagen* §88 is non-technical, and not specifically tied to a property of definitions. In particular, in §88, Frege compares his view of concept-formation with Kant's. Kant's way of forming concepts, he says, "is one of the least fruitful", as it is merely a matter of giving lists of characteristics. This passage is part of Frege's response to the Kantian objection that analytic truths are uninformative, and, in particular, that logic cannot express informative truths. Frege's response is that *on a Kantian view* of logic and of analyticity, analytic judgments are not informative, but, since this view is mistakenly narrow, it does not follow that analytic judgments cannot be informative. By Kantian standards, some of the truths derivable in Frege's logic are informative.[50]

Frege here disagrees with the Kantian view (on his reading) on how to understand logical form. For Frege, the foundational notion of logic is that of an object falling under a concept. In Kant's logic, this relation is not represented. In the paragraph leading up to the above passage, he observes that the Kantian analytic/synthetic distinction is not exhaustive, as it only applies to universal affirmative judgments. Next, he underlines that the "really fruitful definitions in mathematics" are fundamentally different from the definitions of Kant, and that the *Grundlagen* definitions are similar to the former, not the latter. In particular, his view is that while Kant's (analytic) definitions are usable in proofs, what explains why these definitions are less fruitful is their logical form; they only characterize relations between concepts, and do not impose any further structure on the underlying domain. These definitions cannot be used to identify objects within that domain. In contrast, Frege's definitions take the form of descriptions, rather than of lists of characteristics (of concepts), which, he claims, makes this definitions more "organic" (1884 [1953], §88). The use of quantified logic allows him to define names *of objects* rather than just names of concepts, and to define concepts in terms of the objects that fall under them.

That the term "fruitful" is not specifically tied to the proof potential that results from the use of quantification is reinforced by several other passages where Frege appeals to the notion of

[49] Similarly, Horty uses Frege (1880-81 [1979]) to underline that Frege views definition as a kind of concept-construction. Frege's objection to Boole's logic, Horty argues, is that it does not allow for the construction of fruitful concepts because its *definitional techniques* are too limited. And, since in Frege's formalism concepts are constructed "by means of a richer set of definitional techniques," Frege is able to define scientifically fruitful concepts, unlike Boole (Horty 2007, 30). Horty writes: "In Frege's formalism also, new concepts are supposed to be constructed by definition out of old ones, but they are constructed by means of a richer set of definitional techniques. These techniques play an essential role in allowing him to define such fruitful concepts as that of a continuous function, for example" (2007, 30). Fruitful definitions, being definitions of fruitful concepts, are used to introduce terms that "can make new discoveries, new proofs, possible" (2007, 33).

[50] From a Kantian perspective, this shows that these are not logical truths.

fruitfulness.[51] For example, in *Grundlagen* §17 he explains that if arithmetic were analytic, then this would "put an end to. . . the legend of the unfruitfulness [*Unfruchtbarkeit*] of pure logic". For it would show that arithmetic can be developed from principles of logic, and each of those principles, as well as the truths of arithmetic, "would contain concentrated within it a whole series of deductions for future use" (1884 [1953], §17). Frege does *not* appear to think that the fruitfulness of logic is specifically a matter of the use of quantification and its consequences for concept structure, but is, rather, simply a matter of the proof potential of logic.[52]

Similarly, in *Grundlagen* §67 Frege objects to a proposed definition of the concept of Direction on the grounds that it presupposes the principle that "whatever is given to us in the same way is to be reckoned as the same" which, he says, "is a principle so obvious and so unfruitful [*unfruchtbar*] as not to be worth stating. We could not, in fact, draw from it any conclusion which was not the same as one of our premisses" (1884 [1953], §67). As in the previous example, Frege ties the fruitfulness of a principle to its role in generating "significant results".[53] As a final example, in *Grundlagen* §86 Frege discusses Cantor's definitions of Number and of following in a succession, and says: "At any rate, nothing in what I have said is intended to question in any way their legitimacy or their fruitfulness [*Fruchtbarkeit*].[54] On the contrary, I find special reason to welcome in Cantor's investigations an extension of the frontiers of science, because they have led to the construction of a purely arithmetical route to higher transfinite Numbers (powers)". Again, Frege appeals to the notion of fruitfulness in a context where he does not discuss a condition of definitions. These examples show that "fruitful" is not a technical term for Frege, and that he does not specifically tie the notion of fruitfulness (whether it applies to logic, concepts or definitions) to the use of quantification or other logical techniques. And while there are contexts in which Frege uses "fruitful definitions" and "fruitful concepts" interchangeably, such as *Grundlagen* §88, he does not in general confuse the two notions, as we have just seen from his use elsewhere in *Grundlagen*.

7. Conclusion

It is tempting to expect an examination of a notion of philosophical interest to result in a precise characterization of that notion. From this perspective, the question left open is what the precise characterization of fruitfulness is, as a property of definitions. For Frege, however, fruitfulness is not a property that can be explicated in terms of such formal properties of definitions as relate to their logical form. Frege uses the notion as part of his account of definition, but it is not itself a notion that we should analyze by means of a definition.[55] What this means is that we cannot offer a formal characterization of fruitfulness in a fashion similar to the standard characterizations of eliminability and non-creativity. It also means that we cannot inspect a definition in isolation and assess its fruitfulness. There is not an observable property of a definition in virtue of which we can say "this definition is fruitful". But it does not mean that fruitfulness, for Frege, is a trivial notion. What I have tried to show is that we need to focus on the context in which Frege uses his definitions in order to understand his view about fruitfulness and its connection with the analytic/logical definition distinction. I have argued that fruitful definitions are precisely those (logical) def-

[51] Thanks to Jamie Tappenden for directing my attention to these passages, and to this line of response.

[52] Similarly, Frege (1885 [1984]) explains that the logicist development of arithmetic would show that "logic cannot be as unfruitful [*unfruchtbar*] as it may appear on superficial examination" (1885 [1984], 112).

[53] The passage continues: "Why is it, after all, that we are able to make use of identities with such significant results in such diverse fields?" (1884 [1953], §67).

[54] Austin translates this as "fertility".

[55] Tappenden (1995, 436) makes a similar point.

initions that are utilized as analytic definitions in their theory. While definitions themselves are stipulative, Frege thought that they can nonetheless play an important role as explanations of concepts.

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