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Logical Concepts vs. Logical Operations: Two Traditions of Logic Re-revisited

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In what follows, the difference between Frege's and Schröder's understanding of logical connectives will be investigated. It will be argued that Frege thought of logical connectives as concepts, whereas Schröder thought of them as operations. For Frege, logical connectives can themselves be connected. There is no substantial difference between the connectives and the concepts they connect. Frege's distinction between concepts and objects is central to this conception, because it allows a method of concept formation which enables us to form concepts from the logical connectives alone. Schröder in contrast unifies the distinction between concepts and objects (which he calls elements and relatives), but keeps the distinction between logical connectives and what they connect. It will be argued that Frege's particular way of perceiving logical connectives is crucial for his foundational project.

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1. Introduction

In his influential paper "Logic as Calculus and Logic as Language" from 1967, van Heijenoort points out that not all notions of modern logic can be traced back to Frege, but that the Algebra of Logic tradition is similarly important. Löwenheim's (1915) paper was written in the Boole-Schröder tradition and Löwenheim's ideas could never have emerged out of the Frege-Russell tradition. According to van Heijenoort, Löwenheim's 1915 paper is a "cornerstone of modern logic" just like Frege's *Begriffsschrift* (van Heijenoort 1967, 328–29).[1]

In order to distinguish these two traditions, van Heijenoort refers to a dispute between Frege and Schröder about who came closer to realizing the Leibnizian project of a universal characteristic. This project was described in an essay by Trendelenburg (1856), which Frege and Schröder both knew. Trendelenburg describes the two parts of the project as: (1.) Finding the basic concepts and developing a language ("*lingua*") (1856, 48–49) and (2.) developing a calculus to derive sentences from other sentences expressed in this language ("*calculus*") (1856, 54–55).

Frege writes that, unlike Schröder, he wanted to set up not only a calculus, but also a language. Van Heijenoort explains Frege's remark by claiming that Frege's logic, in contrast to Schröder's, had quantifiers, and that it thus had a much greater expressiblity, because it went beyond propositional logic (van Heijenoort 1967, 324–25). In this sense, Schröder's logic is only a *calculus* whereas Frege's is also a *lingua* (van Heijenoort 1967, 325).

Peckhaus (2004a), however, challenges van Heijenoort's classification in his paper "Calculus ratiocinator versus characteristica universalis? The two traditions in logic, revisited". He points out that Schröder also introduced quantifiers into his logic, independently from Frege, in his Lectures on the Algebra of Logic in the 1890s. Thus, Peckhaus concludes that "quantification theory cannot be the criterion for distinguishing the two big traditions in the history of logic" (2004a, 12).[2]

However, although Frege and Schröder each introduced something which is in some way a predecessor of the quantifiers we use in modern logic, they did not identify their quantificational signs as signs for the same thing. Schröder wrote a review of the *Begriffsschrift* in 1880 and claimed that Frege's notion of generality can be adopted in Boolean logic "with minor modifications or extensions" (1880, 91–92). The explanation Schröder gives in that review shows, however, that Schröder did not understand Frege's quantifiers. His "modification" could only capture what can be expressed with one quantifier.[3] It now seems puzzling that Schröder did not even realize his mistake when he introduced his own notion of generality in the 1890s.[4] Frege also never mentioned that Schröder invented something like his quantifiers, even though he wrote a review of Schröder (1890) (Frege 1895 [1984]), the first volume of Schröder's Lectures on

[1]The third cornerstone is Herbrand's thesis from 1929, but I won't focus on that part of the history of logic here.

[2]One has to mention, however, that van Heijenoort (1967) seems to acknowledge at least indirectly that Schröder did at some point introduce quantifiers to his algebra of logic. He calls Löwenheim's logic a "first order predicate calculus" and points out that "Löwenheim uses Schröder's logical notation" (van Heijenoort 1967, 327).

[3]Frege (1882 [1972]) explained this in his reaction to Schröder.

[4]Schröder (1891) first introduced quantification signs, but he had not yet introduced a general notion of relation ("Relative", as he calls them). Schröder (1895) does that and thereby gets technically to the same level as Frege. Furthermore, this paper has a focus on concept formation. What comes closes to what Frege calls concepts are relatives. Thus, we will focus on Schröder (1895).

the Algebra of Logic. It is thus very likely that Frege did read Schröder (1895). Thus, it seems he did not recognize Schröder's quantifiers as comparable to his own.

In what follows, I will suggest a different way to demarcate Frege's logic from Schröder's, which clarifies their different conceptualizations of quantifiers. My suggestion is not limited to quantifiers, but concerns all logical connectives (which are now usually called "logical constants").[5]

For Frege, logical connectives are concepts; for Schröder, they are operations. In "Funktion und Begriff" Frege defines concepts as functions whose values are truth-values and introduces all logical connectives as names of functions, most of them as names of concepts.[6] Since logical connectives denote (logical) concepts, it becomes evident that logical connectives can connect not only non-logical concepts, but also logical connectives themselves. As a result, with Frege's logical notation one can express complete sentences using logical signs only.

Schröder, however, did not have such a project; he thinks of logical connectives as merely "operations", which are to be contrasted with the non-logical "operanda" (1895, 3). In other words: for Schröder, logical connective signs merely connect concepts, which are external to his calculus, and do not express anything themselves.

[5]I use "logical connective" instead of "logical constant" because this is a more neutral name. The name "logical constant" already presupposes some similarity with non-logical vocabulary, because there are also non-logical "constants". Anyway, the word doesn't exactly capture either Frege's or Schröder's perspective. For Frege the logical concepts are not merely connectives, because they have a content of their own. For Schröder the connectives are not exactly logical, because they are part of the "absolute algebra". However, some vocabulary is needed to compare Frege and Schröder and the different perspectives will be worked out in what follows.

[6]Only the value range function and the designation function are not concepts, because their values can be objects other than truth-values. I will speak of "logical concepts", though, because this stresses the similarity of the logical connectives to what is connected. Furthermore these functions play no role in the comparison to Schröder's logic here.

Thus, Peckhaus is right, insofar as the *existence* of a highly developed quantification theory cannot be the criterion for distinguishing the two big traditions in the history of logic. However, Peckhaus does not appreciate the fact that the *conceptualization* of quantifiers, and logical connectives in general, is significantly different in both traditions. So the basic difference is deeper. It is found on a more elementary level.

One central aim of the paper is to show how central Frege's conceptual understanding of logical connectives, his consequent distinction between concepts of different arities and his particular way of concept formation are for his project to show that arithmetic is in fact a part of logic. Thereby it should be appreciated the differences to the logic tradition Frege sets himself against—the Algebra of Logic.

In Section 6, I will furthermore take my investigation of Schröder into account in order to discuss whether these two logic traditions have and can have a real metaperspective.

2. Frege's Logical Concept Formation

In his *Begriffsschrift* of 1879 Frege distinguishes in § 9 between function and argument. From 1884 on, he also distinguishes between concept and object. A concept is unsaturated, i.e., it has at least one argument place. An object, by contrast, is saturated. This distinction is highly important for Frege: in *Grundlagen der Arithmetik* from 1884 one of the three fundamental principles of his inquiries is "never to lose sight of the distinction between concept and object" (Frege 1884 [1953], x). Frege also starts to distinguish between concepts of different order and arity, i.e., the kind and number of argument places.

Grundlagen der Arithmetik is also the publication where Frege starts to reinterpret logical connectives as concepts: He calls existence a second-order *concept* and identity a relation (i.e., a binary *concept*) (Frege 1884 [1953], §54, §65).

This is of particular importance for his project to derive arithmetic from logic. In the *Grundlagen der Arithmetik* Frege also sketches his goal of expressing arithmetical sentences with logical signs only, in order to prove the logical nature of arithmetic. Thus, unlike other logicians before him, Frege does not use logical signs merely to connect non-logical concepts. For Frege, logical connectives are names of logical concepts, and logical sentences can be expressed using these concepts alone. This will be of particular importance for the following discussion. However, it took Frege several years to do away with the seemingly natural distinction between the logical connectives and the conceptual content that they connect. In "Booles rechnende Logik und die Begriffsschrift" from the early 1880s, he distinguishes between the "logical cement" and the "building blocks" (Frege 1880-81 [1979], 13). He obviously did not yet perceive logical connectives as concepts at that point, for otherwise this analogy would make no sense, since the logical concepts would be both at the same time: cement and building blocks.

Even though Frege started to call some logical connectives concepts already in *Grundlagen der Arithmetik*, it took him until 1891 to reinterpret all logical connectives as concept names. In *Funktion und Begriff* he was finally able to give a purely conceptual account of all logical connectives. He could only do this with the help of his idea that sentences denote truth-values. He was then able to define concepts as functions whose values are truth-values. (Generality, for example, is from this point of view a function which maps first-order concepts onto truth-values.) Then, naturally, sentential connectives like the conditional and negation, which were not yet perceived as concepts in the *Grundlagen der Arithmetik*, turn out to be functions which map (single or pairs of) truth-values onto truth-values, and are thus also a particular kind of concepts.

This reinterpretation of his logical signs in the 1890s also had an impact on the technical realization of his concept script: Frege had to apply the distinction between concepts of different order and arity to the logical connectives themselves. Accordingly, he had to rework his concept script of 1879. The result can be found in *Grundgesetze der Arithmetik*. I won't explain how Frege reworked every concept script sign here; instead, I will illustrate what I mean using the example of the concavity.[7]

In the concept script of 1879 Frege introduces the concavity as *one* sign. This sign can be used to express generality over non-function arguments as well as over unary or binary functions (which are conceived as arguments in this context):

$$-\!\!\overset{\mathfrak{a}}{\smile}\!\!-\, f(\mathfrak{a}),\ -\!\!\overset{\mathfrak{f}}{\smile}\!\!-\, \mathfrak{f}(a),\ -\!\!\overset{\mathfrak{f}}{\smile}\!\!-\, \mathfrak{f}(a,b)$$

However, as soon as Frege takes logical signs to be names for concepts, he is forced to distinguish *three* different concepts here, because in the three contexts the concavity sign is used to denote different kinds of concepts. Thus, instead of merely one concavity sign used in three different ways, we find three concept names in the *Grundgesetze der Arithmetik*, all of which contain the concavity sign

$$-\!\!\smile\!\!-$$

First, there is the second-order concept under which all those first-order concepts fall which are true for every object:

$$-\!\!\overset{\mathfrak{a}}{\smile}\!\!-\, \Phi(\mathfrak{a})$$

In this formula "Φ" indicates the empty space for a first-order concept.

Second, the sign which expresses generality over concepts (and functions in general) denotes a third-order concept. This third-order concept holds for all second-order concepts that hold for all *unary* first-order concepts. Frege introduces a new sign to indicate the empty space for such a second-order concept. This

[7] For a comprehensive explanation see Rohr (2020, chapter 1).

is the letter μ occurring in the following expression (Frege 2013, §24):

$$\overset{\mathfrak{f}}{\smile}\ \mu_{\beta}(\mathfrak{f}(\beta))$$

Finally, there is a third-order concept which holds for any second-order concept that holds for all *binary* first-order concepts (Frege 2013, §24):

$$\overset{\mathfrak{f}}{\smile}\ \mu_{\beta\gamma}(\mathfrak{f}(\beta,\gamma))$$

Thus, from the one concavity sign of the concept script of 1879, three distinct concept names emerge in the version of 1893.

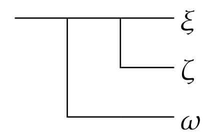
The clear distinction of names of concepts of different order and arity enables Frege to set up a new method of concept formation.

The procedure can be generally explained as follows. Concepts are connected by inserting (object- or concept-) names into the empty spaces of a given concept. One can then obtain a new concept by removing concept and object names from this complex name. The procedure of inserting and removing can then start again.

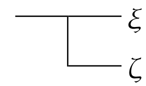
The distinctions between saturated and unsaturated expressions and concepts of different order and arity are central to this kind of concept formation, because they are needed to specify which names can be inserted into an empty space in order to obtain a well-formed expression.

The classical explanation of concept formation can be found in Goldfarb (2001) and Ricketts (2010). However, they focus on the case where we can assume a sentence of ordinary language as given. When Frege realizes his logicist project, however, he does not start with ordinary language sentences, but he only assumes a few logical concepts (and functions) as given. The basic difference is that we always have to start with inserting and not with removing concept names. The formation of new concepts from given ones is explained in *Grundgesetze der Arithmetik*.

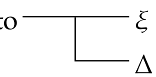
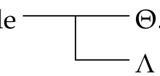
In §12 of his *Grundgesetze der Arithmetik* Frege indicates, for example, how the name of the ternary first-order concept:



can be formed utilizing the concept name:



He writes,

> One can insert insert into  any proper name for "ξ", even for example . Thus we obtain
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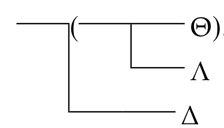
> 
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> wherein we can *fuse* the horizontals:
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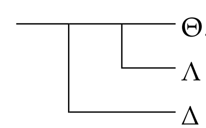
> 
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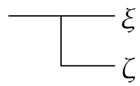
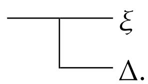
> (Frege 2013, §12).

However, in Frege's explanation, we have in the end a *sentence*, not the corresponding ternary *concept*, because Θ, Λ, and Δ stand for sentences. In order to indicate argument places Frege uses small Greek letters, here ξ, ζ and ω. Likewise in the beginning of Frege's explanation there occurs a Δ. As a result we have a unary instead of a binary concept. Thus, in order to explain how the binary concept name of the implication can be used to form a ternary concept, one needs to add two more steps: one at the beginning, in which the name of a sentence is put into an empty space, and one at the end, in which the three names of sentences are taken out of the formula and leave three empty spaces. Hence, a step-by-step explanation of the formation of the ternary concept out of the binary according to the *Grundgesetze der Arithmetik* would be as follows.

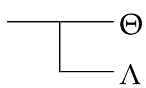
By filling the argument place, "ζ", of



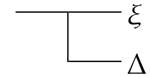
with a proper name, here Δ,[8] we obtain the concept name:



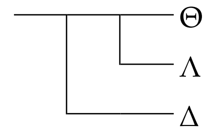
By filling in both argument places, "ζ" and "ξ", with respectively the two proper names Λ and Θ, we can also obtain:



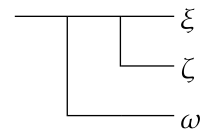
This is a proper name. Thus, we can fill it into the argument place indicated by "ξ" of the concept name



thereby obtaining (after the fusion of the horizontal):



Finally, one removes all three proper names, obtaining the ternary concept name:



Here logical connectives were themselves connected in order to form a new concept. This shows that there is no essential difference for Frege between connectives on the one hand and concepts on the other hand, but connectives *are* (logical) concepts. Logical concepts like the conditional can both connect other contents, and be part of what is connected. The concept script signs are, so to speak, not just part of the *calculus* to connect concepts, but basic signs of the *lingua* which expresses concepts.

In the next two sections we will see that Frege's way of perceiving logical connectives and of forming concepts is significantly different from Schröder's. In Schröder's system, logical connectives only ever connect antecedently given, non-logical concepts. Section 3 describes the basic concepts of Schröder's system. Section 4 discusses how Schröder's system differs from Frege's. In Section 5, we will see which consequences this has for the expressability of logic.

[8]Frege never discusses the justification for using such object names. One has to build them from concept names. As Thiel (1975, 154) suggests, one could build

$$\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}$$

as a name of the truth-value truth, which is an object name according to Frege's *Grundgesetze der Arithmetik*.

3. Schröder's Algebra of Relatives

In contrast to Frege, Schröder neither distinguishes between saturated and unsaturated expressions, nor between concepts of different order and arity.

However, at first glance there seems to be a distinction similar to that of concept and object: In his *Logik der Relative* Schröder introduces a distinction between *elements* on the one hand, and *relatives* on the other. Elements are denoted by capital Latin letters (1895, 4):

$$A, B, C \ldots$$

and relatives are denoted by small Latin letters:

$$a, b, c \ldots$$

Relatives can occur alone or as part of so-called relative coefficients, which have one or more indices:

$$a_i, b_{ij}, c_{ijk} \ldots$$

The indices $i, j \ldots$ are used like variables. They can be substituted by the elements (1895, 7). Thus, for example

$$a_{AB}$$

could be read, in modern terms, as "A is in a-relation to B" or "$a(A, B)$". These expressions may take the value 0 or 1 (1895, 42).

However, for Schröder the elements are not a class of entities *sui generis*, but are themselves relatives.

For Schröder, a class is simply the objects it contains. Thus, he does not distinguish between a class containing only one object and the object it contains. Hence, a relative which contains only one element is identical with this relative.[9] For that reason, elements are (unary) relatives. In the next section, we will see how Schröder provides a unified interpretation of elements and unary relatives with binary relatives.

Another consequence of this conception is that the empty class becomes a very odd object within Schröder's logic. In *Logik der Relative* he presents the following formula:

$$0 =$$

According to Schröder, this is a "complete equation" (*vollständige Gleichung*). On the right hand side of the equation is "literally 'nothing' " (*buchstäblich "nichts"*), which is just what 0 stands for (1895, 26).

In addition to the sign 0 Schröder introduces three more "constant logical relatives" 1, 1′ and 0′. Schröder calls constant logical relatives "modules". (1895, 25–27)

The relative 1 is called the "domain of thought" (*Denkbereich*). Schröder sometimes explicitly distinguishes between domains of different arity[10] written 1^1, 1^2, 1^3 etc. (1895, § 2) The unary domain of thought, 1^1 (*Denkbereich erster Ordnung*), is the sum of all elements, for example:

$$1^1 = A + B + C + D$$

The binary domain of thought, 1^2 (*Denkbereich zweiter Ordnung*), is the sum of all pairs of elements. In modern terminology it is the Cartesian product of 1^1 with itself. Thus, if for example

$$1^1 = A + B,$$

then

$$1^2 = A : A + A : B + B : A + B : B$$

[9]This follows from his formula: $a = \sum_{hk} a_{hk}(h : k)$ (1895, 24). The signs occurring in this formula will be explained in what follows.

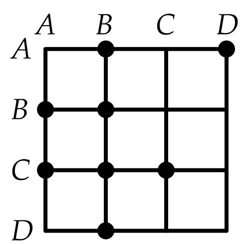
[10]Schröder speaks in this context of the "order" (*Ordnung*). To avoid confusions of order between Frege's sense and Schröder's, I will adopt Frege's usage here and will speak of arity. This also fits Schröder's talk of "unary" (*uninäre*), "binary" (*binäre*) and "ternary" (*ternäre*) relatives. See e.g., Schröder (1895, 1–16).

From this explanation, it should be clear that

$$A : A, A : B, \ldots$$

denote ordered pairs. Schröder (1895, 10) calls them "pairs of elements" (*Elementenpaare*) or "individual binary relatives" (*individuelle binäre Relative*).

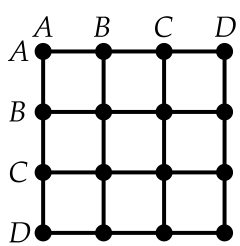
Every relative is defined extensionally by the elements of the domain which it contains. Schröder develops a graphical notation for representing relatives. His first example is the following binary relative (1895, 44):



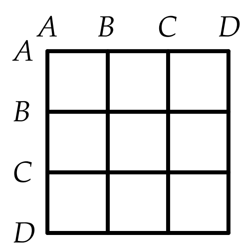
Here the elements of the domain are A, B, C and D. The binary relative represented by this matrix (call it r) would be written in Schröder's linear notation as follows:

$$r = A : B + A : D + B : A + B : B + C : A + C : B + C : C + D : B$$

Similarly, the domain 1 and the empty relation 0 can be represented as follows:

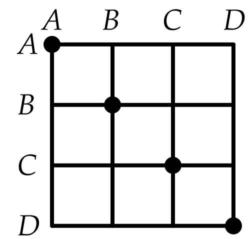


module 1

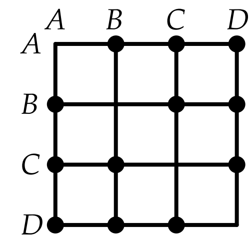


module 0

The constant unary relatives 1′ ("*Einsap*") and 0′ ("*Nullap*")[11] are the relations to which belongs, respectively, every pair of identical elements and every pair of non-identical elements. Thus, they can be represented as follows:



module 1′



module 0′

Schröder's system also contains a number of symbols that are recognizable predecessors of those in contemporary logical languages. Relatives can be connected by the following well-known sentential connectives, which Schröder calls "operations" (1895, 18, 29):

1. the "identical sum", written "+", which is Schröder's inclusive disjunction

2. the "identical product", written "·", which is Schröder's conjunction[12]

3. the negation, written by a stroke above the negated relative "$\bar{a}$"

Schröder also uses the identity sign, which is generally used like ordinary identity, but sometimes also as a biconditional. Occasionally both interpretations have to be applied to a single formula, as the following example shows (Schröder 1895, 119):

$$(a = 1)(b = 1) = (ab = 1)$$

[11] "Eins" and "Null" mean one and zero. "Ap" is an abbreviation of *"Apostroph"* (apostrophe).

[12] As usual in arithmetic, one can write "ab" instead of "$a \cdot b$".

Here the third identity sign from the left should be read as a biconditional, because it holds between expressions which can only take the values "1" and "0". The other identity signs articulate identity between relatives. Since relatives can be different from "1" and "0", these identity signs cannot be read as biconditionals. In Frege's concept script, we could not meaningfully express such a formula. In modern logic, we would read it as a metalogical statement: "a" is true and "b" is true if and only if "a and b" is true. In Schröder's system, the sentence can be read in a similar way: If and only if "$a = 1$" takes the value 1 and "$b = 1$" takes the value 1, then "$ab = 1$" takes the value 1.

There is also a sign which expresses that one relative is subsumed under another one:[13]

$$\Subset$$

The following formula thus expresses that all elements which belong to a also belong to b:

$$a \Subset b$$

Furthermore, Schröder's logic contains two predecessors of our quantifiers:

1. the "product sign", written "$\prod$", which corresponds to the universal quantifier
2. the "sum sign", written "$\sum$", which corresponds to the existential quantifier

Schröder, however, does not count them as "basic operations" (1895, 29). When he explains formulas containing the product and sum sign, he just talks of "identical sum" and "identical product".[14] Thus, just as in arithmetic, Π and Σ are simply shorthand for long iterations of the ordinary binary sum and product signs.

[13] The "*Subsumtionszeichen*" is first introduced in Schröder (1873, 28–29).

[14] See e.g., Schröder (1891, 26) and Schröder (1895, 8).

Expressions containing the quantifiers are generally explained as follows. In an expression like

$$\sum_u f(u)$$

u takes all the values of the domain 1. $f(u)$, according to Schröder (1895, 35), is an expression which contains u as well as constant relatives connected by the logical operations.

However, $\sum$ cannot always be accurately translated as an existential quantifier. Nor can it be always be verbally rendered by "there is".[15] This can be seen in Schröder's definitions for 1^1, 1^2, $1'$ and $0'$. The domain of thoughts is defined as the "identical sum" of all elements or, in the binary domain of thoughts, of all pairs of elements (1895, 8, 10):

$$1^1 = \sum_i i$$

$$1^2 = \sum_{ij} (i : j)$$

The relative modules are defined as the identical sum of all identical ($1'$) or all non-identical ($0'$) pairs of elements (1895, 24–26):

$$1' = \sum_{ij} (i = j)(i : j) = \sum_i (i : i)$$

$$0' = \sum_{ij} (i \neq j)(i : j)$$

Here $\sum$ instead corresponds to the union of sets $\bigcup$. 1^1 is the union of all elements, 1^2 is the union of all pairs of elements, $1'$ is the union of all identical pairs of elements and $0'$ is the union of all non identical pairs of elements.

[15] This is also pointed out, and confirmed via several examples, in Badesa (2004, chapter 2).

In the beginning of this section we saw that Schröder distinguishes between relatives and relative coefficients. With this knowledge about Schröder's operations at hand, we can define relatives as identical sums of some elements of the domain (Schröder 1895, 8). In modern terminology one would say that relatives are classes. The identical sum and the identical product connect relatives. If a and b are relatives, $a + b$ is also a relative.

Relative coefficients can also be connected by these identical operations. Recall that relative coefficients like a_i and b_{ij} always equate to 0 or 1, if we substitute the indices, $i, j \ldots$, by elements. Respectively, the same holds for the identical sum and the identical product of relative coefficients. $a_{ij} + b_{ij}$ always equates to 0 or 1, when the indices are substituted by elements. (Whereas $a + b$ can equate to 0 or 1, but also to any other relative).[16] Relative coefficients thus come closer to what we call concepts in our modern terminology.

The relationship between (binary) relatives and relative coefficients (with two indices) is clarified in the following formula (Schröder 1895, 22):[17]

$$a = \sum_{ij} a_{ij}(i : j)$$

The relative a_{ij} coefficient takes the value 1 if the pair of elements substituted for i and j belongs to a, otherwise it takes the value 0. Thus, we gain the definition of a as a sum of element pairs.

There are also three additional operations which are only defined between relative coefficients. These are called "relative" operations, in contrast to the operations introduced above, which Schröder calls "identical". The relative operations can be defined by means of the identical operations in the following way:[18]

[16]See also Badesa (2004, 37).

[17]The formulas for relatives of other arities would be defined analogously.

[18]The first three formulas can be found at Schröder (1895, 29) and the fourth at Schröder (1895, 24).

1. the "relative sum", written "$\dagger$": $(a \dagger b)_{ij} = \prod_h (a_{ih} + b_{hj})$
2. the "relative product", written ";": $(a ; b)_{ij} = \sum_h (a_{ih} b_{hj})$
3. the "converse", written "$\breve{a}$": $\breve{a}_{ij} = a_{ji}$, a converse relative can be defined by: $\breve{a} = \sum_{ij} \breve{a}_{ij}(i : j)$

Despite this possibility of defining the relative operations by the identical ones,[19] however, Schröder (1895, 29) calls all six operations "basic".

We now have the technical details to understand Schröder's unification of elements with binary relatives and of unary relatives with binary relatives, which will be presented in the following section.

4. Unification of Elements and Relatives

We have now seen that Schröder's logic contains expressions which we can recognize as predecessors of our contemporary quantifier and relation symbols. But for Schröder, the introduction of these expressions leads neither to a principled distinction between elements and relations (which would correspond to Frege's distinction between concept and object) nor to a distinction between relatives of different arity (*"Ordnung"*).

In his introduction to *Algebra und Logik der Relative* Schröder explains that his theory of relatives provides methods which enable us to reinterpret relatives of a particular arity into relatives of a different arity:

> Finally, however, it should be pointed out in advance that the theory of relatives will provide the possibility and a procedure, in order to *reinterpret* expressions, as well as relations, formulas or sentences,

[19]Schröder does not introduce the equations on the right hand side as definitions, but just as true statements.

of relatives of a particular arity from that shared domain into a domain *of different arity* (Schröder 1895, 15–16).[20]

Later in the book, Schröder shows how an element can be perceived as a binary relative (1895, 24):

$$i = \sum_{hk} i_{hk}(h : k)$$

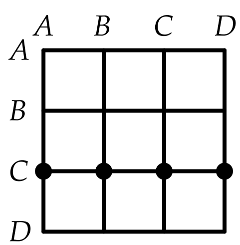
In order to understand this equation we need the following formulas (1895, 25):

$$i_{hk} = 1'_{ih}$$
$$1'_{ij} = (i = j)$$

Thus,

$$i = \sum_{hk}(i = h)(h : k).$$

This means that the element i is identical with the sum of all element pairs which have i as their first element. In the graphical notation presented above elements are presented as binary relatives with exactly one full row.[21] The element C, for example, would be presented as:



C binary

[20] In the German original: "Endlich aber soll im voraus darauf hingewiesen werden, dass die Theorie der Relative die Möglichkeit schaffen und ein Verfahren aufstellen wird, um Ausdrücke, sowohl als Relationen, Formeln oder Sätze, von Relativen einer bestimmten Ordnung aus diesem ihrem gemeinsamen Denkbereich *umzudeuten* in einen Denkbereich *von anderer Ordnung*."

[21] Schröder himself uses this graphical explanation and uses the word *"Vollreihe"* (full row); see Schröder (1895, 140–42).

Thus, there is no sharp difference between elements and (binary) relatives, since a sign for an element can always be reinterpreted as a sign for a binary relative, using the formulas above. This stands in sharp contrast to Frege's clear distinction between concepts of different order and arity. I will return to this important difference later.

As explained earlier, a unary relative is defined as the sum of the elements it contains. Since elements can be redefined as binary relatives with exactly one full row and unary relatives are sums of elements, unary relatives have to be redefined as binary relatives where each row is either completely full or completely empty. The full rows thereby correspond to the elements of the unary relative.

Formally, Schröder's (1895, 140) reinterpretation works as follows:

$$(a\,;1)_{ij} = \sum_h a_{ih} 1_{hj} = \sum_h a_{ih}$$

He later writes explicitly (1895, 464):

$$a_i = (a\,;1)_{ij}$$

Thus:

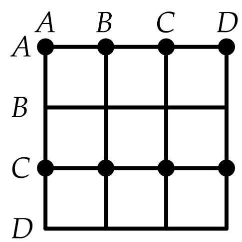
$$a_i = \sum_h a_{ih}$$

The relative coefficient a_i has only one index, a_{ij} has two. So, is the relative a a sum of elements or of pairs of elements? Schröder (1895, 141) explains that if one interprets a as binary, one can fill up up all non-empty rows of a, forming "full rows" in order to attribute a meaning to a_i which is consistent with the binary understanding of a. This procedure of filling up the rows is formally expressed by $a\,;1$ in the formula $a_i = (a\,;1)_{ij}$.

Thus, $a_i = 1$ if and only if for all j, $a_{ij} = 1$. This is expressed by the formula

$$a_i = \sum_h a_{ih}$$

If for example, in the unary interpretation of s_i, $s = A+C$ holds and our domain again contains A, B, C and D as the only elements, then the binary interpretation of the relative belonging to the relative coefficient s_i would be given graphically represented as follows:



s_i *binary*

Note that Schröder's formulas here are equations in the normal object language. So we have binary relatives on both sides of the equation $a_i = \sum_h a_{ih}$. Otherwise, the equation would not be true. We need, in Schröder's own words, to "redefine" (*"umdeuten"*) the relatives as relatives of higher arity (*"Ordnung"*).

It is noteworthy that Frege, in contrast to Schröder, does not have to reinterpret the unary concept as a binary. With his concept formation explained in Section 2 of this paper, Frege can form a unary concept from a binary concept $R(x, y)$ and the second-order unary concept of existence (which is itself formed from the second-order unary concept of generality and the unary first-order concept of negation) by filling in the empty space of the latter concept, existence, with the former one, $R(x, y)$.

Schröder and Frege thus have two opposed strategies for handling logical combinations of relatives of different arity: Frege takes this difference as absolute and utilizes it for concept formation in the way sketched above, while Schröder reinterprets relatives of different arity into the same arity. If Schröder had not done that, he would have had to define his logical operations for every (combination of) arities. In what follows, we will see what light this difference sheds on the fundamental preconditions of Frege's foundational project.

5. Concept Formation in Frege and Schröder

In the last section it was shown that Schröder tries to overcome the absolute distinction between relatives of different arity. Schröder preserves another distinction, however, which Frege finally abolishes in 1891: the distinction between logical signs and non-logical ones. For Schröder there are no such things as logical concepts in the Fregean sense. The logical connectives are operations. All concepts (which he calls "relatives") must be taken from outside. For him the *lingua* is not part of, but separate from, his *calculus*.[22] The same holds for Leibniz's own attempts for a universal characteristic. The idea that logical connectives are themselves concepts was alien to Leibniz.

As we saw, relatives are sums of elements in Schröder's logic. Logical signs, on the other hand, are names of operations. The signs for identical sum and identical product are signs for the operations to build the union and the intersection of relatives. Negation is the operation to build the complement to a relative. Schröder never made any attempt to define these operations as relatives.

Thus, we can sum up: for Frege the fundamental distinctions are (1) the distinction between saturated and unsaturated expressions and (2) the more fine-grained distinctions between unsaturated expressions according to the kind and number of their empty spaces (that is, their order and arity). Schröder on the other hand takes the difference between relatives and operations as fundamental. He does not stress this distinction, because he seemingly takes it for granted, but he does occasionally mention it. His first book on logic from 1877 is called *Operationskreis des*

[22] In the foreword of the first volume of *Vorlesung über die Algebra der Logik* (1890), Schröder sketches how such a language would look and says that it is a task for philosophers to work this out.

Logikkalkuls. In this book he introduces sum, product and negation as operations to calculate with concepts. In the introduction to the first volume of his *Vorlesung über die Algebra der Logik*, he explains that "our whole system of concepts" can be built if all concepts are formed out of "basic concepts" by connecting them with "basic operations" (1890, 93). Here he mentions that the "concepts of these operations will be partly counted as basic concepts in a certain sense." However, he never really clarifies what a "concept of an operation" is, and the word "concept" does not play a central role in Schröder (1895). Thus the remark only seems to show that Schröder was not really clear about the relationship between the logical connectives and what they connect. At the beginning of *Algebra und Logik der Relative* Schröder talks of "operanda" and "operation" (1895, 3).

Relatives are not seen as unsaturated expressions. As we saw above, one finds the expression $f(u)$ in Schröder's writings, which is defined as "an expression which is built by operations... and contains u itself and some other relatives". However, Schröder does not distinguish clearly between the functional expression $f(\ )$ and its argument u. He writes that "the function $f(u)$ is itself a binary relative" (1895, 35), but he never explains what $f(\ )$ itself is. The saturated/unsaturated distinction is totally alien to Schröder. And since it is totally alien for Schröder to think of relatives as unsaturated expressions, they are for him totally different from logical connectives.

This is important in order to understand the difference in possibilities for concept formation in Frege's and Schröder's systems, especially in the possibility of purely logical concept formation. Frege can build an infinite number of new concept names from the *logical* vocabulary alone. For Frege logical connectives are concepts and concepts can be formed by removing and inserting concept names. This method of concept formation can be iterated. Let for example $R(x_1, x_2)$ be a binary logical concept. Then Frege can insert into the empty space marked by x_2 the object name $R(a, b)$ (which denotes a truth-value, which is a particular kind of object). Thereby we gain $R(x_1, R(a, b))$. In another step we can think of a and b as substitutable. We then get $R(x_1, R(x_2, x_3))$, which is a ternary concept.[23] In an analogous way we can obtain $R(x_1, R(x_2, R(x_3, x_4)))$, and so on.

Schröder in contrast cannot form any concepts from his connectives alone at all, because the connectives are for him not concepts, but operations. And even if we presuppose some relatives as given, the Fregean methods of concept formation are not possible in Schröders algebra of logic. We cannot analogously insert r_{AB} into r_{ij} to obtain $r_{ir_{AB}}$ and then by removing A, B obtain $r_{ir_{jk}}$. The first step would not work, because i and j can only be substituted by elements such as A, B, C (1895, 7). The expression r_{AB}, however, does equate to either 0 or 1, which, as we saw above, are "literally nothing" and the sum of all elements. In either case we do not have an element. Thus, $r_{ir_{AB}}$ is not a meaningful expression. For the same reason $r_{ir_{jk}}$ is not a meaningful expression. If one substitutes j and k in r_{jk} by elements, one gets an expression which equates to 1 or 0, which are no elements. Thus, relative coefficients cannot fulfill the same task as indices.

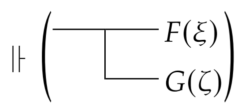
Here again we see the difference in concept formation which is deeply connected with the different understanding of logical connectives. Schröder connects relatives by operations, Frege by inserting concepts into others and removing them and, if necessary, iterating this process. As we have just seen, the Fregean way of concept formation is unavailable in Schröder. And of course Schröder's way of concept formation is unavailable in Frege as well, because there are no operations in Frege's logic.

We also have seen in the last section that Schröder reinterprets concepts of different arity within one formula in such a way that all relatives have the same arity. So even by logical operations we cannot obtain relatives of other arities.

[23] Frege shows this with the example of the conditional in *Grundgesetze der Arithmetik*, as we saw in Section 2 above.

So unlike Frege, who can form infinitely many concepts from a finite number of (logical) concepts, Schröder can only form a finite number of new concepts out of a finite number of given concepts. We even can quantify this more precisely. Let's limit the algebra of logic for a moment to the sentential connectives: logical sum, logical product and negation. We can then obtain only 2^{2^n} new concepts from n given concepts, because there are 2^{2^n} n-ary logical connectives in propositional logic.

The difference between Frege and Schröder in their attitudes toward arities can be explained by a simple example. Let us assume that we have two unary predicates, F(ξ) and G(ξ), or in Schröder's notation a_i and b_i. Frege can now define a binary predicate in the following way:



$$= H(\xi, \zeta)$$ [24]

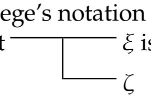
Here Frege gains a concept, which is of different arity from the concepts used to define it.[25] Thus, this new concept is not contained in the 16 possibilities of connecting $F(\xi)$ and $G(\xi)$ (note the difference from $G(\zeta)$!) by propositional connectives.

Analogously we could make the following definition in Schröder's logic (for simplicity I now use a different connective):

$$c_{ij} = a_i b_j$$

However, in order to gain a binary relative c one must reinterpret a and b as binary relatives in the way sketched in the last section. For example, let our elements again be A, B, C, and D, and let $a = A + B$ and $b = B + D$. If we do not reinterpret a and b in a binary way, we first get the result

[24] ⊩ indicates a definition in Frege's notation (2013, §27).

[25] Though of course the concept  is binary. In any case, Frege can combine concepts of different order and arity.

$$c = B$$

But we want to get the following result:

$$c = (A : B) + (B : B) + (A : D) + (B : D)$$

We get the intended result in the following way. First we have to reinterpret a_i and b_j as binary using the formulas presented in the last section. We then get:

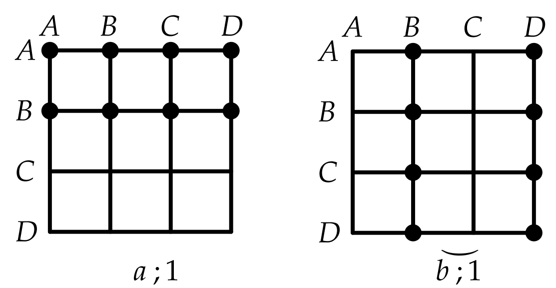
$$\begin{aligned} a\,;1 &= (A : A) + (A : B) + (A : C) + (A : D) + \\ &\quad (B : A) + (B : B) + (B : C) + (B : D) \end{aligned}$$

$$\begin{aligned} b\,;1 &= (B : A) + (B : B) + (B : C) + (B : D) + \\ &\quad (D : A) + (D : B) + (D : C) + (D : D) \end{aligned}$$

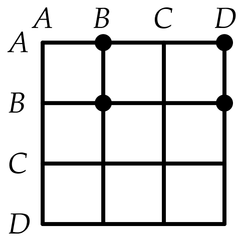
Since in our formula we have b_j and j is the second index in the defined relative coefficient a_{ij}, we need the converse of $b;1$, which is the following:[26]

$$\begin{aligned} \breve{\overline{b\,;1}} = \breve{b}\,;1 &= (A : B) + (A : D) + (B : B) + (B : D) + \\ &\quad (C : B) + (C : D) + (D : A) + (D : D) \end{aligned}$$

Graphically, this can be represented in the following way:



[26] For the first part of the equation see Schröder (1895, 85).



c

In this graphical notation any instance of

$$c_{ij} = a_i b_j \left(= (a\,;1)_{ij}(\breve{b\,;1})_{ij}\right)$$

can be verified. For example, $a_A = 1$ (row A is full in a) and $\breve{b}_A = 0$ (column A is empty in $\breve{b}$) and also $c_{AA} = 0$ (the pair $A : A$ is not in c). Thus, $1 \cdot 0 = 0$, which is correct according to the Algebra of Logic.

In this presentation it is also obvious that the binary relative c is just one of the 2^{2^n} (here $n = 2$) possible combinations of a and b, which are here also interpreted as *binary* relatives. Despite the occurrences of relative coefficients with different numbers of coefficients, we never really step from the unary to the binary in Schröder's logic. We are limited to the 2^{2^n} possibilities for combining n concepts of the same arity.

Even if we add $\sum$ and $\prod$ to the connectives under consideration, this does not change the result that Schröder cannot form concepts of different arity from the concepts used to form it. In a definition of the form:

$$\Vdash \left(\overset{\mathfrak{a}}{\smile} G(\mathfrak{a}, \xi)\right) = F(\xi)$$

Frege does form a unary concept utilizing a binary one (and two unaries, since the concavity and negation are for Frege concepts as well).

In formulas of the form:

$$a_i = \sum_h b_{ih}$$

especially in the particular case of this:

$$a_i = \sum_h a_{ih}$$

which is, as we have seen in the last section, according to Schröder a true sentence in the algebra of logic, Schröder interprets all involved concepts as binary. Schröder does not really state something about the relationship between a binary and a unary concept. The fact that a_i has one and a_{ij} has two indices does not mean that they are unary and binary concepts. The superficial resemblance of relatives like a_i and b_{ij} in Schröder's logic and concepts like $F(\xi)$ and $G(\xi, \zeta)$ in Freges logic is no proof of a deeper similarity of the way these notions are understood by both logicians.

Schröder's logic is, thus, completely unsuitable for Frege's goal of building purely logical concepts, a purpose which is crucial for Frege in order to prove the logical nature of arithmetic. First, for Schröder the connectives of his algebra of logic are not concepts. Second, Schröder's logic cannot be used to build an infinite number of new concepts from a small number of given concepts, because he cannot form concepts of different arity. But this is exactly what Frege needs to do in order to form all arithmetical concepts just from the small logical vocabulary of the Begriffsschrift.

Schröder has a completely different idea of what a *lingua* is. For Schröder, the concepts of the *lingua* are not logical concepts—because there are no such things for him—but concepts taken from other branches of science.

One can contrast this with Frege's understanding of the *lingua* more explicitly. For Schröder, as for Leibniz, *lingua* and *calculus* are two completely separate parts of the universal *characteris-*

tic. For Frege, the signs of the *calculus* are also the signs of the concepts of the *lingua*—at least in arithmetic.

The whole idea that arithmetic is indeed logic and that this shows the analytic a priori character of arithmetic is completely alien to Schröder. For Schröder, logic does not even have the same status. For Frege, logic is the most general science, while for Schröder it is embedded into absolute algebra (Peckhaus 2004b, 596–97). In summary, these both contemporaries have very different ideas of the nature and purpose of logic.

6. Metalogic in the Nineteenth Century?

In "Logic as Calculus and Logic as Language" van Heijenoort draws the attention to the fact that there existed another important logic tradition in 19th and early 20th century besides the Frege-Russell school, namely the Algebra of Logic. He pointed out that Boole already had the idea of changing universes; an idea central to modern model theory. Thus, it seems to be no surprise that Löwenheim's famous 1915 paper was written in Schröder's notation.[27]

However, in this last section, it will be argued in the light of the preceding investigation that Schröder and even Löwenheim lack a real metaperspective. More precisely, they cannot have a metaperspective, because the lack of the modern distinction between object- and metalanguage is central to the algebraic logicians approach.

Van Heijenoort (1967, 325) points out correctly that the relative 1 resembles the modern domain, because it contains all elements and can be changed at will. However, we have seen that "1", the domain of thought, is not part of the metalanguage, but is itself simply a relative, which is used *within* the calculus (Schröder 1895, 464):

$$a_i = (a\,;1)_{ij}$$

[27]Dreben and van Heijenoort (1986) also set Löwenheim in the context of Boole and Schröder and stress their importance for model theory.

Even stating which elements the universal class contains is something that is expressed within the calculus (1895, 5):

$$1^1 = A + B + C + D \ldots$$

Similarly, there is no such a thing as an assignment function. The assignment is rather expressed within the object language, as can be demonstrated with the following formula (1895, 43):

$$a = A : B + A : D + B : A + B : B + C : A + C : B + C : C + D : B$$

Thus, there is no separation between a formula and its interpretation.

Schröder did not introduce the domain of thought in order to interpret previously given formulas; he needs it in order to set up his calculus in the first place. Relatives are extensionally defined, and the universe of discourse is just another relative. But since "1" is used within this calculus and the assignment is also given within the calculus, there is no formal separation of the object language and a metalanguage within which a model can be specified. Moreover, this missing separation is not an inessential feature of Schröder's calculus; it is central to it. If one tries to read all formulas in which "1", "0", an element, or an element pair occur as being part of the metalanguage, most formulas in Schröder's work would be classified as belonging to the metalanguage. But which (logical) language would this metalanguage be the metalanguage *of*?

As a consequence, one has to be very cautious in attributing modern metalogical notions to the algebraic logicians. Goldfarb claims that in the Algebra of Logic

> the following sort of question is investigated: given an equation between two expressions of the calculus, can that equation be satisfied in various domains—that is, are there relations on the domain that make the equation true? This is like our notion of satisfiability of logical formulas (Goldfarb 1979, 354).

Indeed there are notions such as 0 and 1, that have different meanings. Thus, one can think about such questions such as "If $0 = 0'$, how many elements do we have?"[28] Our notion of satisfiability, however, requires a clear distinction between the formula and its interpretation. Since we do not find this distinction in Schröder,[29] it would be a mistake to see our notion of satisfiability as appearing in Schröder's work. Thus, what algebraic logicians do is only in a very weak sense of the word "like" our notion of satisfiability.

As we have seen in the previous sections, even the question of which arity a given relative has can only be answered by looking at the formula to which it belongs.

Let us think about possible interpretations of the relatives occurring in the following set of formulas:

1. $a = A + B$
2. $c = A : B + B : B + A : D + B : D$
3. $c_{ij} = a_i + b_i$

According to the first two formulas a and b are of different arity. As we have seen in the last section, however, in the last formula all relatives have to be interpreted as being of the same arity. If we try to interpret a_i as being of higher arity, we have to use the formula $a_i = (a\,;1)_{ij}$ presented in Section 4. Since a is explicitly given as a unary relative, however, $a\,;1$ is only a unary predicate as well. Only if one already interprets the elements A, B as binary relatives, a becomes binary as well. However, c then becomes a sum of pairs of binary relatives. Thus, the third formula again cannot be interpreted consistently. Thus, not all arbitrary sets of formula can be interpreted consistently.

[28] Löwenheim (1915, 449) pointed out that this holds only if we have exactly one element.

[29] Goldfarb notes that Schröder confuses set-theoretic and sentential interpretations and that Löwenheim has no model-theoretic notion of logical consequence.

The problem here is that formulas like the first and third would usually not be considered object language formulas. But Schröder has no other resources to express assignments for relatives, the chosen domain, and so on.

In his book *The Birth of Model Theory*, Badesa points out explicitly that Schröder lacks our modern separation between object and metalanguage:

> [N]one of the distinctions that today separate syntax from semantics are present in the logic of relatives. There is no precise notion of formal language and, naturally enough, no distinction is made between object language and metalanguage (Badesa 2004, 65).

In this book on Löwenheim's proof of the so-called Löwenheim-Skolem theorem, Badesa also argues that the same holds even for Löwenheim. In his paper, Löwenheim shows how a countable model can be built for a formula which is known to be satisfiable in an uncountable model. In order to do this, Löwenheim substitutes indices like i, j, k etc. by natural numbers 1, 2, 3 etc., in an iterative procedure. However, it is unclear whether these numbers are—in modern terminology—just uninterpreted symbols, which may denote different elements in different domains, or if they are names of elements of a given domain. This subtle point turns out to be highly important: If they are just uninterpreted names, Löwenheim proved the weak version. If these constant names are interpreted within the presupposed uncountable model, in which the formula is satisfied by hypothesis, Löwenheim proved the strong version of the Löwenheim-Skolem-theorem.[30] Löwenheim lacks the vocabulary in order to clearly express this difference between both version, because he expresses himself in Schröder's algebra of logic. As we have seen, the difference between a name and what it denotes cannot be made clearly, because object and metalogic are not distinguished.

[30] Badesa (2004) argues that Löwenheim indeed proved the strong version (with some minor mistakes), contradicting van Heijenoort, who argues that Löwenheim proved only the weak version.

Furthermore, Löwenheim only proved the theorem for single formulas. The reason for this might be that he, like Schröder, was not always able to interpret an arbitrary set of formulas consistently.

It was in Skolem's work that the difference between the weak and the strong version became clear. In Skolem (1920), he published a proof for the strong version, and in Skolem (1923), he also presented an alternative proof for the weak version. He also was able to prove the theorem for sets of formulas, not just single formulas. Skolem studied the Algebra of Logic, but was also influenced by the Frege-Russell tradition[31] and Dedekind's work.[32] So he was not a full-blooded algebraic logician. Thus, it seems that the history of the late nineteenth century and its influence on our modern concept of logic is still in need of further clarification.

7. Summary

The results of this paper can be briefly summarized as follows:

1. For Frege logical signs are signs of concepts, while for Schröder they are signs of operations.

2. Frege makes a sharp distinction between concepts of different order and arity. Schröder, in contrast, expends a lot of effort to unify relatives of different arity.

3. The number of concepts which can be formed in Schröder's logic depends on the number of concepts presupposed. In Frege's logic, the iterative method of concept formation allows forming infinitely many concepts of different order and arity just from the finite set of given logical concepts.

4. Though Schröder's logic was undoubtedly highly important for the birth of model theory, there is no distinction between meta- and object language.

[31]He refers to *Principia Mathematica* several times in the writings around 1920.

[32]Skolem (1920) uses Dedekind's chains (*Ketten*) to construct a countable submodel.

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