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Solving the Conjunction Problem of Russell's *Principles of Mathematics*

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The quantification theory of propositions in Russell's *Principles of Mathematics* has been the subject of an intensive study and in reconstruction has been found to be complete with respect to analogs of the truths of modern quantification theory. A difficulty arises in the reconstruction, however, because it presents universally quantified exportations of five of Russell's axioms. This paper investigates whether a formal system can be found that is more faithful to Russell's original prose. Russell offers axioms that are universally quantified implications that have antecedent clauses that are conjunctions. The presence of conjunctions as antecedent clauses seems to doom the theory from the onset, it will be found that there is no way to prove conjunctions so that, after universal instantiation, one can detach the needed antecedent clauses. Amalgamating two of Russell's axioms, this paper overcomes the difficulty.

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1. Introduction

It is not well known that Russell's *Principles of Mathematics* (*PoM*) offered a viable quantification theory. Extracting the theory from the work is not easy, however. Byrd (1989) pioneered the way, and my paper "Logic in Russell's *Principles of Mathematics*" (1996) discussed the inference rules, axioms and definitions of the logical particles in great detail. I argued that though Russell's presentation of the system was informal and scattered throughout the early parts of the work, the system can be reconstructed in a way that captures a genuine quantification theory which is semantically complete with respect to analogs of the truths of modern quantification theory.

It is of utmost importance to begin by getting straight on the p's and q's of the formal system of *PoM*. Russell explicitly says that in order to say "p is a proposition" one is to write "$p \supset p$".[1] The letters "p" and "q" and "r," etc., are, therefore, not special variables for Russellian propositions. They are individual variables no different from "x" and "y" and "z," etc. This is obvious from the formulation Russell gives of his axioms Ax1–Ax10. The first few are these:

Ax1:	$p \supset q \;.{\supset_{p,q}}.\; p \supset q$
Ax2:	$p \supset q \;.{\supset_{p,q}}.\; p \supset p$
Ax3:	$p \supset q \;.{\supset_{p,q}}.\; q \supset q$
Ax5:	$(p \supset p)(q \supset q) \;.{\supset_{p,q}}.\; pq \supset p$

[1] This was noticed in Griffin (1980) and Cocchiarella (1980).

Had he intended the letters p, q, etc., to stand for propositions only, there would be no need to have the clause "$p \supset p$" in axiom Ax5. Indeed, the absence of such a clause from Ax1–Ax3 would have been impossible to express if he had used special proposition letters. Of course, by modern syntax, expressions such as "$x \supset x$" and "$x \supset y$" are sure to look ill-formed. But the syntax of *PoM* is not the modern syntax of quantification theory. It is quite different. The horseshoe sign is a relation sign which is flanked by terms to form wffs. The relation sign "$\supset$" is for a relation of *implication* and must be flanked by terms and not wffs. To facilitate this, I use nominalizing brackets so that any wff φ of the object-language can be transformed into a term "$\{\varphi\}$". Russell simply assumed readers of *PoM* would understand the implicit nominalizing transformations. Dropping the nominalizing brackets for convenience, and using dots symmetrically for punctuation we can write $x \;.{\supset}.\; y \supset z$, instead of the more demanding $x \supset \{y \supset z\}$. I shall use α and β for any terms, whether individual variables or nominalized wffs.

The differences between term and wff must be kept straight if one is to have any hope of conducting an intelligible study of *PoM*. Recall that we are using letters α and β for terms, whether individual variables or nominalized wffs; and we use the letters φ and ψ as schematic for wffs which when in subject positions have undergone nominalization to terms $\{\varphi\}$ and $\{\psi\}$. The difference is of central importance when it comes to the statement of rules of inference. We must never slight the distinction between a wff and a term. Modus ponens is this:

$$\text{From } \varphi \text{ and } \{\varphi\} \supset \{\psi\}, \text{ infer } \psi.$$

The rule of modus ponens is not to be conflated with any of the following unintelligible rules:

$$\text{From } \alpha \text{ and } \alpha \supset \beta, \text{ infer } \beta.$$
$$\text{From } x \text{ and } x \supset y, \text{ infer } y.$$
$$\text{From } p \text{ and } p \supset q, \text{ infer } q.$$

Perhaps the last is most apt to mislead because the modern quantification theory uses P and Q etc. as statement letters. As we noted, in Russell's *PoM*, every lower case letter of the English alphabet is an individual variable and thus the letters p, q and r etc. are on a par with the letters x, y and z. The same point applies to any derived rules of *PoM*. For instance, using juxtaposition $\alpha\beta$ for conjunction, the derived rule of simplification is this:

$$\text{From } \{\varphi\}\beta, \text{ infer } \varphi.$$

It is not to be conflated with the following unintelligible rules:

$$\text{From } \alpha\beta, \text{ infer } \alpha.$$
$$\text{From } xy, \text{ infer } x.$$
$$\text{From } pq, \text{ infer } p.$$

It takes a bit of getting used to. But the transition to a proper reading of the notations of *PoM* is ultimately not very difficult.

A difficulty arises, however, with respect to my original study in Landini (1996) because it adopts exportations of axioms Ax5–Ax10 of *PoM*. Russell set out the theory in an informal prose and did not use formal symbols. This leaves some room for interpretation of his intent. But Russell's prose strongly suggests that, where his axioms Ax5–Ax10 are concerned, he intended universally generalized axioms whose antecedent clauses are conjunctions.[2] For example, we find that Russell states his fifth axiom as follows, noting in a footnote that "... the implications denoted by *if* and *then*, in these axioms, are formal, while those denoted by *implies* are material" (*PoM* 16):

If p implies p and q implies q, then pq implies p.

Puting this in symbols in a literal way, one arrives at the following:

$$\text{Ax5:} \qquad (x \supset x)(y \supset y) \mathbin{.\!\supset_{x,y}\!.} xy \supset x$$

The juxtaposition $(x \supset x)(y \supset y)$ as well as xy express conjunction in PoM, and thus the above is naturally called *simplification*. The antecedent clause is needed because conjunction applies only to propositions. When x is not a proposition, the conjunction xy obviously doesn't imply x. One must never arrive at a single letter on a line of proof. Only a wff that indicates a proposition can be isolated on a line of proof. In my original paper, I restated the fifth axiom as if it were as follows:

$$^{L}\text{Ax5:} \qquad x \supset x :\supset_{x,y}: y \supset y \mathbin{.\supset.} xy \supset x$$

Accordingly, where φ is a wff for which we can prove $\varphi \supset \varphi$ we can do a universal instantiation and detach. Similarly, we can detach when we have $\psi \supset \psi$. The question before us is whether exported forms such as ^{L}Ax5–^{L}Ax10 are essential for the formal workability of the theory. This matter is far from trivial. The presence of conjunctions in the antecedent clauses would seem to doom the logical system of the historical *PoM* from the onset. There is no way in the system to deduce conjunctions! I call this the conjunction problem of *PoM*.

Once the conjunction problem is noticed, it is natural enough to ask whether Peano had the problem as well. After all, it is well-known that Russell's philosophical logic began from his examination of Peano's work. If we look at Peano's *Formulaire de mathématiques*, vol. II (1897; pp. V and 32 §1 Notes) we find the following quite illuminating passage:

15. $a, b, c \ \varepsilon$ Cls $.\supset::$
$x \ \varepsilon \ a \mathbin{.\supset_x.} x \ \varepsilon \ b \mathbin{.\supset.} x \ \varepsilon \ c \mathbin{.\!\cdot\!.\!=:} x \ \varepsilon \ a \ . \ x \ \varepsilon \ b \mathbin{.\supset_x.} x \ \varepsilon \ c$ Df

And on p. 11, we find the following:

[2] This was pointed out by Milan Soutor (a graduate student at Charles University, Prague) who was then on a Fulbright and visiting at the University of Iowa in 2014–15. Soutor's excellent critical concerns were what led to this paper.

$\cdot 1\ \ a, b, c\ \varepsilon\ \mathrm{Cls} : x\ \varepsilon\ a\ .\ (x; y)\ \varepsilon\ b\ .\supset_{x,y}.\ (x; y)\ \varepsilon\ c\ :\supset\therefore$
$x\ \varepsilon\ a\ .\supset_x : (x; y)\ \varepsilon\ b\ .\supset_y .\ (x; y)\ \varepsilon\ c$ Export

And we find as well:

$\cdot 2\ \ a, b, c\ \varepsilon\ \mathrm{Cls} : x\ \varepsilon\ a\ .\supset_x : (x; y)\ \varepsilon\ b\ .\supset_y .\ (x, y)\ \varepsilon\ c\ \therefore\supset :$
$x\ \varepsilon\ a\ .\ (x; y)\ \varepsilon\ b\ .\supset_{x,y}.\ (x; y)\ \varepsilon\ c$ Import
"Import" signifie faire l'opération inverse de "exporter".

This passage reveals that Peano had attempted to build exportation (and importation) into the rules (and definitions) involved in his quantification theory itself. Had this worked, it would avoid the conjunction problem altogether. Of course, it could work as a primitive new rule, but *not* as a part of a definition. It is important, therefore, to investigate whether a more historically accurate account of the logic of *PoM* is workable or whether exported restatements of the axioms 5–10 are required. I do not think that Russell committed the error that Peano committed with exportation. Accordingly, in what follows, I will first offer an error theory, explaining how it is that Russell did not notice the conjunction problem. I then show how one can avoid the conjunction problem by simply amalgamating Russell's second and third axioms into one. The needed amendment is therefore quite minimal.

2. An Error Theory

How is it that Russell missed the conjunction problem of *PoM*? We need an error theory, since on the face of it the problem is so salient that it is difficult to imagine it could have been missed. We are drawn immediately, of course, to the definitions of the logical particles given in *PoM*. The definitions of the logical particles that Russell adopted in *PoM* pose various difficulties discussed in Byrd (1989). Russell presented conditional definitions. In the case of negation, he writes:

> Hence we proceed to the definition of negation: not p is equivalent to the assertion that p implies all propositions, *i.e.*, that "r implies r" implies "p implies r" whatever r may be. (*PoM* 18)

In Landini (1996), I avoided the problem, putting:

$$\sim\alpha =_{\mathrm{df}} z \supset z\ .\supset_z .\ \alpha \supset z$$

This applies to any term α, and since both x and $\{\varphi\}$ are terms, we get both of the following instances:

$$\sim x =_{\mathrm{df}} z \supset z\ .\supset_z .\ x \supset z$$
$$\sim\varphi =_{\mathrm{df}} z \supset z\ .\supset_z .\ \varphi \supset z,$$

where φ is any wff of the formal language. For disjunction and conjunction, Landini (1996) avoids Russell's conditional definition by adopting the following:

$$\alpha \vee \beta =_{\mathrm{df}} \alpha \supset \beta\ .\supset .\ \beta$$
$$\alpha\beta =_{\mathrm{df}} \sim(\alpha \supset \sim\beta)$$

These hold for any terms α and β and thus respectively enable a definition of $x \vee y$ and xy as readily as for $\varphi \vee \psi$ and $\varphi\psi$, where φ and ψ are any wffs of *PoM*'s formal language. We needn't be detained by the definition of disjunction. But in defining conjunction ("logical product"), the fact that Russell offered a conditional definition might be of importance for the conjunction problem. Thus, it is worth taking a close look. He writes:

> If p implies p then if q implies q, pq (the logical product of p and q) means that if p implies that q implies r, then r is true. In other words, if p and q are propositions, then their joint assertion is equivalent to saying that every proposition is true which is such that the first implies that the second implies it. (*PoM* 16)

As Byrd points out, there is a slip that is corrected by Russell himself in the adjoining sentence of clarification. Russell read "r" as if it stood only for propositions and then immediately corrects himself. In Landini (1996), I corrected and avoided the conditional definition as follows:

$$\alpha\beta =_{\text{df}} z \supset z \,.\!\supset_z\!.\, (\alpha \,.\!\supset\!.\, \beta \supset z :\supset: z)$$

This yields a definition both for xy and for $\varphi\psi$. The latter is:

$$\varphi\psi =_{\text{df}} z \supset z. \,.\!\supset_z\!.\, (\varphi \,.\!\supset\!.\, \psi \supset z :\supset: z)$$

Unfortunately, even with the rule of conditional proof, one cannot use this to arrive at a conjunction. Let's try:

1. φ	premise
2. ψ	premise
3. $z \supset z$	assumption for cp
4. $\varphi \,.\!\supset\!.\, \psi \supset z$	assumption for cp
5. $\psi \supset z$	1, 4, mp

This cannot succeed. We cannot use 2 and 5 in a modus ponens since no single variable may occur on a line of proof.

We take a step toward finding an error theory, however, by realizing the the above is on the right track. In a 1904 letter to Frege, Russell contemplated an alternative definition of tilde (and accordingly a definition of conjunction) which, in fact, avoids the conjunction problem entirely. The alternative definitions for tilde are these:

$$\sim\alpha =_{\text{df}} \alpha \supset f$$
$$f =_{\text{df}} (x)(x \supset x \,.\!\supset\!.\, x)$$

Now imagine that conjunction were defined as follows:

$$\alpha\beta =_{\text{df}} \sim(\alpha \supset \sim\beta)$$

Paired with the definitions for tilde, the definition of conjunction becomes this:

$$\alpha\beta =_{\text{df}} \alpha \,.\!\supset\!.\, \beta \supset f :\supset: f$$ [3]

[3] It must be understood that in the system of Russell's 1906, "The Theory of Implication," we find $(x)(x \supset x)$ as a logical truth rather than $(x)(x \supset x \,.\!\supset\!.\, x)$ as is required in *PoM*. In the 1906 system, it is viable to adopt: $\sim p =_{\text{df}} (x)(p \supset x)$.

One can now arrive at a derived rule of conjunction as follows:

DR (conj): From φ, ψ infer $\varphi\psi$.

φ	
ψ	
$\varphi \,.\!\supset\!.\, \psi \supset f$	assumption for cp
$\psi \supset f$	mp
f	mp
$\varphi \,.\!\supset\!.\, \psi \supset f :\supset: f$	cp
$\varphi\psi$	df(conj)

As an error theory, therefore, it is plausible to imagine that Russell had just this sort of idea in mind in *PoM*. That is, he may have thought that the alternative definitions of conjunction and tilde are deducible from the rules of quantification theory because equivalent to the original. If so, he would thereby imagine that a derived rule of conjunction should be deducible by means of his original definitions of conjunction and tilde. This could have caused him to fail to see the conjunction problem of *PoM*. Indeed, we noted that in *PoM* Russell did notice that in his conditional definition of conjunction, he misconstrues his variable "r" as if it were restricted to propositions. We find a similar slip in a letter of 24 May 1903 that he wrote to Frege (see Frege 1980, 159). Russell offers the following definition:

$$\sim\alpha =_{\text{df}} \alpha \supset (r)r$$

The slip is that the technically illicit "$(r)r$" is used instead of the formally correct "$(r)(r \supset r \,.\!\supset\!.\, r)$." The proper definition is this:

$$\sim\alpha =_{\text{df}} \alpha \supset (r)(r \supset r \,.\!\supset\!.\, r)$$

The difference is significant, since the variable "r" is not a special variable for propositions but just another individual variable. The sloppy use of "$(r)r$" might very well be the source of Russell's not noticing the conjunction problem of *PoM*. In that same letter, Russell takes $(x)\varphi x$ as a primitive notation and puts:

$$\varphi x \supset_x \psi x =_{\text{df}} (x)(\varphi x \supset \psi x)$$

He also has the following quantification rules (q-rules):

$$\text{From } (x)(p \supset \varphi x) \text{ infer } p \supset (x)\varphi x$$
$$\text{From } p \supset (x)\varphi x \text{ infer } (x)(p \supset \varphi x),$$

where x is not free in p. Now if we were to apply the new definition of tilde to the definition of conjunction, we arrive at the following:

$$\alpha\beta =_{\mathrm{df}} (r)(r \supset r :\supset: \alpha \supset (x)(x \supset x .\supset. \beta \supset x) .\supset. r)$$

Let us imagine that Russell slipped into misconstruing the above as:

$$\alpha\beta =_{\mathrm{df}} (r)(\alpha \supset (r)(\beta \supset r) .\supset. r)$$

An instance is then the following:

$$\varphi\psi =_{\mathrm{df}} (r)(\varphi \supset (r)(\psi \supset r) .\supset. r)$$

Using the slip in the above, Russell might imagine legitimating a derived rule of conjunction as follows:

1.	φ	premise
2.	ψ	premise
3.	$\varphi \supset (r)(\psi \supset r)$	assumption for cp
4.	$(r)(\psi \supset r)$	1, 3, mp
5.	$\psi \supset (r)r$	4, q-rule
6.	$(r)r$	2, 5, mp
7.	$\varphi \supset (r)(\psi \supset r) .\supset. (r)r$	3–6, cp
8.	$(r)(\varphi \supset (r)(\psi \supset r) .\supset. r)$	7, q-rule
9.	$\varphi\psi$	8, df(conj)

This looks as though it works! But the error is revealed once the illicit "$(r)r$" is replaced by the proper expression "$(r)(r \supset r .\supset. r)$." We get:

1.	φ	premise
2.	ψ	premise
3.	$\varphi \supset (r)(r \supset r .\supset. \psi \supset r)$	assumption for cp
4.	$(r)(r \supset r .\supset. \psi \supset r)$	1, 3, mp
5.	$\psi \supset (r)(r \supset r .\supset. r)$	4, q-rules
6.	$(r)(r \supset r .\supset. r)$	2, 5, mp
7.	$\varphi \supset (r)(r \supset r .\supset. \psi \supset r) .\supset. (r)(r \supset r .\supset. r)$	3–6, cp
8.	$(r)(r \supset r :\supset: \varphi \supset (r)(r \supset r .\supset. \psi \supset r) .\supset. r)$	7, q-rules
9.	$\varphi\psi$	8, df(conj)

The above stalls at line 5, for want of the needed import and export rules in *PoM*. All the same, this best explains how Russell missed the conjunction problem of *PoM*.

The plan ahead is to rectify this mistake. But before we get into the nitty-gritty details of proofs, it is worth considering whether perhaps Russell's sometime rather loose discussion of axioms and rules of inference might provide an error theory too. I find, in general, that Russell's infelicitous discussion of axioms *versus* rules of inference is an unjust source of criticism of his early work. In *PoM*, he is quite clear about the distinction between inference rule modus ponens, for which is reserved the word "*Therefore*," from a statement of "implication," whether formal implication (universal) or not. He discussed this explicitly. Moreover, I believe that Russell was aware that every axiom gives rise to derived rules of inference. Now there is an axiomatic approach to definition that introduces new signs into the object-language of a theory by means of axioms assuring the non-creativity and eliminability of the new sign. If one thinks of Russell's *PoM* as adopting an axiomatic approach to definitions of the logical particles, then they too will give rise to derived inference rules. That fact certainly doesn't blur the distinction between an axiom and an inference rule. In any case, holding that Russell blurred the distinction cannot diagnose why he missed the conjunction problem. On the axiomatic approach, definitions are introduced as axioms that are biconditionals (i.e., the conjunction of two conditionals) which provide for non-creativity and eliminability. But such an approach to definition will only make the conjunction problem worse. To use the various definitions one would already have to solve the conjunction problem (so that one can arrive at a biconditional). Of course, one might evade this by introducing separate axioms of definition for each conditional of what would otherwise be a biconditional definition of the sign (e.g., for conjunction). This departs too much from what is actually in *PoM* which introduced definitions using "if and only if", a locution we have eliminated in favor of $=_{\mathrm{df}}$ so that our fully stipulative definitions (as convenient notations) permit immediate replacement without appeal to any deduction. In any event, an error theory according to which Russell's definitions are axioms introducing derived inference rules, reduces to the above error theory.

3. *PoM* versus ^{L}*PoM*

Proofs in the system are made somewhat easy because *PoM* implicitly adopts a rule of conditional proof. In Landini (1996), I brought to the fore that Russell was endeavoring to improve Peano's system of "formal implication" and adopted a rule of conditional proof. The evidence I offered is indirect, relying on Peano's letter to Frege which was concerned to know when it is legitimate to conduct a universal generalization. He wrote:

> The indices to the sign ⊃ satisfy laws which have not yet been sufficiently studied. This theory, already abstruse in itself, becomes even more so unless the rules are accompanied by examples. (Peano 1889a, §18)

Frege's reply to Peano noted the the entirety of the rules were well stated in his 1879 *Begriffsschrift* and that he knows no reason they should be said to be "abstruse" (Frege 1897, 247). By allowing conditional proof, Peano (and Russell) encountered difficulties that Frege did not. Frege's axiomatic quantification theory has a universal generalization rule that is an (analog of) this:

$$\text{From} \vdash p \supset \varphi y \text{ infer} \vdash p \supset (x).\varphi x,$$

where x is not free in p. Frege's axiomatic approach allows universal generalization only on theses. In contrast, Peano and Russell must formulate the rule of universal generalization in such a way that reveals the rules governing precisely when one may generalize *within* the scope of an assumption. As I pointed out, this issue arises in Russell 1906 paper "The Theory of Implication," when he writes (Russell 1906, 195):

> *7.11 What is true of any is true of all.

He then goes on to add:

> If φy is true however y is chosen, then $(x).\varphi x$ is true.

As we now know, in modern quantification theory one must not generalize a variable which occurs free in the assumption line of a conditional proof—if one is working within the scope of that assumption. This is precisely what Russell is trying to express. Though I didn't use conditional proof in Landini (1996), there is no good reason to deprive ourselves of it of here. The use of conditional proof greatly facilitates the ease at which derivations can be made in the theory.

Russell's *PoM* does not offer a separate a quantifier-free subsystem. The formal implications, following Peano, are universally generalized wffs. In later work of 1904, Russell would adopt definitions such as:

$$\alpha \supset_x \beta =_{\mathrm{df}} (x)(\alpha \supset \beta)$$

But in *PoM*, Russell says that he adopts Peano's signs $\supset_x$ (formal implication) and ⊃ (implication) as primitives. Formally, it is the signs "⊃" and "$\supset_{x,y_1,\ldots,y_n}$," that are primitive in *PoM*. We needn't follow this practice and shall adopt Russell's later definition. Moreover, we shall follow the rules I set out in Landini (1996) where I presented the textual evidence that Russell has a rule of universal generalization and a rule of universal instantiation. The following are the implicit rules of inference:

Universal Instantiation (ui)
From $\alpha \supset_{x,y1,\ldots,yn} \beta$, infer $\alpha* \supset_{y1,\ldots,yn} \beta*$,

where $\alpha*$ and $\beta*$ are exactly like α and β respectively except containing δ at all free occurrences of x in α and in β.

Universal Generalization (ug)
From $\alpha \supset_{y1,\ldots,yn} \beta$ infer $\alpha \supset_{x,y1,\ldots,yn} \beta$.

Quantifier Distribution (qd)
From $\varphi \ .\!\supset_{x,y1,\ldots,yn}. \ \psi \supset \chi$ infer $\varphi \ .\!\supset_{y1,\ldots,yn}. \ \psi \supset_x \chi$,

where x is not free in the wff φ. The addition of the rule of conditional proof (cp) completes the rules of the system.

Now Taking Russell's informal prose seriously, the original axioms are these:

Ax1:	$x \supset y \ .\!\supset_{x,y}. \ x \supset y$
Ax2:	$x \supset y \ .\!\supset_{x,y}. \ x \supset x$
Ax3:	$x \supset y \ .\!\supset_{x,y}. \ y \supset y$
Ax4 (modus ponens):	From φ and $\{\varphi\} \supset \{\psi\}$, infer ψ.

I have left the nominalizing brackets so that the nominalizing transform involved is made salient.

Ax5 (simp):	$(x \supset x)(y \supset y) \ .\!\supset_{x,y}. \ xy \supset x$
Ax6 (syll):	$(x \supset y)(y \supset z) \ .\!\supset_{x,y}. \ x \supset z$
Ax7 (imp):	$(y \supset y)(z \supset z) \ .\!\supset_{x,y}. \ (x \ .\!\supset\!. \ y \supset z \ :\!\supset\!: \ xy \supset z)$
Ax8 (exp):	$(x \supset x)(y \supset y) \ .\!\supset_{x,y}. \ (xy \supset z \ :\!\supset\!: \ x \ .\!\supset\!. \ y \supset z)$
Ax9 (comp):	$(x \supset y)(x \supset z) \ .\!\supset_{x,y}. \ x \supset yz$
Ax10 (reduction):	$(x \supset x)(y \supset y) \ .\!\supset_{x,y}. \ (x \supset y \ .\!\supset\!. \ x \ :\!\supset\!: \ x)$

In my original formal reconstruction, however, the axioms Ax5–Ax10 are presented as follows:

^{L}Ax5 (simp):	$x \supset x \ :\!\supset_{x,y}: \ y \supset y \ .\!\supset\!. \ xy \supset x$
^{L}Ax6 (syll):	$x \supset y \ :\!\supset_{x,y}: \ y \supset z \ .\!\supset\!. \ x \supset z$
^{L}Ax7 (imp):	$y \supset y \ :\!\supset_{x,y,z}: \ z \supset z \ .\!\supset\!. \ (x \ .\!\supset\!. \ y \supset z \ :\!\supset\!: \ xy \supset z)$
^{L}Ax8 (exp):	$x \supset x \ :\!\supset_{x,y,z}: \ y \supset y \ .\!\supset\!. \ (xy \supset z \ :\!\supset\!: \ x \ .\!\supset\!. \ y \supset z)$
^{L}Ax9 (comp):	$x \supset y \ :\!\supset_{x,y,z}: \ x \supset z \ .\!\supset\!. \ x \supset yz$
^{L}Ax10 (reduction):	$x \supset x \ :\!\supset_{x,y}: \ y \supset y \ .\!\supset\!. \ (x \supset y \ .\!\supset\!. \ x \ :\!\supset\!: \ x)$

As we can see ^{L}Ax5–^{L}Ax10 are exportations Ax5–Ax10 of *PoM*.

Observe that any wff of the form $\alpha \supset \beta$ is such that in the system of *PoM*, we can use ui and Ax1 to get the theorem:

$$\vdash \alpha \supset \beta \ .\!\supset\!. \ \alpha \supset \beta$$

Thus, given the definition of "$\alpha \vee \beta$," it follows from Ax1 that we have:

$$\text{(disj prop):} \qquad \vdash \alpha \vee \beta \ .\!\supset\!. \ \alpha \vee \beta$$

Moreover, any quantified wff (all of which are of the form $\alpha \supset_{x,y_1,\ldots,y_n} \beta$) is such that in the system of *PoM*, we can use conditional proof and the rule ui to get the theorem:

$$\vdash \alpha \supset_{x,y_1,\ldots,y_n} \beta \ .\!\supset\!. \ \alpha \supset_{y_1,\ldots,y_n} \beta$$

Using this together with Ax2, we get:

(univ prop):	$\vdash \alpha \supset_{x,y_1,\ldots,y_n} \beta \ .\!\supset\!. \ \alpha \supset_{x,y_1,\ldots,y_n} \beta$
(conj prop):	$\vdash \alpha\beta \supset \alpha\beta$
(neg prop):	$\vdash {\sim}\alpha \supset {\sim}\alpha$

The difference between Russell's *PoM* and the system ^{L}*PoM*, i.e., my original reconstruction of *PoM* is clear enough. The question is whether one can be more faithful to *PoM* without undermining it as a workable quantification theory.

4. The Conjunction Problem of *PoM*

The difficulty we are facing is that even with the apparatus of conditional proof, nothing in the system of *PoM* enables one to arrive at a conjunction. Hence, we shall not be able to apply the rule of modus ponens after we have applied the inference rule of universal instantiation. To be sure, with conditional proof, we can assume conjunctions as antecedent clauses, but clearly this procedure will never yield theorems that do not have conjunctions as antecedents. And there certainly are such valid wffs in *PoM*. The conjunction problem of *PoM* does not arise in Landini (1996) because the reconstruction takes as axioms exportations of Russell's axioms.

To see what is at stake, notice how the exportations adopted in Landini (1996) avoid the conjunction problem entirely. It enables proof of theorems such as the following that don't have conjunctive antecedents:

^{L}*PoM* (add): $\vdash x \supset x :\supset_{x,y}: y \supset y .\supset. (x .\supset. y \supset x)$

Proof

1. Assume $x \supset x$
2. Assume $y \supset y$
3. $x \supset x :\supset: y \supset y .\supset. xy \supset x$ — ^{L}Ax5 (simp), ui, mp
4. $y \supset y .\supset. xy \supset x$ — 1, 3, mp
5. $xy \supset x$ — 2, 4, mp
6. $x \supset x :\supset: y \supset y .\supset. (xy \supset x :\supset: x .\supset. y \supset x)$ — ^{L}Ax8 (exp), ui, mp
7. $y \supset y .\supset. (xy \supset x :\supset: x .\supset. y \supset x)$ — 1, 6, mp
8. $xy \supset x :\supset: x .\supset. y \supset x$ — 2, 7, mp
9. $x .\supset. y \supset x$ — 5, 8, mp
10. $y \supset y .\supset. (x .\supset. y \supset x)$ — 2–9, cp
11. $x \supset x :\supset: y \supset y .\supset. (x .\supset. y \supset x)$ — 1–10, cp
12. $x \supset x :\supset_{x,y}: y \supset y .\supset. (x .\supset. y \supset x)$ — 11, ug

Conditional proof makes the technique easy. Now one might imagine that in the original *PoM*, one can arrive at the above by exporting, having first used conditional proof to arrive at:

PoM (add): $\vdash (x \supset x)(y \supset y) :\supset_{x,y}: x .\supset. y \supset x$

But one cannot apply the exportation axiom. Thus, we see the catastrophe that is the conjunction problem of *PoM*.

It is worth pointing out that in ^{L}*PoM* we can also prove the following:

^{L}*PoM* (simp2): $\vdash x \supset x :\supset_{x,y}: y \supset y .\supset. xy \supset y$

Proof

1. Assume $x \supset x$
2. Assume $y \supset y$
3. $y \supset y .\supset. y \supset y :\supset: x \supset x .\supset. (y \supset y :\supset: x .\supset. y \supset y)$ — ^{L}*PoM* (add), ui, mp
4. $y \supset y .\supset. y \supset y$ — ^{L}Ax1, ui, mp
5. $x \supset x .\supset. (y \supset y :\supset: x .\supset. y \supset y)$ — 3, 4, mp
6. $y \supset y :\supset: x .\supset. y \supset y$ — 1, 5, mp
7. $x .\supset. y \supset y$ — 2, 6, mp
8. $y \supset y :\supset: y \supset y .\supset. (x .\supset. y \supset y :\supset: xy \supset y)$ — ^{L}Ax 7 (imp), ui, mp
9. $y \supset y .\supset. (x .\supset. y \supset y :\supset: xy \supset y)$ — 2, 8, mp
10. $x .\supset. y \supset y :\supset: xy \supset x$ — 2, 9, mp
11. $xy \supset y$ — 7, 10, mp
12. $y \supset y .\supset. xy \supset y$ — 2–11, cp
13. $x \supset x :\supset: y \supset y .\supset. xy \supset y$ — 1–12, cp
14. $x \supset x :\supset_{x,y}: y \supset y .\supset. xy \supset y$ — 13, ug

In this way, we arrive at the important result of the commutation of conjunction. We have the following proof:

^{L}*PoM* (comm conj): $\vdash x \supset x :\supset_{x,y}: y \supset y .\supset. xy \supset yx$

Proof

1. Assume $x \supset x$
2. Assume $y \supset y$
3. $x \supset x :\supset: y \supset y .\supset. (xy \supset x)$ — ^{L}Ax5 (simp), ui, mp
4. $x \supset x :\supset: y \supset y .\supset. (xy \supset y)$ — ^{L}*PoM* (simp2), ui, mp
5. $y \supset y .\supset. (xy \supset x)$ — 1, 3, mp
6. $xy \supset x$ — 2, 5, mp
7. $y \supset y .\supset. (xy \supset y)$ — 1, 4, mp
8. $xy \supset y$ — 2, 7, mp
9. $xy \supset y :\supset: xy \supset x .\supset. xy \supset yx$ — ^{L}Ax9 (comp), ui, mp
10. $xy \supset x .\supset. xy \supset yx$ — 8, 9, mp
11. $xy \supset yx$ — 6, 11, mp
12. $y \supset y .\supset. xy \supset yx$ — 2–11, cp
13. $x \supset x :\supset: y \supset y .\supset. xy \supset yx$ — 1–12, cp
14. $x \supset x :\supset_{x,y}: y \supset y .\supset. xy \supset yx$ — 13, ug

Recall that since a wff of conjunction is a universally quantified wff, we have:

$$\text{(conj prop):} \quad \vdash xy \supset xy$$

This holds in *PoM* as well as in ^{L}*PoM*. With this in place, we can see that from ^{L}Ax8 alone we get:

$$^{8}PoM\ \text{(conj)}: \vdash x \supset x :\supset_{x,y}: y \supset y .\supset. (x .\supset. y \supset xy)$$

Proof

1. Assume $x \supset x$
2. Assume $y \supset y$
3. $x \supset x :\supset: y \supset y .\supset. (xy \supset xy :\supset: x .\supset. y \supset xy)$ — ^{L}Ax8 (exp), ui, mp
4. $y \supset y .\supset. (xy \supset xy :\supset: x .\supset. y \supset xy)$ — 1, 3, mp
5. $xy \supset xy :\supset: x .\supset. y \supset xy$ — 2, 4, mp
6. $xy \supset xy$ — (conj prop)
7. $x .\supset. y \supset xy$ — 5, 6, mp
8. $y \supset y .\supset. (x .\supset. y \supset xy)$ — 2–7, cp
9. $x \supset x :\supset: y \supset y .\supset. (x .\supset. y \supset xy)$ — 1–8, cp
10. $x \supset x :\supset_{x,y}: y \supset y .\supset. (x .\supset. y \supset xy)$ — 9, ug

Now from our theorem (Conjunction) we have:

$$\vdash \varphi :\supset: \psi .\supset. \varphi\psi$$

As we can see, ^{8}*PoM* (conj) is a very important result. It gives us the derived rule:

$$\text{DR (conj):} \qquad \text{From } \varphi, \psi \text{ infer } \varphi\psi.$$

Once we see this, we can notice that it is not necessary to adopt all of the exported axioms ^{L}Ax5–^{L}Ax10. It is sufficient to alter only Russell's Ax8 of Exportation. That is, we need only adopt ^{L}Ax8, leaving the others as is. Naturally, I call this system by the name ^{8}*PoM*. From this change alone we can arrive at the needed conjunctions and avoid the conjunction problem of *PoM*. With this in place, we can derive as theorems the exported forms of axioms Ax5–Ax10 of *PoM*.

It will be noted that with these theorems of ^{L}*PoM* in place, we see that we need depart from the original *PoM* only by abandoning Ax8 (exp) in favor of ^{L}Ax8 (exp). The other axioms can remain intact. This offers a way of solving the conjunction problem of *PoM*. But it does not seem much better from a historical standpoint than just replacing all of Russell's axioms Ax5–Ax10 in favor of ^{L}Ax5–^{L}Ax10. A better solution is desirable.

5. Solution: Amalgamating Ax2 and Ax3

The best way to solve the conjunction problem in keeping with the historical *PoM* is to drop Russell's Ax2 and Ax3 in favor of a new axiom that amalgamates the two. This approach adopts the following axiom on behalf of *PoM*:

$$\text{Ax2/3:} \qquad x \supset y .\supset_{x,y}. (x \supset x)(y \supset y)$$

Let us use the name ^{a}*PoM* for the system of *PoM* altered by this change to Ax2/3. This is the only change needed to solve the conjunction problem. As we shall see, in ^{a}PoM we can arrive at Ax2 and Ax3 as theorems. At the same time, we arrive at theorems ^{L}Ax5–^{L}Ax10.

To see this, first note that (conj prop) is a theorem of ^{a}PoM since that system has conditional proof. With this, we can prove the following:

^{a}PoM (a): $\vdash x \supset y :\supset_{x,y}: x .\supset. y \supset xy$

Proof

1.	Assume $x \supset y$	
2.	$x \supset y .\supset. (x \supset x)(y \supset y)$	Ax2/3, ui, mp
3.	$(x \supset x)(y \supset y)$	1, 2, mp
4.	$(x \supset x)(y \supset y) .\supset. (xy \supset xy :\supset: x .\supset. y \supset xy)$	Ax8 (exp), ui, mp
5.	$xy \supset xy :\supset: x .\supset. y \supset xy$	3, 4, mp
6.	$xy \supset xy$	(conj prop)
7.	$x .\supset. y \supset xy$	5, 6, mp
8.	$x \supset y :\supset: x .\supset. y \supset xy$	1–7, cp
9.	$x \supset y :\supset_{x,y}: x .\supset. y \supset xy$	9, ug

By applying ^{a}PoM (a), we have:

^{a}PoM (aa): $\vdash \varphi \supset \psi :\supset: \varphi .\supset. \psi \supset \varphi\psi$

Hence we arrive at a derived rule of conjunction:

DR (conj): From φ, ψ , infer $\varphi\psi$.

This follows because we have conditional proof in the system. If we have ψ on a line of proof, then by conditional proof, we arrive at $\varphi \supset \psi$. Thus we arrive at our derived rule from modus ponens (three times) on ^{a}PoM (aa) to get $\varphi\psi$. Note that from conditional proof, the following theorems are immediate:

^{a}PoM (a1): $\vdash x \supset x .\supset. x \supset x$

^{a}PoM (a2): $\vdash x \supset x .\supset. x \supset x :\supset: x \supset x .\supset. x \supset x$

Thus, by using these with our DR (conj), we get theorems such as the following:

^{a}PoM (b1): $\vdash (x \supset x .\supset. x \supset x)(x \supset x .\supset. x \supset x)$

^{a}PoM (b2): $\vdash (x \supset x .\supset. x \supset x)(y \supset y .\supset. y \supset y)$

From ^{a}PoM (a1) and Ax5 it is easy to see that we arrive at:

^{a}PoM (simp-p): $\vdash (x \supset x)(y \supset y) .\supset. x \supset x$

Indeed, we can now see that, just as in the system ^{8}PoM, we will now be able to prove in ^{a}PoM theorems ^{L}Ax5–^{L}Ax10. Let us call these ^{a}PoM (expAx5)–^{a}PoM (expAx10). The conjunction problem is solved.

Our next task is to show that in the system ^{a}PoM we can prove as theorems Ax2 and Ax3 of *PoM*. Let us begin with Ax2. Since we have ^{L}Ax6 (syll), we can readily prove Ax2 as a theorem in the system ^{a}PoM. We have the following:

^{a}PoM Ax2: $\vdash x \supset y .\supset_{x,y}. x \supset x$

Proof

1.	$x \supset y .\supset. (x \supset x)(y \supset y)$	Ax2/3, ui
2.	$(x \supset x)(y \supset y) .\supset. x \supset x$	^{a}PoM (simp-p)
3.	$x \supset y .\supset_{x,y}. x \supset x$	1, 2, ^{L}Ax6 (syll), ui, mp, ug

Next we need a proof of Ax3. To get this result, we first prove a lemma:

^{a}PoM (simp-q): $\vdash (x \supset x)(y \supset y) .\supset_{x,y}. y \supset y$

Proof

1.	$(y \supset y .\supset. y \supset y)(y \supset y .\supset. y \supset y)$	^{a}PoM (b1)
2.	Assume $x \supset x$	
3.	$y \supset y .\supset. y \supset y$	Ax1, ui, mp
4.	$x \supset x :\supset: y \supset y .\supset. y \supset y$	2–3, cp

5. $(y \supset y \,.\!\supset\!.\, y \supset y)(y \supset y \,.\!\supset\!.\, y \supset y) \,.\!\supset\!.$
$((x \supset x :\!\supset\!: y \supset y \,.\!\supset\!.\, y \supset y) \supset ((x \supset x)(y \supset y) \,.\!\supset\!.\, y \supset y))$ Ax7 (imp), ui, mp
6. $(x \supset x :\!\supset\!: y \supset y \,.\!\supset\!.\, y \supset y) \supset ((x \supset x)(y \supset y) \,.\!\supset\!.\, y \supset y)$ 1, 5, mp
7. $(x \supset x)(y \supset y) \,.\!\supset\!.\, y \supset y$ 4, 6, mp
8. $(x \supset x)(y \supset y) \,.\!\supset_{x,y}\!.\, y \supset y$ 7, ug

From this we finally arrive at Russell's Ax3. We have:

^{a}*PoM* Ax3: $\vdash x \supset y \,.\!\supset_{x,y}\!.\, y \supset y$

Proof

1. $x \supset y \,.\!\supset\!.\, (x \supset x)(y \supset y)$ Ax2/3, ui, mp
2. $(x \supset x)(y \supset y) \,.\!\supset\!.\, y \supset y$ ^{a}*PoM* (simp-q) , ui, mp
3. $x \supset y.\, .\!\supset_{x,y}\!.\, y \supset y$ 1, 2, ^{L}Ax6 (syll), ui, mp, ug

Thus we see that we have proved all of ^{L}Ax5–^{L}Ax10 as well as Ax2 and Ax3 from the system of ^{a}*PoM*. I cannot imagine any better way to solve the conjunction problem of *PoM*.

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