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On the Curious Calculi of Wittgenstein and Spencer Brown

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In his *Tractatus*, Wittgenstein sets out what he calls his N-operator notation which can be used to calculate whether an expression is a tautology. In his *Laws of Form*, George Spencer Brown offers what he calls a "primary algebra" for such calculation. Both systems are perplexing. But comparing two blurry images can reduce noise, producing a focus. This paper reveals that Spencer Brown independently rediscovered the quantifier-free part of the N-operator calculus. The comparison sheds a flood light on each and from the letters of correspondence we shall find that Russell, as one might have surmised, was a catalyst for both.

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1. Introduction

George Spencer Brown studied at Trinity College, Cambridge, between 1947 and 1952. He had colorful careers as mathematician, engineer, psychologist, educational practitioner, author, and poet. His many ideas have become favorites of those given to the fantastic, spiritual and occult, but also those formally minded academics doing the hard work of searching for an empirical science that embraces the existence of biological self-representing systems (see, e.g., Varela 1979). In his book *Laws of Form* (*LofF*; 1971) we find the best characterization of his many ideas and we are explicitly told that the work was developed in collaboration with his brother D. J. Spencer Brown (*LofF*, 99). References to his brother occur in several passages. In the opening explanations of its Preface, George writes:

> Apart from the standard university logic problems, which the calculus published in this text renders so easy that we need not trouble ourselves further with them, perhaps the most significant thing, from the mathematical angle, that it enables us to do is to use complex values in the algebra of logic. They are the analogs, in ordinary algebra, to complex numbers $a + b\sqrt{-1}$. My brother and I had been using their Boolean counterparts in practical engineering for several years before realizing what they were. (*LofF*, ix)

Later in the book, George refers to a "private communication" with D. J. Spencer Brown (*LofF*, 85). George's autobiography tells us his brother D. J. Spencer Brown died of a pulmonary embolism and it has pictures of the two together when they were young

boys. These many bits of evidence suggest *Laws of Form* was produced by collaboration of the two brothers.

In volume 20 of the 1969 issue of the *British Journal for the Philosophy of Science*, there is an advertisement for Spencer Brown's *Laws of Form*. The ad was this:

> The theme of this book is that a universe comes into being when space is severed or taken apart. By tracing the way such a severance is represented, the author reconstructs the basic forms underlying linguistic, mathematical, physical and biological science.

> 'In this book, G. Spencer Brown has succeed in doing what, in mathematics, is very rare indeed. He has revealed a new calculus of great power and simplicity. I congratulate him.' Bertrand Russell

In the Preface of the book, Spencer Brown writes (*LofF*, xiii): "The exploration on which this work rests was begun towards the end of 1959. The subsequent record of it owes much, in the early stages, to the friendship and encouragement of Lord Russell, who was one of the few men at the beginning who could see a value in what I proposed to do." There are many letters of correspondence between Russell and George Spencer Brown (and also with David Spencer Brown). George had written to him explaining that his new discoveries were not being properly understood by referees and he hoped to get help in having it evaluated by a logician "of sufficient genius" to understand it. In his *Autobiography*, Russell explains:

> In 1965, a young mathematician G. Spencer Brown, pressed me to go over his work, since, he said he could find no one else who he thought could understand it. As I thought well of what little of his work I had previously seen, and since I feel great sympathy for those who are trying to gain attention for their fresh and unknown work against the odds of established indifference, I agreed to discuss it with him. But as the time drew near for his arrival, I became convinced that I should be quite unable to cope with it and with his new system of notation. I was filled with dread. But when he came and I heard his explanations, I found that I could get into step again and follow his work. I greatly enjoyed those few days, especially as

his work was both original and, it seemed to me, excellent. (Russell 1967–69, vol. 3, 238)

This corroborates the content of the advertisement of the book.

On the dust jacket of the Julian Press edition of *Laws of Form* the quotation of Russell's comment reappears. The quote is attributed to Russell by Lancelot Law Whyte. Quine, however, thinks Russell's praise was intended to be quite measured. He recalls that his last correspondence from Russell, then aged 95, was the following:

> My last word from Russell came in January 1967. It was this:
>
>> I enclose a paper by G. Spencer Brown, which I have given one careful reading, but no more. I am very lazy at the moment, but thought I should draw Spencer Brown's work to your attention.
>
> I looked into Spencer Brown's work and was not moved by it. When his little book *Laws of Form* came out two years later, it bore this blurb from Russell:
>
>> Reveals a new calculus of great power and simplicity.
>
> At first glance the blurb as printed looks longer and more extravagant; one must look sharp for the quotation mark. (Quine 2008, 110)

Evaluating the work was not easy for anyone, and certainly it was not easy for Russell. On reverse of the dust jacket of the 1971 edition Alan Watts has the audacity to write: ". . . this book is surely the most wonderful contribution to Western Philosophy since Wittgenstein's *Tractatus*." In his autobiography (2004), George goes much further, writing:

> About the Author: George Spencer-Brown is best known for writing *Laws of Form*, described by Bertrand Russell as the only completely unique book in the history of literature.

That almost reads as if some sort of inside joke. Surely no evidence will be found that Russell said or held that. But without question Russell's autobiography reveals that he had found the work interesting and worthy of careful attention and certainly not deserved of the rather quick dismissal (which it was receiving). He was sympathetic to George Spencer Brown's work and understood his frustration in not finding an open minded person for its evaluation. It was Russell who first suggested that Quine might do it justice.

Whatever the contributions may have been in the eyes of Russell, it is interesting that he came to preside over both the *Tractatus* and *Laws of Form*, and the respective aphorisms of each remain as engaging as ever. Unlike the *Tractatus*, Spencer Brown's *Laws of Form* doesn't begin with "The world is all that is the case." It begins with "A universe comes into being when a space is severed or taken apart" (*LofF*, v). Spencer Brown is certainly sympathetic with Tractarian ideas, but seems to want his own twist. In an unsigned document entitled "Preface" (among the letters of correspondence with Russell) there is a curious statement of the nature of philosophy. We find:

> Wittgenstein used to say, What can be said, can be said clearly. I am saying, What must be said, must be said wrong.
>
> . . . Whenever we say anything in a way, we do so at the expense of other ways. There are more than two sides to every question, and to give all the answers would be an endless, as well as a useless process. The more fully we describe, the less clear is our description; and the clearer, the emptier.

This echoes Wittgenstein's Tractarian thesis that logic and mathematics are *shown* (in practices of calculation) and thereby they do not consist of a body of truths which can be *said*. With respect to mathematics we find the following in *Laws of Form*:

> It might be helpful at this stage to realize that the primary form of mathematical communication is not description, but injunction. In this respect it is comparable with practical art forms like cookery, in which the taste of a cake, although literally indescribable, can be conveyed to a reader in the form of a set of injunctions called a recipe. Music is a similar art form . . . He [Wittgenstein] notes elsewhere that the mathematician, descriptively speaking, says noth-

ing. The same may be said of the composer, who, if he were to attempt a *description* (i.e., a limitation) of the set of ecstasies apparent *through* (i.e., unlimited by) his *composition*, would fail miserably and necessarily. But neither the composer nor the mathematician must, for this reason, be silent. (*LofF*, 77)

The Primary Algebra of Spencer Brown's book *Laws of Form* has been thought to be an algebra for logic based on the Sheffer stroke (Grattan-Guinness 2000, 557; Banaschewski 1977; Mequire 2003). So also had Wittgenstein's N-operator notation. This paper argues that this is mistaken and that the two systems of calculation are simply notational variants of one another.

It is not surprising that this has not been noticed. Both systems are obscure; both breach syntactic well-formedness as we know it. Both were produced by eccentrics apt to worry that the genius of their work cannot be understood except by genius.[1] Unfortunately, the letters of correspondence between Russell and Spencer Brown do not contain evidence of recognition on the part of Russell that Spencer Brown's Primary Algebra is a rediscovery of Wittgenstein's N-operator. But comparing two blurry images can reduce noise, producing a focus. By examining the two obscure calculi of Wittgenstein and the Spencer Browns, a sort of clarity results.

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[1]In what follows we shall find ample quotes to this effect by Spencer Brown. With respect to Wittgenstein, in the Preface to his *Tractatus* (Wittgenstein 1922), he remarks: "Perhaps this book will be understood by someone who has himself already had the thoughts that are expressed in it." In a letter to Russell of 6 May 1920 he responds to Reclam's rejecting the publication of his work by saying that if it is of the highest rank it doesn't matter; and comparing it to Kant's *Critique of Pure Reason*, he says he is indifferent to "whether it's printed in twenty or a hundred years sooner or later" (Wittgenstein 1974, 88). In a letter to Russell of 13 March 1919 Wittgenstein wrote that the "short remarks" (aphorisms) of his *Tractatus* (then called his *Logisch-Philsophische Abhandlung*) will not be understandable even to Russell without a previous explanation and that "This of course means that *nobody* will understand it." See Russell's *Autobiography* (1967–69, vol. 2, 162).

2. Brother David

In uncovering the historical origins and background of George Spencer Brown's discovery of his primary algebra, letters of correspondence with Russell are very illuminating. But there is an oddity in the letters that must be addressed first and foremost. The letters are sometimes addressed as if they were between Russell and David Spencer Brown, George's brother. In a letter of 3 January 1961, we find:

> You will be interested to know that my brother and I, working together have discovered a proof of the Four Colour Theorem. This was my third attempt to solve the mystery, and, at last, successful.

Is this letter from George or is it from D. J. Spencer Brown?

Russell didn't reply until 27 January 1962 apologizing that he was ill with a severe attack of flu. Russell addresses his reply to "Spencer Brown" and remarks:

> It is very interesting that you and your brother have discovered a proof of the four colour theorem. It is worrying that no competent person has been found to examine your logical work. I am inclined to think that Quine might be the best person if he were willing. If you agree, I am willing to write to him and say that so far as I can judge your work is of great importance. I suggest him because I think he would appreciate the work. I do not think your situation is so very unusual among mathematicians and logicians who have afterwards become famous. Take, for example, Grassmann and Frege, and, in a different line, Mendel. We both hope that you are having a restful and delightful holiday at Las Palmas and send you our best wishes.

The four color problem is to prove that with exactly four distinct colors any contiguous partition of a plane into distinct regions can be colored in such a way that no two adjacent regions have the same color. Had George and David solved it, this would be a quite significant achievement indeed. Russell's reply is a bit strange in that it does not react with sufficient surprise and

congratulations at such a momentous mathematical result. This may suggest that Russell understood that the two were given to hyperbole. On the other hand, perhaps he interpreted them as having taken themselves to have found an important lemma which might direct further research. The proof we accept today is a computer assisted proof which individually checks 1,936 maps. The proof was given in 1976 by Kenneth Appel and Wolfgang Haken. In *Laws of Form*, George Spencer Brown offers some clarification:

> D. J. Spencer Brown and I found evidence, in unpublished work undertaken in 1962–5, suggesting that both the four-colour problem and Goldbach's theorem are undecidable with a proof structure confined to Boolean equations of the first degree, but decidable if we are prepared to avail ourselves of equations of higher degree. (*LofF*, xxi)

In a letter to the editor of *Nature* dated 1976, George speaks of the "late" D. J. Spencer Brown's work on the four color problem, attempting to clarify what exactly had been his contribution. It seems, then, that according to George, his brother D. J. Spencer Brown died in 1976.

Now Russell met with George on more than one occasion. We have a letter of 30 May 1962 in which Russell writes:

> Many thanks for your letter and for the enclosed typescript which I have looked at but not yet studied carefully. My trouble is that, a, I am quite stale as regards mathematical logic, and, b, I am so busy that I cannot find time to give a great deal of attention to it.

On 8 January 1965 he repeated his lament:

> I am very glad to hear that your work has gone so well and am anxious to understand it. I shall be very glad of a visit from you to give you a chance to explain your work. . . . I suppose you realize that I am stale about mathematical logic which I have hardly worked on at all for the last fifty years. You will find me slow to appreciate new points, and I am likely to cause you impatience so that your work of explanation may take longer than you expect. However, I

shall hope that I may be able to understand your work within some finite time.

Russell's early 1962 suggestion of Quine as a reviewer was brought up again by a letter of 6 January 1967 signed with "David." We find:

> Dear Bertie
>
> Something tells me you will be rather pleased with this enclosed demonstration that Sheffer failed to prove the independence of his postulates.
>
> You once offered to plead my cause with Quine, and I think he is much more likely to accept such a supplication in a paper such as this one . . . I should therefore be most grateful if you would recommend this paper to his attention and send it to him . . .
>
> I very much hope we may meet again soon.

The letter is signed by hand as "David" with "G. Spencer Brown" typed beneath. Quine was contacted. On 13 January 1967, Russell wrote the following to David:

> I have given your work one careful reading, but not enough. I am sending one of your two copies to Quine with a note. I am desperately busy and can't at the moment do more.

In a letter of 20 September 1967, Russell's wife Edith wrote:

> Dear David:
>
> I have just recounted our telephone conversation to Bertie and he is very pleased that you are willing to come here again to explain further . . . He is extremely anxious, as I told you, to understand and wants very much to be able to support your work entirely. It will be a pleasure to see you again quite apart from mathematics and we hope that you can come soon.

But by 3 October 1967, Christopher Farley (Russell's secretary) wrote a letter which reads:

Dear Mr. Spencer-Brown,

Lord Russell has asked me to write to you. He feels, to his regret, that he will be unable to devote more attention to your mathematical work. This is, of course, a great disappointment to him but he feels that it is time for him to try to cut down the number of commitments which he has taken on. As you know, he has taken the greatest interest in your work and feels strongly that it should be brought to the attention of those who are competent to assess its merits. He believes it would be a mistake for him to immerse himself in such work at his advanced age.

As we know, Quine's assessment was not very favorable.

It is odd to find letters signed "David" above the typed "G. Spencer Brown." (This occurs again in a typed letter of 26 September 1967.) In any case, it seems clear enough that George was going by the name "David" in these letters and that Russell had come to identify the two. There is a letter of 19 September 1967 signed by David with a note at the top in Russell's hand-writing which says: "from G. Spencer Brown." As we see, late in their correspondences, David has familiarity enough to address his letters with "Dear Bertie" rather than "Dear Lord Russell" which had been his former practice. And we do find some of Russell's replies addressed with "Dear David." (See also the letter of 1 April 1965.) In a letter of 28 September addressed with "Dear David" there is a note at the top in Russell's hand that reads "to G. Spencer Brown." The handwriting in letters signed by David and those signed by George are strikingly similar.

It seems safe, therefore, to say that the "David" of the correspondences and the meetings with Russell is George himself. There is a 23 May 1964 letter from David which opens as follows:

Dear Lord Russell: I am writing from the Mathematical Laboratory at Cambridge, and am enclosing duplicate copy of my Laws of Thought as far as it goes, which is far enough to demonstrate the validity of the propositions of *Principia Mathematica*. . . . The algebra is unique in containing fewer initials than any other such algebra (e.g. two equivalence initials to Sheffer's three).

David goes on to write notations of what would be the primary algebra of George's book *Laws of Form*. Here is a facsimile of notations on the last page:

(We shall demonstrate how this reveals the *wff* to be a tautology anon.) Spencer Brown goes on to say: "I hope this letter is helpful, but I think perhaps the best thing would be for me to come up to Wales to be on hand when you want to look at the work . . . I am most grateful to you for offering to act as referee in this application, and I am enclosing a copy of the printed notice so that you shall see what it is. Also a carbon copy of my curriculum vitae." The letter is cursively signed by hand as David. The curriculum vitae he sent, however, is that of George Spencer Brown.

Now it would not be untoward if George were to publish his work as "George Spencer Brown" even though he prefers his friends to call him "David." This might explain matters. But confusion sets in when we find in George's autobiography that "D. J. Spencer Brown" stands for "David John Spencer Brown" (Spencer Brown 2004, 64). There is an article by D. J. Spencer Brown and J. C. Miller (1966) in which D. J.'s address is given as:

c/o 28b Porchester Terrace, London, W.2.

This address appears at the bottom of George's CV and it is the address from which David wrote to Russell repeatedly in 1966–67.

In light of this evidence, one begins to wonder whether there really were two brothers. Perhaps there were and George just

liked to call himself "David" in expression of the closeness he had with his brother. But the identity of the George (= David) of the letters is clear enough. And so it is clear that it was indeed George, and not a brother David, who earnestly wanted to write a biography of Russell. In a typed letter of 28 February 1961, we find:

> My suggestion about your biography was made quite seriously, and I hope you will consider it so. I think I realize the magnitude of the task, and I think also that I could record your achievements with more understanding than others I know.

The letter is signed by George. But in a 4 April 1961 letter addressed to "Lady Russell," we find:

> I'm very glad that you should think well of my proposal to write Lord Russell's biography. . . . I think it is important that he should have a biographer who will record his life with . . . his philosophy with understanding and his achievements with appreciation. And I think, even with all my faults and weaknesses, that I could do this. What my agent & my publisher think remains to be seen, but I will tell you when they do.

This letter is signed David Spencer Brown. There is also a captivating letter of 9 July 1961 concerning the nature of the would-be biography. David writes:

> I see that I have touched upon a point on which I had meant to remain silent. But it has now become plain that I regard my own place in the history of mathematics as almost as assured as yours, and that I must therefore proceed to apologize for this conceit.

It is worth quoting at length from this entertaining letter. David continues as follows:

> I am forced, therefore, into this untimely admission only because it is, to me, the primary reason for wishing to write your life. My desire to do so stands or falls, therefore, on the public reaction to my mathematical work over the past twelve months. During this time (i.e., since I just wrote to you, in some confusion, about the single operator) I have discovered so much, and in such variety, that

> I cannot help feeling, even if only for statistical reasons, that some of it is likely to endure. If not, then we are deluding ourselves, and I shall not, in my opinion, be worthy of being your biographer and will gladly relinquish my place to someone better qualified to take it.
>
> Only a diamond can cut a diamond, and only a master can interpret another master. I hold so strongly to this that my one continuous waking nightmare is the thought of my own work being interpreted in the minds of non-masters. Prejudicial rejection, uncritical acceptance, it is the same death each time: for in neither attitude can there be any respect. It is only through the masters that the masters live on. For even when the masters are wrong, one is still safe: for if a man is a master, he will always find men to contradict him, because they will always find men to listen; but if a man is a nobody, who will bother to contradict what he says.

The letter is as amusing now as it must have been to Russell. But he nonetheless took the plan for a biography seriously.

In a letter of 3 July 1961 Russell expresses a concern that his papers would be held up for an indefinite time and that it is of central importance not to delay or conflict with the publication of his autobiography. Russell notes that a satisfactory arrangement would have to be made with Unwin and Longman and that "inextricable difficulties" will arise if Unwin is not the publisher of a would-be Spencer Brown critical assessment of Russell's philosophical work together with a volume of letters. Russell explains that Unwin is "one of my literary executors and will feel, I am sure, that he as a prescriptive right as regards anything that I have written." David Spencer Brown had been entertaining the idea of using the publisher Longmans who had published George's *Probability and Scientific Inference* in 1957. (See letter of 28 June 1961.) David refers to Longmans has "my own" publisher. Ultimately, negotiations broke down. In a letter of 8 April 1962, Russell writes that "the two publishers seem to have agreed that it shouldn't be done until after my death, and I do not see what further can be done about it at the moment." The biography was never commissioned and never written.

George Spencer Brown was born 2 April 1923 and died 25 August 2016. An engaging obituary can be found in the *Telegraph* (2016). At the end of the obituary, we find:

CORRECTION: This obituary, now amended, originally referred to George Spencer-Brown's "invented, and later disinvented" brother, David J Spencer-Brown. We have been told by someone who knew George Spencer-Brown that his brother existed and was not 'invented' as suggested. We are happy to make this clear.

3. All and Only Logical Equivalents are to Have One and the Same Expression

Spencer Brown's work has remained obscure in spite of quite a lot of important reconstructive work that has gone into explaining, extending and applying what he might have been up to in *Laws of Form*.[2] What follows focuses only on his *Primary* (logical) *Algebra* and *Primary Arithmetic*. By comparing his work to the Tractarian N-operator we shall find that a new clarity in each emerges that is not otherwise noticed. This comparison is justified by the following thesis: both Wittgenstein and Spencer Brown were (independently) seeking to find a notation for quantification theory with identity in which all and only logical equivalents have one and the *same* notation.

In what follows, I'll offer ample evidence of this, though it has been often neglected in the vast and ever growing literature on Wittgenstein. There can be no significant doubt whatsoever that, at the very least, this is what Wittgenstein was seeking in his work in 1912–1916 when dialoging with Russell over the foundations of logic and arithmetic. I don't wish to get into the fray with Wittgenstein scholars who may wish to contest that, while this is true of *some* of Wittgenstein's early work, it is not what his *Tractatus* was up to. We can put such disputes aside for the purpose

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[2]See Richard Shoup's website (2013) devoted to George Spencer Brown, his work, and related ideas. It includes an extensive bibliography of the secondary literature on *Laws of Form.* See also Banaschewski (1977) and Engstrom (1999).

of this paper, deferring to the arguments to be found in my book *Wittgenstein's Apprenticeship with Russell* (Landini 2007) that there was no *substantive* change in his thinking and that the famous Tractarian *Doctrine of Showing* is embedded in the early work of Wittgenstein's 1913 *Notes on Logic* as much as in the late. The point is simply that Wittgenstein's N-operator was introduced and designed for one and only one reason: namely, to offer a notation in which all and only logical equivalents of quantification theory with identity have one and the same notation. It is precisely this that he hoped would establish his fundamental idea that logic and arithmetic do not consist in a body of truths, but consist in practices of calculation of operations. Wittgenstein's Tractarian view is that *logic* (a practice of calculation of whether an expression is a tautology) and *arithmetic* (a practice of calculating the correctness of equations on basis of recursive functions defined with numeral exponents) both find their common source in calculation of operations. We find (Wittgenstein 1922):

6.22 The logic of the world, which is shown in tautologies by the propositions of logic, is shown in equations by mathematics.

6.234 Mathematics is a method of logic.

Wittgenstein's form of logicism does not "reduce" arithmetic to logic. On Wittgenstein's view, both inference with tautologies (sometimes called "logic") and inference with equations ("arithmetic") are practices of calculation outcomes of operations. It is here that the unity resides. That is, *Tractarian Logicism* is the thesis that there is no difference in kind between the operations of logic (the N-operation) and the operations involved in the equations of arithmetic. Waismann reports that Wittgenstein gave the following explanation of what it is that mathematics and logic have in common:

What is right about Russell's idea is that in mathematics as well as in logic, we are dealing with systems. Both systems are due to operations. What is wrong about it is the attempt of constructing mathematics as part of logic. (Waismann 1979, 218)

The logical method is that of the calculation of the internal properties of operations that are *shown* in their signs. Such calculations occur with the N-operator and with operators of arithmetic equations with numerical exponents. Calculation of logical tautologyhood consists in working with the rules of the N-operator (function) while calculation of arithmetic correctness consists in working with rules governing operations (functions) recursively defined.

Unfortunately, it is exceedingly rare to find anyone working on the *Tractatus* that noticed that there are *rules* governing the practice of calculating tautologyhood by means of the N-operator. A survey of the literature reveals only Anscombe (1959), who in attempting to explain the rules imagined that Wittgenstein's discussion of the *general form of a truth-function (proposition)*, i.e.,

6 $[\overline{p}, \overline{\xi}, N(\overline{\xi})]$

is best explained by appealing to the notion, used so often in mathematics, of a function (operation) giving the general term of a consecutive (*successive*) series. For example we have

$$0, 1, 4, 9, 16, x^2$$

The function (operation) x^2 cited here gives the general term of this successive (linear) and consecutive series. This works because the natural numbers are naturally ordered consecutively and hence we apply the operation to each in its consecutive order. Anscombe tries to do the same for the truth-functions, imagining the bases upon which N is to operate to be implicitly ordered consecutively and the application of N to bases to be governed by implicit rules. But unfortunately, there is no consecutive order at all of the bases and her laudable project failed. She was, however, on the right track. Wittgenstein's notion of the general form is connected to his project of calculating of truth-functions using his N-operator implicitly governed by implicit rules. Wittgenstein writes:

6.002 If we are given the general form according to which propositions are constructed, then with it we are also given the general form according to which one proposition can be generated out of another by means of an operation.

The Tractarian N-operator rules emerge rather clearly when one actually uses its expressive resources, as Wittgenstein intended, as a pictorial method whereby tautologies are *shown* in the symbolism alone. Its aim was clearly to extend the successes of the quantifier-free notation to quantification theory (with identity, when use of identity is meaningful). That is, the N-operator notation was intended to preserve the pictorial feature of the Tractarian truth-tabular notation which enables all and only logical equivalents (of propositional logic) to have one and the same representation.

Any historically faithful account of the *Tractatus* must respect *showing* as its fundamental idea—and in consideration of its elucidation that logic is not a genuine science Wittgenstein thought that he must find a decision procedure—in the form of an expressive notation in which all and only logical equivalents of quantification theory (with identity) have one and the same representation, and *show* how such a representational system enables rules to recognize tautologies. There is a vast amount of evidence for this interpretation and for its evolution in Wittgenstein's letters and notes. For instance, in his 1914 notes dictated to Moore, we find:

Internal relations . . . can't be expressed in propositions, but are all shown in symbols themselves, and can be exhibited systematically in tautologies. (Wittgenstein 1979, 116)

In a letter of November 1913 to Russell, he wrote:

That they all follow from *one* proposition is clear because *one symbolic rule* is sufficient to recognize each of them as true or false. And this is the one symbolic rule: write the proposition down in the *ab*-notation, trace all the connections (of poles) from the outside to the inside poles: Then if the *b*-pole is connected to such *groups*

of inside poles only as contain opposite poles of one proposition, then the whole proposition is a true logical proposition. If on the other hand this is the case with the *a*-pole, the proposition is false and logical [logically false]. If finally, neither is the case, the proposition may be true or false, but it is in no case logical. (Wittgenstein 1979, 125)

Wittgenstein's *ab*-notation is just a variant of his Tractarian *tf*-notation and it is clearly being used as a decision procedure to calculate whether a given *wff* is a tautology.

The *tf*-notation picture of the truth-conditions "$q \supset r :\supset: p \lor q \,.\supset.\, r \lor p$" is as follows:
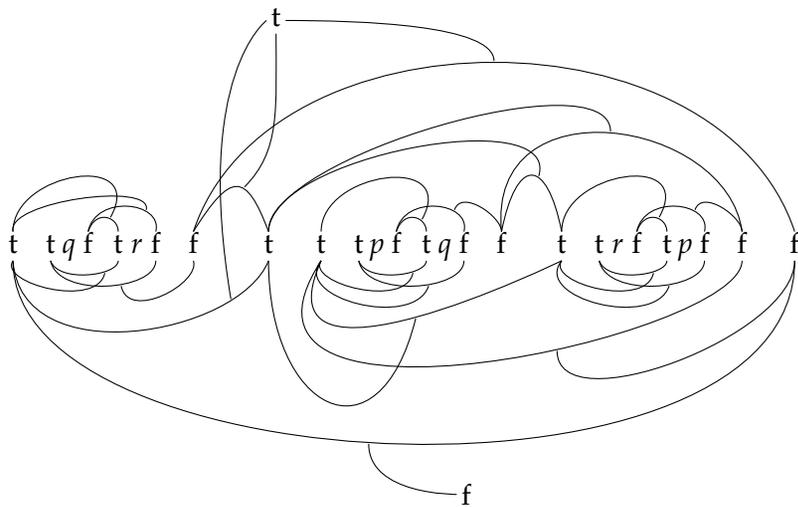
The *tf*-notation nicely *shows* that this is a tautology. By following the *f*-pole of the whole, one arrives at the *t*-pole of $q \supset r$ and the *f*-pole of $p \lor q \,.\supset.\, r \lor p$. But this requires the *t*-pole of $p \lor q$ and the *f*-pole of $r \lor p$. These, in turn, require the *f*-pole of *r* and the *f*-pole of *p*. But the *f*-pole of *r* requires the *f*-pole of *q* because we are committed to the *t*-pole of $q \supset r$. And this is impossible. The formula thereby *shows* itself to be a tautology.[3]

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[3]Something quite similar to the *ab*-notation was independently discovered much later by Martin Gardner (1982), who calls it a system of "shuttles."

There can be no significant doubt that at this time extending the *tf*-notation (and truth-tabular) decision procedure for propositional logic to quantification theory with identity (expressed with exclusive quantifiers) was precisely what Wittgenstein intended to do in further work. The letter goes on to explicitly say that this can be extended to quantification theory and identity. In the very same letter of Nov. 1913 he wrote:

Of course the rule I have given applies first of all only for what you call elementary [quantifier-free] propositions. But it is easy to see that it must also apply to all others. For consider your two Pps in the theory of apparent variables *9.1 and *9.11.... it becomes obvious that the special cases of these two Pps like those of all the previous ones become tautologous if you apply the ab notation. The ab Notation for Identity is not yet clear enough to show this clearly but it is obvious that such a notation can be made up....I can sum up by saying that a logical proposition is one of the special cases of which are either tautologous—and then the proposition is true—or self-contradictory (as I shall call it) and then it is false. And the ab notation simply shows directly which of these two it is (if any). (Wittgenstein 1979, 126)

Did he abandon this further work? Certainly not. And there can be no other apparatus besides the N-operator notation of the *Tractatus* that could be this further work.

The Tractarian N-operator has been something of a mystery. Prior to the views set for in my book *Wittgenstein's Apprenticeship with Russell*, few imagined that it might contain, as does Wittgenstein's *ab*-notation (*tf*-notation), rules for the practice of calculation of tautologyhood. I set forth the following five N-operator rules of calculation of sameness:

(N_1) $N(\xi_1, \ldots, \xi_n) = N(\xi_i, \ldots, \xi_j), 1 \leq i \leq n$, and $1 \leq j \leq n$.

(N_2) $N(\ldots \xi, \ldots, \xi \ldots) = N(\ldots \xi \ldots)$.

(N_3) $N(\ldots NN(\xi_1, \ldots, \xi_n) \ldots) = N(\ldots \xi_1, \ldots, \xi_n, \ldots)$.

(N_4) $N(\ldots N(\ldots \xi, \ldots, N\xi, \ldots) \ldots) = N(\ldots)$.

(N_5) $N(p, N(\xi_1, \ldots, \xi_n)) = NN(N(p, N\xi_1), \ldots, N(p, N\xi_n))$.

Rules (N_1)–(N_5) above are not genuine identities but are rules governing the practice of calculation. Now, admittedly, the *Tractatus* does not explicitly give all the above rules of *sameness*. One has to work a bit to notice them. But there are ample hints in the following:

> 5.501 When a bracketed expression has propositions as its terms—and the order of the terms inside the brackets is indifferent—then I indicate it by a sign of the form '($\overline{\xi}$).' 'ξ' is a variable whose values are terms of the bracketed expression and the bar over the variable indicates that it is the representative of all its values in the brackets.
>
> (E.g., if ξ has the three values P, Q, R, then
>
> $$(\overline{\xi}) = (P, Q, R).)$$
>
> What the values of the variables are is something that is stipulated.
>
> The stipulation is a description of the propositions that have the variable as their representative.
>
> How the description of the terms of the bracketed expression is produced is not essential.
>
> We *can* distinguish three kinds of description: 1. Direct enumeration, in which case we can simply substitute for the variable the constants that are its values; 2. giving a function fx whose values for all values of x are the propositions to be described; 3. giving a formal law that governs the construction of the propositions, in which case the bracketed expression has as its members all the terms of a series of forms.

The first rule (N_1) is simply that the order doesn't matter. It is almost explicitly given in the above passage. The rule (N_2) is clear enough. Wittgenstein writes:

> 5.51 If $\overline{\xi}$ has only one value, then $N(\overline{\xi}) = \sim p$ (not p); if it has two values, then $N(\overline{\xi}) = \sim p . \sim q$ (neither p nor q).

This strongly suggests that $N(p) = N(p, p)$ and thus also $N(p) = N(p, p, p)$. Rule (N_1) tells us that arguments to N may be *permuted*. Rule (N_2) permits the *deletion of repeated* arguments.

Wittgenstein's intent in using his expression $\overline{\overline{\xi}}$ is precisely to justify such transformations on grounds that, when properly understood, the notations are in some sense the *same*. The other rules of N-operator notational sameness find their evidence here. The idea is that all the transformations are legitimated by their being, according to Wittgenstein, one and the same N-notation expression. Their "sameness," of course, lies in their being instances of the N-notation $N(\overline{\xi})$. Wittgenstein glosses this with the following:

> 3.16 What values a propositional variable may take is something that is stipulated. The stipulation of values *is* the variable.
>
> 3.317 To stipulate values for a propositional variable *is to give the propositions* whose common characteristic the variable is. The stipulation is a description of those propositions. The stipulation will therefore be concerned only with symbols, not with their meaning. And the *only* thing essential to the stipulation is *that it is merely a description of symbols and states nothing about what is signified.* How the description of the propositions is produced is not essential.

The identifications made by the rules are largely straightforward. Obviously, we can legitimately add an N to both sides of any among the identities (N_1)–(N_5). For convenience, $NN(\xi_1, \ldots, \xi_n)$ is used instead of $N(N(\xi_1, \ldots, \xi_n))$. Rule ($N_3$) enables *deletion of internal double* N's. It says that if $NN(\xi_1, \ldots, \xi_n)$ is an argument to the N-operator, then we can delete this argument and add $\xi_1, \ldots, \xi_n$ as new arguments to an N-operator. Rule (N_4) tells us that we may *delete* any argument of the form $N(\ldots \xi, \ldots, N\xi, \ldots)$. Finally, Rule ($N_5$) assures *distributed* forms. What explains this distribution is the fact that they both have the form

$$NN(N(p, \xi)).$$

This occurs because the notation can also be read as

$$NN(N(p, \xi), \ldots, N(p, \xi)).$$

The difference lies only in the designation of values of ξ. The one form yields

$$NN(N(p, Nq), N(p, Nr))$$

when ξ is regarded as the list Nq, Nr. The form yields

$$NN(N(p, N(q, r))),$$

when ξ is assigned to $N(q, r)$. These rules are so obviously built into the bar notation, $N(\overline{\xi})$, that I doubt that any rival interpretation of the *Tractatus* can reasonably reject their existence even if they would hope to reject their import. That is, no careful historian of the *Tractatus* could fail to acknowledge that these rules hold.

It should be noted that the account given in my book *Wittgenstein's Apprenticeship with Russell* offers the only account of this development. Other interpretations may prefer to simply reject the thesis that this further work must be embedded somewhere in the *Tractatus*, holding that Wittgenstein abandoned his early view. But (to my mind) such an interpretation is perverse. If it were true, it would be exceedingly odd, indeed, that *any* of the N-operator rules (as articulated below) should exist at all! That there are such rules is itself corroboration of the view that Wittgenstein came to think that rules of sameness of the Tractarian N-operator notation extend (improve) his *ab*-notation (*tf*-notation), realizing his clearly stated goal (in letters and work notes) of finding a notation to capture quantification theory with identity in which all and only logical equivalents have one and the same notation. My work reveals how the N-operator notation, with its Boolean expansions using free *exclusive variables* ranging over an arbitrarily large *n*-element domain, was supposed to realize that goal. This is an effort at history, not an effort at a formal modern logical reconstruction of some Tractarian ideas.

The account of the N-operator in *Wittgenstein's Apprenticeship with Russell* is not intended as a formal system of reconstruction. It is not, as it were, what might be called an "equational

axiomatization of the finitary part of the Tractarian system," and it cannot be regarded as on a par with the several such enhanced formal reconstructions offered e.g., by Geach (1981) and Soames (1983). My work rejects the historical account of the N-operator offered by Anscombe (1959), and it also rejects the historical account offered in Fogelin (1976, 1982). But in the spirit of offering a historically faithful account, it is intended as explanation of precisely how the N-operator representations were intended by Wittgenstein to reveal that the practice of logic consists in the calculation of tautologyhood using outcomes of N-operation. Unlike Anscombe and Fogelin and indeed all other historical accounts, I explain precisely how the N-operator was supposed to take over where the *ab*-notation (*tf*-notation) and the truth-tabular representations left off—an account that explains Wittgenstein's clear intent that all and only logical equivalents of quantification theory with identity are to have one and the same expression.[4] There is no other account in the literature on the N-operator that does this—if only for the simple reason that no other account in the literature noticed that this was a central desideratum.

My account of the N-operator offers an explanation of precisely why Wittgenstein departed from the *tf*-notation (*ab*-notation) and his truth-tabular representations in his efforts to express the *wffs* of quantification theory with identity in such a way that all and only logical equivalents have one and the same expression. My account explains that Wittgenstein introduced his N-operator because he knew that his truth-table and *tf*-notation

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[4]It is important not to conflate the appendix of Landini (2007) with its historical account of the N-operator of the *Tractatus*. This appendix is intended to formulate a *modern* formal axiomatic system of exclusive quantifiers that requires *infinite* domains for quantification and which excludes pseudo-expressions such as "$(x)(x = x)$". In contrast, Wehmeier (2004) and Rogers and Wehmeier (2012) are concerned to formulate a system of exclusive quantifiers that permits domains of any non-empty cardinality and which does not exclude "$(x)(x = x)$". None of these systems can be regarded as efforts to capture the historical *Tractatus*. They are all endeavors to develop different formal systems of exclusive quantifiers.

(or *ab*-notations) cannot do justice to quantification theory (with identity). Obviously, an increase in the size of the domain of quantification will upturn any truth-tabular representation. Venn had already discovered a pictorial notation in which all and only logical equivalents (of propositional logic) have one and the same expression. Why not use a Venn propositional diagram? Such a diagram is a picture of truth conditions and it has the salient feature of representing all and only logical equivalents (of propositional logic) in one and the same way. Venn's propositional diagrams are such that all tautologies are depicted in one and the same way. Nothing is shaded. Contradictions shade every area, and contingent statements have some areas shaded and some areas unshaded. Now in fact, in the *Tractatus*, Wittgenstein does have an analog of this. He emulates Venn's picture with the expression with a truth-tabular representation. Consider the following pictures of "$p \,.\!\supset\! q \supset r$":

(t,t,t,t,t,f,t)(p, q, r).
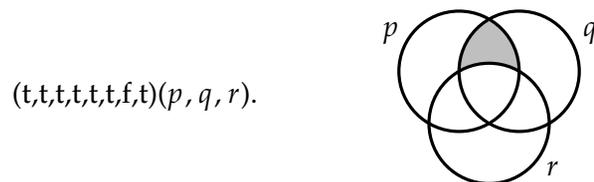
The truth-tabular expression obviously captures precisely what Venn's propositional diagram captures. All and only logical equivalents of the propositional logic have one and the same notation in terms of their truth conditions. The familiar truth-table is this:

	p	q	r	$p \,.\!\supset\! q \supset r$
1	t	t	t	t
2	f	t	t	t
3	t	f	t	t
4	f	f	t	t
5	t	t	f	f
6	f	t	f	t
7	t	f	f	t
8	f	f	f	t

Tautologies of p, q, r, all have the same representation as (t,t,t,t,t,t,t,t)(p, q, r). Why, then did Wittgenstein introduce his N-notation? What advantage does it have over truth-tabular representations? The answers are clear. The truth-tabular expressions cannot be extended to cover quantification. Quantificational expressions cannot have their truth conditions depicted by a truth-tabular expression. Wittgenstein wanted a notation that captures quantification theory (with identity built into exclusive quantifiers). The truth-tabular notations (or equivalently, Venn propositional notations) cannot be extended to quantification theory. When the domain of quantification enlarges, there will be an unstable movement of the t's and f's and there is no schematic way to stabilize them. Now in the *Tractatus*, Wittgenstein wrote:

> 5.502 So instead of '(----T)(ξ,....)' I write N($\overline{\xi}$). N($\overline{\xi}$) is the negation of all the values of the variable ξ.

This should not be read as saying that the N-notation is itself a truth-tabular notation! If it were, it would offer no benefit at all. The proper interpretation is that the word "instead" in the passage indicates that Wittgenstein is *moving on* from his notations *truth-tabular*, *ab*-notation, *tf*-notation, in a way that he hopes recovers their main feature (i.e., of expressing all and only logical equivalents in one and the same way) but can apply itself to quantification theory with identity (where proper).

Now let us postpone the discussion of quantification theory in N-notation. It is important to first understand how the N-operator recovers the features of the truth-tabular representations, i.e., how was it supposed to recover the feature that all and only logical equivalents of quantifier-free *wffs* have one and the *same* representation. Now it is quite easy to translate from propositional logical notations into Wittgenstein's N-notation. First translate into the dyadic Sheffer dagger. We next just replace each dagger with an N and replace each negation sign with an N. The two-placed Sheffer dagger $p \downarrow q$ could be writ-

ten as $\downarrow(p,q)$ and this parallels Wittgenstein's notation $N(p,q)$. We can, thus, translate from dagger notation into N-operator notation by simply replacing occurrences of the dagger and occurrences of $\sim$ with occurrences of N. After all, $\sim p =_{\text{df}} p \downarrow p$ and hence we have

$$\sim p = p \downarrow p = \downarrow(p,p) = N(p,p) = Np.$$

This uses both a dyadic $N(\xi_1, \xi_2)$ and a monadic $N\xi$. The dagger was proved by Post (1921) to be adequate to express every dyadic truth-function. Sheffer (1913) showed that the dagger and stroke are each separately adequate to express any truth-function expressible in *Principia Mathematica*. He did not show the expressive adequacy of *Principia's* signs $\sim$ and $\vee$. In any case, for any propositional formula of a language with modern logical signs, $\bullet$, $\vee$, $\sim$, $\supset$, we can express the formula in dagger notation. The following chart may help see the relationships:

Modern	Dagger	N-notation	
$\sim p$	$p \downarrow p$	$N(p,p)$	Np
$p \supset q$	$\sim(\sim p \downarrow q)$	$N(N(N(p,p),q),N(N(p,p),q))$	$NN(Np,q)$
$p \vee q$	$\sim(p \downarrow q)$	$N(N(p,q),N(p,q))$	$NN(p,q)$
$p \bullet q$	$\sim p \downarrow \sim q$	$N(N(p,p),N(q,q))$	$N(Np,Nq)$

Obviously, the N-operator notation is expressively adequate for the truth-functions of propositional logic.

The parallel with the dyadic dagger, however, can be misleading. The differences between Wittgenstein's N-operator notation and the Sheffer dagger are quite significant. We see that Wittgenstein allows $N\xi$, as well as a dyadic $N(\xi_1, \xi_2)$, and even

$$N(\xi_1, \xi_2, \ldots, \xi_n)$$

which has *n*-many argument places. (As we shall see, the notations of Spencer Brown's calculus do the same.) And one may well object that it is syntactically ill-formed. Is the *Tractatus* introducing *several* syntactically distinct N-operator symbols? Does

he mean there can be several different N-operator expressions $N(\xi), N(\xi_1, \xi_2), N(\xi_1, \xi_2, \xi_3)$ and so on? Wittgenstein's intent is not explicit. But the likely answer is that Wittgenstein intends for the syntactic expression $N(\overline{\xi})$ of the N-operator to be one-placed. Wittgenstein allows, however, that we can place in the position of $\overline{\xi}$ a list, or a recipe, or a schema, which determines what are to be the base(s) of the operation. This makes it appear, for example, that when $\overline{\xi}$ is assigned to the list "p,q,r" we may write "$N(p,q,r)$" as if we had a three-placed expression "$N(\xi_1, \xi_2, \xi_3)$," with "p," "q" and "r" in the respective positions. Wittgenstein's intent breaches syntax as we know it, and it is uncomfortable to have to get along with a wink and a nod. In any event, on Wittgenstein's view, the positions of p and q in $N(p,q)$ are supposedly unordered—the recipe or list not determining any order. Thus $N(p,q)$ is to be *the same* as $N(q,p)$. Moreover, we see that $N(p,p)$ collapses to Np. And this is general. Repetitions collapse.

These results are the key features of Wittgenstein's hope of showing that all and only logical equivalents have one and the same N-notation. And to show this was of utmost importance to him, for it is precisely this that is to reveal that there is no science of logic. One can calculate tautologyhood immediately from the N-notation symbols alone, which is supposed to be the distinctive feature of logical propositions.

6.113 The peculiar mark of logical propositions is that one can recognize that they are true from the symbols alone and this fact contains in itself the whole philosophy of logic. And so too it is a very important fact that the truth or falsity of non-logical propositions cannot be recognized from the proposition alone.

The expressive adequacy of the N-notation for the *wffs* of propositional logic is clear since it can express any *wff* in disjunctive normal form; and, as noted, Post (1921) proved this is expressively adequate for the propositional truth-functions. If we interpret the N-operator as picturing truth-conditions and follow

TLP 5.51, then all and only tautologies are pictured by

$$\mathrm{NN}(\ldots \xi, \ldots \mathrm{N}\xi, \ldots).$$

The *soundness* of the system of N-operator rules is easily revealed by the fact that each rule is semantically valid. That is any interpretation that models the left side of the identity of an N-rule also models the right side (and vice versa). This establishes consistency of the rules as well. That the system is a decision procedure for tautologyhood is easily demonstrated as well. If a formula has the above form it clearly renders the truth-conditions of a tautology. Suppose a formula A is a tautology. A formula in conjunctive normal form[5] is tautologous if and only if every conjunct is a disjunction containing some formula and its negation. Thus the conjunctive normal form of A can be expressed as follows:

$$p_1 \vee \sim p_1 \vee c_1 \;:\bullet:\; p_2 \vee \sim p_2 \vee c_2 \;:\bullet:, \ldots, :\bullet:\; p_n \vee \sim p_n \vee c_n.$$

When the truth-conditions of this are pictured in N-notation, the result is:

$$\mathrm{N}(\mathrm{N}(p_1, \mathrm{N}p_1, c_1), \mathrm{N}(p_2, \mathrm{N}p_2, c_2) :, \ldots, : \mathrm{N}(p_n, \mathrm{N}p_n, c_n)).$$

By rule (N_4) we can eliminate all of the components except one, and arrive at:

$$\mathrm{NN}(p_i, \mathrm{N}p_i, c_i),$$

where $1 \leq i \leq n$. This has form $\mathrm{NN}(\ldots \xi, \ldots \mathrm{N}\xi, \ldots)$. In this way, calculation by means of the N-operator forms a decision procedure for tautologies that is complete.

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[5]A formula is in conjunctive normal form if and only if it is a conjunction (possibly degenerate), each conjunct of which is a disjunction (again possibly degenerate) of propositional letters or their negations. A single propositional letter or its negation or a disjunction of such is a degenerate conjunction. Every tautology in conjunctive normal form has the form $\mathrm{NN}(\ldots \xi, \ldots \mathrm{N}\xi, \ldots)$ in N-notation.

Upon first receiving the *Tractatus*, Russell wrote to Wittgenstein asking several questions. In his remarks, Russell reveals that he thinks of the N-operator as a sort of generalized Sheffer dagger. We saw that at *TLP* 6 Wittgenstein gave the general form of a truth-function (proposition). On 13 August 1919, Russell asks the following question about this passage:

"General truth-function '$[\overline{p}, \overline{\xi}, \mathrm{N}(\overline{\xi})]$.'"

Yes, this is *one* way. But could one not do equally well by making $\mathrm{N}(\overline{\xi})$ mean "at least one value of ξ is false," just as one can do equally well with $\sim p \vee \sim q$ and with $\sim p \bullet \sim q$ as fundamental? I feel as if the duality of generality and existence persisted covertly in your system. (McGuinness and von Wright 1990, 108)

Wittgenstein's reply was this:

You are right that $\mathrm{N}(\xi)$ may also be made to mean $\sim p \vee \sim q \vee \sim r \vee \ldots$ But this doesn't matter! I suppose you don't understand the notation $\overline{\xi}$. It does not mean "for all values of $\xi \ldots$". (Wittgenstein 1979, 131)

Rules (N_1)–(N_5) are equally valid whether we regard N as akin to the Sheffer dagger or its dual the Sheffer stroke. The truth-conditions of tautologies will be pictured differently, of course. If we interpret the N-operator as picturing truth-conditions akin to the stroke, then all and only tautologies have their truth-conditions pictured by the form:

$$\mathrm{N}(\ldots \xi, \ldots \mathrm{N}\xi, \ldots).$$

Perhaps it was because (N_1)–(N_5) apply equally on either approach that Wittgenstein felt justified in his reply that the existence of the dual operator doesn't matter. In any event, if we accept (N_1)–(N_5) as central to Wittgenstein's conception of N-operator symbolism, we can see how it recovers the central feature of truth-table and Venn-diagrammatical representations. The propositional calculus is, on Wittgenstein's account, a calculus of equations performed by the N-operation.

To illustrate the sameness of the N-notation picturing of truth-conditions, consider the *wffs* "$p \mathbin{.\supset.} q \supset r$" and "$p \mathbin{.\supset.} r \vee \sim q$" which are logically equivalent. Now with the Sheffer dagger we have:

$$p \mathbin{.\supset.} q \supset r \qquad\qquad p \mathbin{.\supset.} r \vee \sim q$$
$$\sim(\sim p \downarrow \sim(\sim q \downarrow r)) \qquad \sim(\sim p \downarrow \sim(r \downarrow \sim q))$$
$$\sim\downarrow(\sim p, \sim\downarrow(\sim q, r)) \qquad \sim\downarrow(\sim p, \sim\downarrow(r, \sim q))$$
$$\mathrm{NN}(\mathrm{N}p, \mathrm{NN}(\mathrm{N}q, r)) \quad \mathrm{NN}(\mathrm{N}p, \mathrm{NN}(r, \mathrm{N}q))$$
$$\mathrm{NN}(\mathrm{N}p, \mathrm{N}q, r) \qquad \mathrm{NN}(\mathrm{N}p, r, \mathrm{N}q).$$

These are regarded as the same by Wittgenstein by the N-operator rules of calculating sameness.

The rules (N_1)–(N_5) are used to calculate whether a given propositional formula is a tautology. Let us consider the following two examples:

$$p \mathbin{.\supset.} q \supset p$$

$$\sim(\sim p \downarrow \sim(\sim q \downarrow p))$$
$$\mathrm{NN}(\mathrm{N}p, \mathrm{NN}(\mathrm{N}q, p))$$
$$\mathrm{NN}(\mathrm{N}p, \mathrm{N}q, p) \qquad\qquad\qquad \text{by (N_3)}$$

Tautology

$$p \supset q \mathbin{.\supset.} p \mathbin{:\supset:} p$$

$$\sim(\sim p \downarrow q) \mathbin{.\downarrow.} p) \mathbin{:\downarrow:} p)$$
$$\mathrm{NN}(\mathrm{N}(\mathrm{N}(\mathrm{N}p, q), p), p)$$
$$\mathrm{NN}(\mathrm{N}(p, \mathrm{N}(\mathrm{N}p, q)), p) \qquad \text{by (N_1)}$$
$$\mathrm{NN}(\mathrm{NN}(\mathrm{N}(p, \mathrm{NN}p), \mathrm{N}(p, \mathrm{N}q)), p) \quad \text{by (N_5)}$$
$$\mathrm{NN}(\mathrm{N}(p, p), \mathrm{N}(p, \mathrm{N}q), p) \qquad \text{by (N_3)}$$
$$\mathrm{NN}(\mathrm{N}p, \mathrm{N}(p, \mathrm{N}q), p) \qquad\quad \text{by (N_2)}$$

Tautology

These are examples of calculation by means of the N-operator rules applied to propositional logic.[6] The have the same form

⁶The rules cannot apply to the expression of exclusive quantifiers in N-notation. See Landini (2007).

$\mathrm{NN}(\ldots \xi, \ldots \mathrm{N}\xi, \ldots)$ of a tautology. It should be noted that the expressions in the notation of *Principia* and in the notation of the Sheffer dagger we have formulas. The N-operator notation, however, yields *terms* (pictures) which can flank an identity sign and be manipulated by rules for substituting identities.

## 4. Primary Algebra and Primary Arithmetic

Once we see from Wittgenstein's letters and work notes precisely what he is up to, we may search for help in the work of logicians such as Peirce and Venn and others who have had a similar idea. In searching, one will eventually land, as I did, upon Spencer Brown's book *Laws of Form*. As a preliminary, let us observe that *Laws of Form* parallels the *Tractatus* in a very important way. Of course, several references to Tractarian ideas are given in the work, and oddly the N-operator is not explicitly discussed. So, ultimately, we shall find it fair to regard it as having been independently rediscovered by Spencer Brown. But the point is that Spencer Brown's primary algebra, like Wittgenstein's N-operator, endeavors to present all and only propositional logical equivalents in one and the same notation. Like Wittgenstein's *Tractatus*, he maintains that the practice of calculation marks what is the essence of both the practice of logic (in determining whether a proposition is a tautology) and the practice of arithmetic in determining the correctness of an equation. He writes:

> In examining the interpretation as thus set out, we at once see two sources of power which are both unavailable to the standard sentential calculus. They are, notably, the condensation of a number of representative forms into one form, and the ability to proceed, where required, beyond logic through the primary arithmetic.

> Regarding the first of these sources, we may take, for the purpose of illustration, the forms for logical conjunction. In the sentential calculus they are

$$a \bullet b$$

$$b \bullet a$$

$$\sim(\sim a \lor \sim b)$$

$$\sim(\sim b \lor \sim a)$$

$$\sim(a \supset \sim b)$$

$$\sim(b \supset \sim a)$$

Each of these six distinct expressions is written, in the primary algebra in one and only one way,
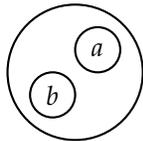
This is a proper simplification, since the object of making such sentences correspond with these symbols is not representation, but calculation. Thus, by the mere principle of avoiding an unnecessary prolixity in the representative form, we make the process of calculation considerably less troublesome.

But the power granted to us through this simplicity, although great, is itself small compared with the power available through the connexion of the primary algebra with its arithmetic. For this faculty enables us to dispense with a whole set of lengthy and tedious calculations, and also with their no less troublesome alternatives, such as the exhaustive (and mathematically weak) procedures of truth tabulation, and the graphical (and thus mathematically unsophisticated) methods of Venn diagrams and their modern equivalents. (*LofF*, 115)

Clearly, Spencer Brown is under the spell of the *Tractatus*. Wittgenstein said in the Preface of his *Tractatus*, "Perhaps this book will be understood only by someone who has himself already had the thoughts that are expressed in it—or at least similar thoughts." It seems clear enough that George Spencer Brown was having similar thoughts. He is searching, just as was Wittgenstein, for a notation in which all and only logical equivalents have (in some sense) *one and the same* representation. The connection is so close that it is difficult to imagine him coming up with it independently.

We have now to reveal that the rules embedded in Wittgenstein's N-notations very nicely parallel the algebra of Spencer Brown. In fact, the parallel is so close that it is tempting to imagine that Spencer Brown (who admits that he intensely studied the *Tractatus*) perhaps unconsciously lifted the idea from the *Tractatus* itself. One might, then, naturally search for evidence of just this in the many letters of correspondence between Spencer Brown and Russell. Perhaps Russell himself noticed the tight connections. No evidence has yet been found.

In *Laws of Form*, Spencer Brown set out a "primary algebra." The algebra employs signs that seem strange at first blush. We find the following.

The first two constitute the so-called "axioms" of the primary algebra. The latter two are the "axioms" for what Spencer Brown calls his "primary arithmetic."

Spencer Brown's primary algebra is presumably supposed to be independent of the primary arithmetic. But as we shall see in some cases Spencer Brown appeals to the arithmetic in his proofs of theorems. He offers proofs of the following theorems in his primary arithmetic:

$$C_2 \qquad \overline{p\,q\,|\,p} = \overline{p\,|\,q}$$

$$C_3 \qquad \overline{\;\overline{\rule{0pt}{6pt}}\;p} = p$$

$$C_4 \qquad \overline{\overline{p\,|\,q}\,|\,p} = p$$

$$C_5 \qquad p\,p = p$$

$$C_6 \qquad \overline{\overline{p\,|\,q}\,|\,\overline{p\,|\,q}} = p$$

$$C_7 \qquad \overline{\overline{p\,|\,q}\,|\,r} = \overline{p\,r\,|\,\overline{q}\,|\,r}$$

$$C_8 \qquad \overline{\overline{p\,|\,q\,s\,|\,r\,s}} = \overline{\overline{p\,|\,q\,|\,r}\,|\,\overline{p\,|\,s}}$$

Now we find, in comparing Spencer Brown's Primary Algebra with the rules of Tractarian calculation with the N-operator that theorem (C_1) corresponds to N-operator rule (N_3) and theorem (C_5) corresponds to the N-operator rule (N_2). Clearly, Spencer Brown's Transposition ($J2$) corresponds to (N_5) and his Position ($J1$) corresponds to (N_4). We shall have occasion to comment only on (C_1)–(C_5) for it is here that Spencer Brown arrives at results that can be shown to be surprising (indeed illicit) from the perspective of Wittgenstein's Tractarian N-operator. That is, with these successes of comparison between the N-operator rules and Spencer Brown's algebra, it comes as a surprise that Spencer Brown can successfully arrive at (C_1) and (C_5) as theorems. In the last section of this paper we shall see that he was mistaken.

Note that rule (N_1) of the N-operator allows rearranging the ordering of the terms. Spencer Brown does not state this rule explicitly. But it is quite clear that he assumes it. It is explicitly used in the proof of the theorem he calls (C_1), which we shall investigate in detail below (*LofF*, 29). It is also clear from the following passage:

> Sheffer explicitly assumes the restriction of his operator to a binary scope, and also, implicitly, assumes the relevance of the order in which the variables under operation appear. Each of these assumptions is in fact less central to mathematics than is commonly supposed, and neither is necessary at this stage. Sheffer was therefore forced to design his initial equations so ingeniously as to contradict them both... By allowing it to stand, Sheffer's description is rendered practically useless as a calculus. (*LofF*, 109)

In this passage, Spencer Brown explains that though his algebra can be interpreted so that his sign $\overline{p\,q}$ is akin to Sheffer's dagger $p \downarrow q$, his laws transcend Sheffer's work. Rule (N_2) of the N-operator allows iterations to be collapsed and it is supposed to be captured in Spencer Brown's (C_5). Rule (N_3) of the N-operator, which says that the double N's of any embedded occurrence of $NN(\xi_1, \ldots, \xi_n)$ can be dropped leaving $\xi_1, \ldots, \xi_n$, is captured in theorem (C_1). The rule (N_4) is captured by Position and finally rule (N_5) (distribution) is captured by Transposition.

Now the Tractarian rule (N_4) allowing dropping an internal $N(\ldots p, \ldots Np, \ldots)$, is expressed in Spencer Brown's system by the following rule which is stronger than ($J1$):

$$\overline{p\,|\,\overline{p\,q}} = $$

The rule enables dropping a component of the above form. The stronger rule yields the weaker ($J1$) since in the case where q is p, we have $\overline{p\,|\,\overline{p\,p}}$ and so we can use (C_5) to amalgamate $p\ p$ to arrive at ($J1$). Spencer Brown mentions that the stronger rule might have been a starting point but since it is provable in the algebra the weaker rule is adopted (*LofF*, 86). Unfortunately, he does not give a proof. I find that the algebra can prove the stronger rule from ($J1$) together with ($J2$) and (C_5). The proof uses the following lemma:

$$Lemma \quad \overline{p\,|\,\overline{p\,q}} = \overline{q\,|\,\overline{q\,p}}$$
$$demonstration$$
$$\overline{p\,|\,\overline{p\,q}}$$

$$= \overline{\overline{p\ \overline{q}\,|\,q}\ |\ \overline{p\ q}}\qquad\qquad\text{(J1)}$$

$$= \overline{\overline{p}\,|\,p\,|\,\overline{p\ q}\,|\,q\,|\,q}\qquad\qquad\text{(J2)}$$

$$= \overline{p\ \overline{q}\,|\,q}\qquad\qquad\text{(J1)}$$

From this, we have the needed theorem (J1*):

Theorem J1* $\overline{p\,|\,p\ q}\ =$

demonstration

$$\overline{\overline{p\,|\,p\ q}}$$

$$= \overline{p\ q\ \overline{p}}\qquad\qquad\text{rearrange}$$

$$= \overline{\overline{\overline{p\ q\ \overline{p}}\ |\ \overline{p\ q\ \overline{p}}}}\qquad\qquad\text{(C}_5\text{)}$$

$$= \overline{\overline{\overline{p\ q\ \overline{p}}\ |\ \overline{p\ q\ \overline{q}}}}\qquad\qquad\text{Lemma}$$

$$= \overline{\overline{p\ q}\,|\,p\ q}\qquad\qquad\text{(J2)}$$

$$=\qquad\qquad\text{(J1)}$$

Spencer Brown's primary arithmetic and algebra are readily interpretable in terms of the rules (N$_1$)–(N$_5$) that we discovered for Wittgenstein's N-operator. To begin with, it should be noted that Spencer Brown tacitly assumes the following abbreviation

$$p\ q\ =_{\text{df}}\ \overline{\overline{p\ q}}.$$

The N-operator notation does not employ this abbreviation. Nonetheless, translation from expressions of *laws of form* into N-notation is straightforward once we recognize the abbreviation. For example theorem (C$_5$)

$$p\ p = p$$

is transcribed into N-notation as:

$$\text{NN}(p,p) = p.$$

Similarly, theorem (C$_1$)

$$\overline{\overline{p}}\ = p$$

is transformed as

$$\text{NN}(p) = p.$$

The point of (C$_1$) captures the rule (N$_3$) of the N-operator. Namely, it is to allow dropping the double cross, keeping p. At first blush, there seems to be a greater generality in the N-operator rules than in Spencer Brown's theorems. Spencer Brown's principles above are not stated for cases where multiple arguments appear under his cross, but he derives these as "theorems of second-order." He allows the substitution of concatenations of symbols, not just single propositional letters, for his p, q etc. For example, he allows

$$\overline{\overline{p\ q}}\ = p\ q$$

to be an instance of (C$_1$).

We noted that Spencer Brown's account seems to yield the surprise that some of the rules of the N-operator can be theorems of Spencer Brown's primary algebra. There are, however, problems in his derivations of (C$_1$) and (C$_5$) in *Laws of Form*. Our comparison to the Tractarian N-operator reveals this clearly. It is, therefore, only by assuming (C$_1$) and (C$_5$) as axioms together with (J1) and (J2) that Spencer Brown fully captures (N$_3$) and (N$_2$). Spencer Brown relates the following interesting story of the existence of the demonstration of (C$_1$):

I had at first supposed the demonstration of C1 to be impossible from J1 [Position] and J2 [Transposition] as they stand. In 1965 a pupil, Mr. John Dawes, produced a rather long proof to the contrary, so the following year I set the problem to my class as an exercise,

and was rewarded with a most elegant demonstration by Mr. D. A. Utting. I use Mr. Utting's demonstration, slightly modified, in the text. (*LofF*, 89)

This curious story leaves out Mr. Dawes's proof that (C_1) is not independent of (J1) and (J2). Dawes's proof may be lost to history. Utting's proof, embellished to rearranging terms, is as follows:

$$C_1 \quad \overline{\overline{p}} = p$$

demonstration

$$\overline{\overline{p}}$$

$$= \overline{\overline{p}\,\overline{p}\,\overline{p}} \qquad\qquad \text{(J1)}$$

$$= \overline{\overline{p\,\overline{p}}\,\overline{p\,\overline{p}}} \qquad\qquad \text{(J2)}$$

$$= \overline{\overline{p}\,\overline{p}\,\overline{p}\,p} \qquad\qquad \text{rearrange}$$

$$= \overline{\overline{p}\,p} \qquad\qquad \text{(J1)}$$

$$= \overline{\overline{p}\,p\,\overline{p}\,p} \qquad\qquad \text{(J1)}$$

$$= \overline{\overline{p}\,\overline{p}}\,p \qquad\qquad \text{(J2)}$$

$$= p \qquad\qquad \text{(J1)}$$

While we do not have an independence proof of the rules (N_1)–(N_5) of the N-operator (Tang 2018), it remains worth investigating whether Utting's proof is successful. We find that it is not. Here we can see that the second to last line

$$\overline{\overline{p}}\,\overline{\overline{p}}\,p$$

abbreviates

$$\overline{\overline{\overline{p}\,\overline{\overline{p}}\,\overline{p}}}$$

The application of (J1) eliminates the component

$$\overline{\overline{p}\,\overline{\overline{p}}}$$

and thus yields only $\overline{\overline{p}}$.

Since (C_1) has not been demonstrated, the demonstration of (C_5) fails as well. In *Laws of Form*, the demonstration of (C_5) is given as follows:

$$C_5 \quad p\,p = p$$

demonstration

$$p\,p$$

$$= \overline{\overline{p}}\,p \qquad\qquad (C_1)$$

$$= p \qquad\qquad (C_4)$$

The problem in the derivation of (C_5) is obvious because it depends on a flawed derivation of (C_1). There is yet another flaw. Spencer Brown appears to employ the rule of Order from the Primary Arithmetic. This occurs in the application of (C_4). It seems that what is intended is the substitution of $\overline{}$ for q in (C_4) to yield:

$$\overline{p\,\overline{\overline{p}}}\,p = p$$

Then by applying Order, we arrive at $\overline{\overline{p}}\,p = p$. But of course, Order was not part of the Primary Algebra and thus may not be used in the demonstration. Accordingly, we cannot transcribe Spencer Brown's demonstration into N-notation, and we must conclude that (N_2) cannot be produced in this way as a theorem.

Hence, there is a sense in which Spencer Brown captures the full analogs of the rules of the N-operator only if he adds (C_1) and (C_5) as axioms. Alternatively, he may simply pair his Primary Algebra with his Primary Arithmetic to achieve the result.

As we have seen, pairing of the Primary Arithmetic with the Primary Algebra is ultimately what Spencer Brown advocates. In an amusing passage he writes:

> The consequences of this arithmetical availability are sweeping. All forms of primitive implication become redundant, since both they and their derivations are easily constructed from, or tested by reduction to, a single cross. For example, everything in pp. 98–126 of *Principia Mathematica* can be rewritten without formal loss in the one symbol
>
> ⌐
>
> provided, at this stage, the formalities of the calculation and interpretation are implicitly understood, as indeed they are in *Principia*. Allowing some 1500 symbols to the page, this represents a reduction of the mathematical noise-level by a factor of more than 40000. (*LofF*, 117)

After page 126 of *Principia* begins quantification theory. So Spencer Brown is saying that his system can capture the quantifier-free calculus. Since in the introduction to the 1925 second edition to *Principia*, Russell expressed a desire to reform the quantifier-free calculus using Sheffer's new apparatus, we can see why Spencer Brown would naturally assume that he would also be interested in his work.

The technique for calculating by means of the N-operator whether a proposition is a tautology is paralleled by Spencer Brown's system. In contrast, Spencer Brown allows his notations to themselves be substituted for p, q, etc., in propositions and to appear themselves isolated on a line of proof. Accordingly, in calculating whether a proposition is a tautology in *laws of form*, the sequence ends with

⌐

if and only if it is a tautology. We found that, when expressed in N-operator notation, all and only tautologies have the form $NN(\ldots \xi, \ldots N\xi, \ldots)$. The analog of Spencer Brown's last inference would be to use (N_4) to arrive at N. But that is not allowed in N-notation.

Let us take the example given in Spencer Brown's letter to Russell of 23 May 1964 quoted above. Consider the formula below used to illustrate his method.

$$q \supset r :\supset: p \vee q \,.\!\supset.\, r \vee p.$$

Expressed with the Sheffer dagger this is:

$$\sim(\sim q \downarrow r \,.\!\downarrow.\, \sim(p \downarrow q \,.\!\downarrow.\, \sim(r \downarrow p))).$$

Now let us show the calculations in the following chart:

Wittgenstein	Spencer Brown	
$NN(N(Nq, r), NN(N(p, q), NN(r, p)))$	$q \rceil r \rceil p\ q \rceil r\ p \rceil\!\rceil\!\rceil$	
$NN(N(Nq, r), N(p, q), r, p)$	$q \rceil r \rceil p\ q \rceil r\ p$	
	$q \rceil r \rceil r\ p\ q \rceil p$	*rearrange*
$NN(NN(N(Nq, r), r), NN(N(p, q), p))$	$q \rceil r \rceil r \rceil\!\rceil p\ q \rceil p \rceil\!\rceil$	(C_1)
$NN(N(N(r, q), N(r, Nr)),$ $\quad N(N(p, Nq), N(p, Np)))$	$r\ q \rceil r\ r \rceil\!\rceil p\ q \rceil p\ p \rceil\!\rceil$	(J2)
$NN(NN(r, q), NN(p, Nq))$	$r\ q \rceil\quad p\ q \rceil\!\rceil$	(C_1)
$NN(r, q, p, Nq)$	$r\ q\ p\ q \rceil$	(C_1)
	⌐	

As we can see, Spencer Brown's calculus is just a notational variant of Wittgenstein's N-operator.

5. Primary Algebra Yields Primary Arithmetic?

That N-notation is expressively adequate to propositional truth-functions is obvious. It can express anything in disjunctive normal form. We saw that its rules form a decision procedure for tautologies, proved above by the method of conjunctive normal form. Every tautology has the form $NN(\dots p, \dots Np, \dots)$. Similarly, the expressive adequacy of Spencer Brown's system is manifestly obvious since it can express anything in disjunctive normal form. Its consistency and soundness are established by this as well. The success of Spencer Brown's system as a decision procedure for tautologies follows if and only if it captures precisely the rules of the N-operator. It is to that question to which we now turn.

We saw that in the Primary Arithmetic, two rules are adopted. The first allows dropping the double cross, and the second allows amalgamating multiple crosses. They are:

Number ‛‛ = ‛

Order ‛ =

We saw as well that theorem (C_1) is not forthcoming in the Primary Algebra and must be taken as axiomatic. But having done so, $(J1)$, $(J2)$ and (C_1) yield the Primary Arithmetic. This is revealing since Spencer Brown imagines things the other way—as if the Primary Arithmetic is more fundamental than the Primary Algebra. To see this properly, let us arrive at Order and Number as theorems. These are readily provable from the following which follows from (C_1) and $(J1)$. We have:

Theorem J1.1 $\overline{p}\,|\,p =$ ‛

demonstration

$\overline{\overline{\overline{p}\,|\,p}} = \overline{p}\,|\,p$ (C_1)

$\overline{p}\,|\,p = \overline{\overline{p}\,|\,p}$

$\overline{p}\,|\,p =$ ‛ $(J1)$

Number ‛‛ = ‛

demonstration

$\overline{\overline{p}\,|\,p} =$ ‛ (C_1)

‛‛ = ‛ Theorem J1.1

Order $\overline{\overline{}}\,| =$

demonstration

$\overline{\overline{p}\,|\,p}\,| =$ $(J1)$

$\overline{\overline{}}\,| =$ Theorem J1.1

In this respect, we do well to concur with Russell in thinking that logic (in the sense in which logic is algebraic calculation, and not merely the calculation of whether a proposition is tautologous) is prior to arithmetic, and indeed prior to number generally.

Unfortunately, in all this we have to acknowledge that the primary algebra involves, as does the N-operation, a sort of wink and nod when it comes to the syntactic question of how many places, one or any finite number, the operator sign has. Strictly speaking, there should be several operator signs that are interrelated, but this would undermine the claimed austerity of the system, for it would then produce new rules and axioms connecting the different operation signs. In a letter of 27 May 1960, Spencer Brown says: "Since I saw you last, I have made a rather unexpected discovery—that there are at least four infinite classes of universal operators. Two of them sub-classes within the other

two. I know only by hearsay of Zilinsky's [*sic*] (is that his name?) proof that there were only two universal operators, but I think I can . . . where he went wrong. . . . Briefly, it comes from discarding the binary limitations of the scope of logical operators. . . ." Spencer Brown complains (as in several of his letters) that his work continues to be summarily dismissed without it being understood and says that Braithwaite has sent him a letter about it ". . . which was so angry and so childish and so rude that one begins to doubt that he is still in full possession of his faculties." Spencer Brown goes onto say that Braithwaite will not admit that his notation is an advance over that of Sheffer. He says: "I should have thought this was patent." He gives the example

$$\sim a \bullet \sim b \;.\bullet.\; (c \vee d \;.\vee.\; e),$$

noting that Sheffer has to write

$$\sim(c \downarrow d) \downarrow e \;.\downarrow.\; \sim(a \downarrow b),$$

whereas he can write

$$\overline{\overline{c\ d\ e}\ \big|\ a\ b}\,.$$

But the problem Braithwaite saw is that the operator signs employed are syntactically not the same. We have seen this already with Wittgenstein's N-operator notation. The translation of the Sheffer notation yields

$$\mathrm{N}(\mathrm{N}(\mathrm{NN}(c,d),e),\mathrm{NN}(a,b)).$$

In Spencer Brown's notation this is

$$\overline{\overline{\overline{c\ d}\,\big|\,e}\ \big|\ \overline{a\ b}}\,.$$

By the transformation rules for N-notation, this becomes

$$\mathrm{N}(\mathrm{N}(c,d,e),a,b).$$

And in Spencer Brown's calculus, the rule for dropping double hooks yields this

$$\overline{\overline{c\ d\ e}\ \big|\ a\ b}$$

But as Braithwaite correctly observes here we have the occurrence of a new triadic hook. The analog is the triadic "$\mathrm{N}(p_1, p_2, p_3)$". While the transformation rules make for a simplification of the calculus, we cannot properly speak of the syntactic notations being "the same."

Russell's reply to Spencer Brown's letter was sent on 18 June 1960. He remarks that he finds Spencer Brown's results interesting and that he can't imagine what in it should make Braithwaite angry. Russell goes on to say: "I am quite incapable now-a-days of giving much time or thought to logic as other matters absorb me. The work of Zilinsky [*sic*] is quite unknown to me." In 1925 Eustachy Żyliński proved that joint denial $p \mid q$ and alternative denial $p \downarrow q$ are the *only* two propositional dyadic logical particles each adequate on its own to express every truth-function (Żyliński 1925). Russell once knew of Żyliński. In 1929 he had sent him a letter of recommendation for Leon Chwistek that, in part, earned him a position as Professor of Logic at Lwow (see Jadacki 1986). Spencer Brown points out that there are yet others. But this does not contradict Żyliński because his syntax is not confined to dyadic operation signs.

Putting this troublesome feature aside we have revealed, having distilled the rules of N-operator sameness, that we can use them to help understand and evaluate Spencer Brown's work. And conversely, we can use Spencer Brown's work in *Laws of Form* to help in the ongoing quest to better understand how Wittgenstein's Tractarian N-operator (and combinators in general) might apply itself to the emulation of the arithmetic of numbers and to other areas. *Laws of Form* is a fascinating book, rich with many ideas that could help in exploring Tractarian themes. We have found that the two rather inscrutable systems, one from Spencer Brown and one from Wittgenstein, enhance

each other to produce something of a clear picture of what is going on in each. How then do the rules governing the practice of calculating sameness by Wittgenstein's N-notation compare to the rules Spencer Brown uses to calculate? The answer we found is that by the light of N-operator rules, Spencer Brown made some mistakes in his attempt to reduce the number of rules needed. Here again we see the value of the close associations between the systems.

6. N-operational Quantification Theory *with* Identity

Wittgenstein's intention in introducing his N-operator was precisely to extend to quantification theory the result he had already obtained in his ab-notation (*tf*-notation) as applied to the quantifier-free calculus. That is the *sole* reason for his N-operator. All and only logical equivalents of quantification with identity are to have one and the same expression. Spencer Brown wanted to extend his algebra to quantification theory and at once he ran into the problem of how to capture subordinate occurrences of quantifiers and the difference between an existential quantifier and a universal quantifier. His example was (*LofF*, 128):

> Some a are b.
> Some b are c.
> Therefore: Some a are c.

Now this argument is valid if and only if the following is a (generalized) tautology:

> Some a are b • Some b are c .⊃. Some a are c.

Spencer Brown realized, however, that it is valid in a one-element domain, for it is

$$(Ax_1 • Bx_1) • (Bx_1 • Cx_1) .⊃. (Ax_1 • Cx_1).$$

This is a tautology. But *invalid* in a domain greater than one. In a two-element domain it is:

$$(Ax_1 • Bx_1 .\lor. Ax_2 • Bx_2) • (Bx_1 • Cx_1 .\lor. Bx_2 • Cx_2) .⊃.$$
$$(Ax_1 • Cx_1 .\lor. Ax_2 • Cx_2)$$

This is not a tautology. Spencer Brown didn't know, quite, how to capture the difference in his algebra. Again if one were to take it in a one-element domain we get the following:
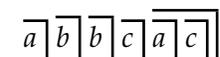
$(a • b) • (b • c) .⊃. (a • c)$

i.e., $Ax • Bx. • .Bx • Cx :⊃: Ax • Cx$

$\sim\{\sim[\sim(\sim Ax \downarrow \sim Bx) .\downarrow. \sim(\sim Bx \downarrow \sim Cx)] :\downarrow: \sim Ax \downarrow \sim Cx\}$

$\text{NN}\{\text{NN}[\text{NN}(\text{N}Ax, \text{N}Bx), \text{NN}(\text{N}Bx, \text{N}Cx)], \text{N}(\text{N}Ax, \text{N}Cx)\}$
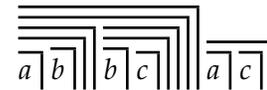
Now by the rules, this yields

$$\text{NN}\{\text{N}Ax, \text{N}Bx, \text{N}Bx, \text{N}Cx, \text{N}(\text{N}Ax, \text{N}Bx)\}.$$

Spencer Brown correctly transforms it as:

This has a tautologous form in N-notation because it is the same as:

$$\text{NN}\{\text{NN}(\text{N}Ax, \text{N}Bx), \text{N}Cx, \text{N}(\text{N}Ax, \text{N}Bx)\}.$$

It is illuminating to see how Wittgenstein was able to get past the obstacle that Spencer Brown saw but failed to resolve.

The key, obviously, is just to embrace such expansion over arbitrarily large domains. There is a wonderful hint in Wittgenstein's notebook entry for 22 May 1915, which, albeit prior to the introduction of the N-operator symbol, explains this procedure. We find:

The mathematical notation "$1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$" together with the dots is an example of that extended generality. A law is given and the terms that are written down serve as an illustration. In this way instead of $(x)fx$ one might write "$fx \bullet fy \ldots$" (Wittgenstein 1979, 49)

The notion of an "operation" is tied by Wittgenstein to the notion of the *dots* (a.k.a. "and so on") rendering a recipe (law) and an illustration of the general form. (This is done in the *Tractatus* by the bar $\overline{\xi}$ notation at 5.501.) Thus, "$(x)\varphi x$" is expressed as an expansion over an arbitrarily large n-element domain:

$$\varphi x_1 \bullet \ldots \bullet \varphi x_n,$$

where $x_1, \ldots, x_n$ are all to be interpreted *exclusively* of one another. I mark this passage, therefore, as the "discovery" (if you will) of the N-operator. Consider then the following:

$$(x)(Fx \bullet Gx).$$

Now this is to have the same representation in N-notation as

$$(x)(Fx) \bullet (x)(Gx).$$

The way N-notation does this is by imagining an arbitrary large n-element domain for quantification. Thus to find N-notations for the above, simply find the Boolean expansions over an n-element domain, where each free variable x_i, where $1 \leq i \leq n$, is intended to be referring to a distinct entity. Next, translate into N-notation. We have:

$$(x)(Fx \bullet Gx)$$
$$Fx_1 \bullet Gx_1 \bullet \ldots \bullet Fx_n \bullet Gx_n$$
$$N(NFx_1, NGx_1, \ldots, NFx_n, NGx_n).$$

Compare the following:

$$(x)(Fx) \bullet (x)(Gx)$$
$$Fx_1 \bullet \ldots \bullet Fx_n \bullet Gx_1 \bullet \ldots \bullet Gx_n$$
$$N(NFx_1, \ldots, NFx_n, NGx_1, \ldots, NGx_n).$$

As we can see, in N-notation these are the *same* (because, as Wittgenstein pointed out at *TLP* 5.501, the order doesn't matter).

Unfortunately for Wittgenstein, (polyadic) predicate logic is not decidable and that entails that his hope that the rules of N-notation still apply even when schematic n is used was quite mistaken. We can, of course, apply the rules to finite subgroups within N($\ldots\ldots$), but there is no schematic way to proceed in general. If there were a way to apply the rules, quantification theory would be decidable. Hence, the *Tractarian* rules cannot be applied for schematic n, but may only be applied when n is fixed. Since the viability of the rules of N-operator sameness is precisely what Wittgenstein hoped to use to establish the result that quantification theory is not a science, Wittgenstein's plan fails to succeed. The kind of representational system Wittgenstein sought for quantificational formulas is impossible.

Curiously, Fogelin (1976, 1982) intuited that since quantification theory is not decidable, the N-notation must fail to be expressively adequate. But as Geach pointed out, he could not make clear what connection there is between its being expressively adequate and its being a decision procedure. My account can. Fogelin mistakenly imagined that a connection lies in that the *Tractatus* cannot independently confine the scope of bound variables and in its demand for *successive* applications of N. Geach (1981) called him to task, finding nothing in the demand of successive application that favors Fogelin's thesis. Geach sees no ground for Fogelin to link expressive adequacy for quantification to the decidability of quantification theory.

Geach held that the *Tractatus* fails to offer a notation that can emulate different scopes for the bound variables of quantification. Geach therefore enhances the account by introducing a minimal class theory and a new notation for confining variables using class notations, but he recognized that in doing so he abandons the letter of the *Tractarian* program which maintains *TLP* 6.031 that classes are superfluous in mathematics. In fact, Fogelin, Geach, and Soames were all unaware of Wittgenstein's

intent to offer a notation in which all and only logical equivalents of quantification theory with identity have one and the *same* expression. Wittgenstein's N-notation (with its exclusive free variables) is expressively adequate for quantification over any fixed, albeit arbitrary large, *n*-element domain. It is expressively adequate for arbitrarily large finite domains, without confining any variable! Consider the following:

$$(x)\sim Fx$$

$$\sim Fx_1 \bullet \ldots \bullet \sim Fx_n$$

$$N(Fx_1, \ldots, NN(Fx_{n-1}, Fx_n))$$

i.e., $N(Fx_1, \ldots, Fx_n)$

$$\sim(\exists x)Fx$$

$$\sim(Fx_1 \vee \ldots \vee Fx_n)$$

$$N(NN(Fx_1, \ldots, Fx_n))$$

i.e., $N(Fx_1, \ldots, Fx_n)$.

These are *the same*, given the rule of sameness according to which internal double NN's drop out. In the latter, no variable's scope has been confined inside the negation sign. It is the fixed *n*-element domain (where *n* is any natural number) that does all the work.

The same goes for the following which is a case of mixed multiply general *wff* that Fogelin was so very concerned about. We have:

$$(x)(\exists y)R(x, y)$$

$$(\exists y)R(x_1, y) \bullet \ldots \bullet (\exists y)R(x_n, y))$$

$$[R(x_1, x_1) \vee \ldots \vee R(x_1, x_n)] \bullet \ldots \bullet$$

$$[R(x_n, x_1) \vee \ldots \vee R(x_n, x_n)]$$

$$N\{NNN[R(x_1, x_1), \ldots, R(x_1, x_n)], \ldots,$$

$$NNN[R(x_n, x_1), \ldots, R(x_n, x_n)]\}$$

$$N\{N[R(x_1, x_1), \ldots, R(x_1, x_n)], \ldots,$$

$$\ldots, N[R(x_n, x_1), \ldots, R(x_n, x_n)]\}.$$

No variables are confined here at all, and all distinct free variables $x_1, \ldots, x_n$ are exclusive of one another. Fogelin, as we noted, was under the impression that all the free variables of N-notation must be confined with precisely the same scope. He is correct in this impression, but he failed to understand that Wittgenstein was expressing quantification over a fixed domain of arbitrarily large size *n*. It is certainly adequate to *that* without being decidable. The connection between expressive adequacy and a decision procedure comes from the application of the rules of sameness. The rules of sameness (N$_1$)–(N$_5$) cannot apply unless one fixes the *n* so that it is no longer acting as a schema. Without the rules, however, Wittgenstein fails to realize his goal of *showing* (elucidating) that quantification theory with identity is not a genuine science. He has not realized his goal of establishing that all there is to quantification theory is the *practice* (= the decision procedure) of calculating tautologyhood.

Finally, let us note that the Tractarian elimination of identity is quite easy as well. In an appendix to *Wittgenstein's Apprenticeship with Russell*, I offered a symbolism that facilitates the translation involving identity (where, and only where legitimate) into N-notation. This has been misunderstood, so let me take the opportunity to note that my symbolism in the appendix was not intended to itself be attributed to Wittgenstein. Consider expressing *Principia*'s *wff*

$$(\exists x)(Fx \bullet (y)(Fy \supset y = x)).$$

My symbolism first puts it into the following notation:

$$(\exists x)(Fx \bullet (y^x)\sim Fy).$$

This says that exactly one entity is *F*. (Literally, it says that there is some entity that is an *F* and every entity distinct from it is

not an *F*.) To find Wittgenstein's N-notation for it, we first take its expansion over an arbitrarily large *n*-element domain. This yields:

$$[Fx_1 \bullet (y^{x_1})\sim Fy] .\vee. \ldots .\vee. [Fx_n \bullet (y^{x_n})\sim Fy]$$
$$\text{i.e., } [Fx_1 \bullet (\sim Fx_2 \bullet \ldots \bullet \sim Fx_n)] .\vee. \ldots .\vee.$$
$$[Fx_n \bullet (\sim Fx_1 \bullet \ldots \bullet \sim Fx_{n-1})].$$

It is easy to then express this in dagger notation and then express it in N-notation. My approach makes it impossible to translate the pseudo-expression "$(x)(x = x)$" into N-notation.

Interestingly, the account of Wittgenstein's elimination of identity in favor of N-notation that I put forth also makes every instance of *Principia*'s infinity statement (its so-called "infin ax") come out as having a tautologous form in N-notation. This is a criterion for adequacy for any proposed transcription of Wittgenstein's notion of eliminating identity in favor of exclusive quantifiers. It is a criterion for adequacy because one must explain the following:

> 5.535 All the problems that Russell's "axiom of infinity" brings with it can be solved at this point. What the axiom of infinity is intended to say would express itself in a language through the existence of an infinite number of names with different meanings.

Wittgenstein held that that infinity is shown in N-notation and that this obviates *Principia's* need to *say* that there are infinitely many entities by writing its pseudo-expression "infin ax" as an antecedent clause of some theorems. The following in translation have the same N-operator form, namely NN($\ldots p, \ldots, Np \ldots$), of a tautology:

$$(\exists x)(Fx \bullet (z^x)\sim Fz) \supset (\exists w)\sim Fw$$

If there is exactly one entity that is an F

then some entity is not an F.

$$(\exists x)(\exists y^x)(Fx \bullet Fy \bullet (z^{x,y})\sim Fz) \supset (\exists w)\sim Fw$$

If there are exactly two distinct entities that are each F

then some entity is not F.

And so on. No rival account of Wittgenstein's N-operator achieves this.

The very language of N-notation *shows* the infinity (potential infinity) of the "domain", as it were, of quantification. It *shows* this by the availability of infinitely many distinct variables expressed by the use of the schematic "*n*" which represents the fact that Wittgenstein's position is antithetical to the modern semantic notion of interpreting a quantifier as ranging over a domain (which by definition has a fixed cardinality). Wittgenstein's intent was to *show* infinity by means of distinct variables available for N-operator expressions. His intent, if put in terms of a semantic interpretation of a formal language over a domain, would have to be expressed by saying that the domain of interpretation must be *infinite*. This is not to disparage formal logical reconstructions of various *Tractarian* ideas concerning exclusive quantifiers. For example, Wehmeier (2004) and Rogers and Wehmeier (2012) have developed formal systems for quantification theory for exclusive quantifiers in such a way that aims to parallel the modern notion that the truths of quantification theory with identity should be invariant truths over any non-empty domain of every cardinality (finite or infinite). Their formal reconstructions do not aim at an exacting history that captures what Wittgenstein achieved in his *Tractatus*. Rogers and Wehmeier allow finite domains of interpretation and they do not endeavor to capture Wittgenstein's clear intent that an expression such as "$(x)(x = x)$" should be a pseudo-proposition that is ungrammatical. Moreover, they certainly do not have the aim of revealing how it is that Wittgenstein thought he had made good on his clear intentions in his letters, work notes and in his *Tractatus* itself, to find a notation for quantification theory *with identity* (where admissible) that expresses

all and only logical equivalents in one and the same way. Spencer Brown did. Working independently (it seems), he found a notation and a system of rules of calculation that is (with the corrections we have given for his oversight concerning the proof of (C_1) and (C_5)) an alternative of Wittgenstein's N-operator notation as applied to quantifier-free *wffs*.

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