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On Operator N and Wittgenstein's Logical Philosophy

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In this paper, I provide a new reading of Wittgenstein's N operator, and of its significance within his early logical philosophy. I thereby aim to resolve a longstanding scholarly controversy concerning the expressive completeness of N. Within the debate between Fogelin and Geach in particular, an apparent dilemma emerged to the effect that we must either concede Fogelin's claim that N is expressively incomplete, or reject certain fundamental tenets within Wittgenstein's logical philosophy. Despite their various points of disagreement, however, Fogelin and Geach nevertheless share several common and problematic assumptions regarding Wittgenstein's logical philosophy, and it is these mistaken assumptions which are the source of the dilemma. Once we recognize and correct these, and other, associated expository errors, it will become clear how to reconcile the expressive completeness of Wittgenstein's N operator, with several commonly recognized features of, and fundamental theses within, the Tractarian logical system.

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1. Introduction

Concerns about the expressive completeness of Wittgenstein's operator N have led some scholars to consider or even suspect that Wittgenstein may have been guilty of an elementary logical blunder (Soames 1983, 578; Fogelin 1987, 82). Other scholars (McGray 2006, 150; Geach 1981, 170) have noted the low probability that, if Wittgenstein had committed an elementary blunder, exceptionally competent reviewers of the day, such as Russell and Ramsey, would have let it pass. Within the interchanges between Fogelin and Geach, in particular, a dilemma emerged whereby it seems we must either accept Fogelin's claim that N is expressively incomplete, or reject certain fundamental theses within Wittgenstein's logical philosophy, such as his repudiation of set theory, and his claim that all meaningful propositions may be expressed *via* a finite number of successive operations. In either case, Wittgenstein would have committed a fairly significant logical blunder. In this paper, I will instead argue that there was no blunder and thus that there was nothing for Russell or Ramsey to let pass. Once we identify and recognize the significance of several interrelated and underappreciated features of Wittgenstein's logical philosophy, and correct a number of associated expository errors, it will become clear how to address Fogelin's concerns about expressive completeness without undermining any of the commonly recognized and fundamental theses which characterize the Tractarian logical system.

In addition to several subsidiary features, there are four primary, significant and interrelated elements of Wittgenstein's logical philosophy that are not fully appreciated by either Geach or Fogelin. These are (1) Wittgenstein's conception of infinity as actual as opposed to potential; (2) Wittgenstein's provision of the N operator for purely philosophical as opposed to practical purposes; (3) Wittgenstein's characterization of elementary propositions as endowed with structure but nevertheless logically independent; and (4) Wittgenstein's characterization of N as a sentential as opposed to quantificational operator. Because they fail to appreciate (1) and (2), it seemed obvious to both Fogelin and Geach (in opposition to (4)) that Wittgenstein intended the N operator to be deployed in consort with some sort of quantificational device, whether it be an open sentence, or a class forming operator, in order to facilitate quantification over an infinite domain. However, careful attention to Wittgenstein's symbolism and explanations shows that he only ever intended N to apply to semantically atomic, elementary propositions, and to N-expressed[1] truth-functions thereof. As per (3), above, elementary propositions have structure, but are nevertheless logically independent. This entails that N may be applied, in the basis case, to semantically atomic sentence letters which lack any internal structure, just as Wittgenstein indicates within his symbol for the general form of a truth-function: $[\bar{p}, \bar{\xi}, \mathrm{N}(\bar{\xi})]$ (TLP 6).[2] N may then subsequently be applied to N-expressed

[1]To say that a truth-function is "N-expressed" means simply that it appears as expressed using the N operator, rather than as it would appear as expressed via any other combination of truth-functional connectives, including the ampersand (conjunction), the tilde (negation), or the wedge (disjunction). Thus, the truth-function known as joint negation may be expressed using the ampersand and the tilde as "$\sim p$ & $\sim q$", or using the N operator as "$\mathrm{N}(p, q)$." In the latter case, the truth-function is said to be "N-expressed" in the intended sense.

[2]When I wish to refer to Wittgenstein's *Tractatus* with indifference to any distinctions between the 1922 Odgen translation, and the 1961 Pears and McGuiness translation, I will simply use "TLP" followed by a proposition number (e.g., TLP 5.501). By contrast, I will use TLP 1922 followed by a proposition number, to refer specifically to the Odgen translation, and TLP

truth-functions of the elementary propositions symbolized by these semantically atomic sentence letters, in order to express all truth-functions, including the truth-functional expansions[3] which correspond to quantified propositions. It can then readily be seen to be expressively complete with regards to classical predicate and propositional logic, provided we recognize and appreciate the significance of (1) and (2).

1961 followed by a proposition number, to refer specifically to the Pears and McGuiness translation.

[3]Throughout this paper I will deploy truth-functional expansions expressed using classical truth-functions such as conjunction and disjunction, for illustrative purposes, as aids to understanding the content of truth-functional expansions expressed using N. Despite the fact that Wittgenstein is critical of Frege and Russell, at TLP 5.521, for introducing generality in association with conjunction and disjunction, our use of disjunction and conjunction to elucidate truth-functional expansions, and generality, will be harmless provided the reader recognizes that these connectives are ultimately to be eliminated from our expressions in favour of a single truth-function, specifically N. Notably, at 5.521 Wittgenstein claims merely that understanding generality in terms of these truth-functions is associated with difficulties; he does not explicitly proscribe it. Indeed, when Wittgenstein also says at 5.521, that he "dissociates the concept *all* from truth-functions," he cannot be proscribing such an elucidatory use of conjunction and disjunction, since N is also a "truth-function" and in that case he would also be proscribing the use of N to elucidate quantification. And aside from being *prima facie* implausible, this would make it very hard to explain why Wittgenstein defines N such that it may take an indefinite number of arguments (see Section 2). It is thus more probable that, by this remark, Wittgenstein means simply that like the sentential connectives, quantifiers can be eliminated in favour of a single truth-functional operator, N, which cannot be applied either to quantifiers, or to other "general" logical operators (e.g., class forming operators), in order to construct well-formed sentences in N notation. Referencing disjunction and conjunction will help us to understand *how* to translate from expressions employing classical truth-functional operators and quantifiers, into expressions employing only N. Wittgenstein himself would seem to be sympathetic to this approach, since he acknowledges that the ideas of disjunction and conjunction are each "embedded" (5.521) within the notions of universal and existential quantification, and because he endorses this very approach to understanding the *Tractatus* account of generality himself on several occasions (e.g., Wittgenstein 2005, 249; cf. Moore 1955, 2; Stern, Rogers and and Citron 2016, 215).

Because most contributors to this debate are, like me, unpersuaded that Wittgenstein is guilty of an elementary blunder, they do not in general accept, as does Fogelin, the idea that the N operator is expressively incomplete. Many, as we shall see, nevertheless do not fully appreciate these integral, and exculpatory features of the Tractarian logical system. In a valiant effort to rescue Wittgenstein from the appearance of incompetence, commentators have been led to devise many ingenious, if ultimately unnecessary technical, notational innovations. An unfortunate consequence of these commendable efforts, however, is that such technical, notational innovations have come to appear to be much more relevant to the understanding of Wittgenstein's proposal than they actually are. A subsidiary goal of this paper will thus to be to recover the deeper and ultimately *philosophical* meaning of the N operator, which has less to do with concerns of notational innovation, as we shall see, than has sometimes been supposed.

In order to more fully appreciate these important lessons it will be helpful to delve first, in Section 2, into the nature of Wittgenstein's N operator and to clear up some misconceptions concerning its application. In that context, focus will be placed on the three different methods of selecting elementary propositions for presentation to the N operator, identified by Wittgenstein at TLP 5.501. Special attention will be paid to explicating Wittgenstein's intended interpretation of the second and third of these methods, and to his related remark at TLP 5.32 that "all truth-functions are the results of successive applications to elementary propositions of a finite number of truth-operations." Subsequently, in Section 3 we can then on this basis look more clearly and deeply into the controversy which has developed in the literature regarding the expressive powers of N (see, e.g., Fogelin 1982, 1987, 78–85; Geach 1981, 1982; Soames 1983; McGray 2006).

As we shall see, while there are many contributors, Fogelin and Geach are the two main combatants in the debate, with

Fogelin arguing the case that Wittgenstein's N operator is expressively incomplete, and Geach setting himself against this claim. This controversy between Fogelin and Geach might seem to present us with a dilemma, in virtue of which we must either accept Fogelin's claim that the N operator is expressively incomplete, or reject certain commonly recognized, and seemingly fundamental theses which characterize the *Tractatus* (including Wittgenstein's repudiation of set theory, and his claim that all propositions are constructible *via* a limited and successive series of steps). In Section 4, however, it will be argued that despite their apparent disagreement, each of Fogelin and Geach share several problematic assumptions relevant to the interpretation of Wittgenstein's N operator. Most importantly, each seems to at least implicitly (and mistakenly) assume both that infinity is understood by Wittgenstein as potential rather than actual, and that the N operator is meant to be of practical, rather than merely philosophical import. As a consequence of these errors, Fogelin and Geach are then each further led to (mistakenly) presume that N sometimes deals with the internal structure of sentences, and that in those cases it functions more like a quantifier than a sentential operator. In Section 5, finally, it will be shown that if we do not make these mistaken and interrelated assumptions, it becomes a very straightforward matter to construct analogues of mixed multiply-general propositions, and other allegedly problematical propositions, using Wittgenstein's N operator notation. In particular, it will be shown how to do so using a method which both evades Fogelin's concerns about expressive completeness, and is also perfectly consistent with all fundamental philosophical commitments of the *Tractatus*.

2. Wittgenstein's Operator N

Wittgenstein offers the N operator, in his *Tractatus*, as a means of constructing all truth-functions *via* successive applications thereof to elementary propositions (TLP 1961, 5.476). It thus features prominently within his symbol for the general form of a truth-function (TLP 1961, 6), with which he claims to provide "the general form of a proposition." As Wittgenstein explains at TLP 5.51, when N is applied to one proposition, it is equivalent to ordinary negation. So $\mathrm{N}(p)$ is equivalent to $\sim p$. When N is applied to a pair of propositions, it is equivalent to "joint" negation. So for instance $\mathrm{N}(p, q)$ would be equivalent to $\sim p \ \& \sim q$. However, unlike the classical propositional connectives, N is defined such that it may apply to more than two propositions at once. Indeed, it can apply to an indefinite number of propositions where each is listed in indifferent order within the brackets following N. So $\mathrm{N}(p, q, r)$ would be equivalent to $((\sim p \ \& \sim q) \ \& \sim r)$ (as would $\mathrm{N}(r, q, p)$). This extension is especially useful, as we shall see, because it enables us to use N to express propositions which would otherwise require recourse to other logical operators, such as Russellian quantifiers. It thus aids Wittgenstein in his attempt to show that N may be used exclusively of any other logical operators to express all meaningful propositions.

So, for example, Wittgenstein says at TLP 5.52 that the proposition $\sim(\exists x) \, fx$ would be expressed in N notation, by taking as its arguments "all values of a function fx for all values of x." In other words, N would take as its arguments each of the substitution instances of fx, e.g., fa, fb, fc, etc., for each value of x, i.e., a, b, c, etc. The inputs, or arguments, to the operation N would thus be the various distinct outputs, or values, which result when each of these individual constants is independently substituted in for "x" in "fx". Importantly, here already it should be clear that what ultimately ends up as an argument to the N operator is not an open sentence which contains a variable, but rather a proposition which results from *replacing* a variable with an individual constant. With this in mind, we can now more closely examine the instructions which Wittgenstein provides at TLP 5.501, for stipulating the values of a variable ξ, which become arguments to the operation N.

Of the three kinds of stipulation Wittgenstein identifies, the first is the most straightforward. It is the sort of stipulation involved in the first three cases we identified above, specifically $\mathrm{N}(p)$, $\mathrm{N}(p, q)$ and $\mathrm{N}(p, q, r)$. In each of these cases the stipulation takes the form of what Wittgenstein calls "direct enumeration" (TLP 5.501). In this case, to arrive at the N-expression we want we simply substitute, for ξ within the expression $\mathrm{N}(\bar{\xi})$, "the constants that are its values" (5.501). In other words, if we want to use the N operator to express $((\sim p \ \& \sim q) \ \& \sim r)$ we simply substitute p, q, and r for ξ in $\mathrm{N}(\bar{\xi})$. The bar on top of the $\bar{\xi}$ simply tells us the N applies to every sentence in the brackets (or that ξ is to represent each of the sentences in brackets; TLP 5.501), i.e., p, q, and r. In order to stipulate the relevant propositions p, q, and r, in this case we can simply and directly enumerate them.

According to the second method, however, as opposed to directly enumerating the propositions we may give "a function fx whose values for all values of x are the propositions to be described" (TLP 5.501). Here it is important to note that what end up as arguments to the N operator are elementary propositions and that these propositions are the *values* which result from replacing the variables within a propositional function such as fx, with individual constants which are values of "x". Because these values are semantically atomic, elementary propositions, they may each in turn be assigned a distinct sentence letter prior to being placed within the brackets under the scope of the operator. We will see in more detail how the mechanics of this procedure are supposed to work in Section 5. In any case, there is thus no suggestion in TLP 5.501, that anything which itself contains a variable ends up under the scope of the N operator. Here the open sentence fx, for example, is used *prior* to the application of N to *stipulate* the propositions which will be the arguments to N. It is used to construct a list of these propositions, or rather to *describe* such a list in lieu of having to actually write it down.

In the case of the third method, as opposed to simply providing a propositional function we instead give a "formal law" which provides instructions for how to select elementary propositions which constitute a series of forms, each of the members of which would then (theoretically) be placed under the scope of the N operator. Here again Wittgenstein provides a method of constructing (or describing in lieu of writing down) a list of substitution instances of a polyadic propositional function, such as xRy. So for instance a "formal law" might generate, as substitution instances of the form xRy, each of the members of a series of forms, e.g., aRb, bRc, cRd, etc. Each of these substitution instances of xRy would be terms within a series of forms the members of which would (in theory) be placed under the operator. Such a method would be useful, for instance, for indicating how one might deploy the N operator to express general propositions dealing with ordered series or ordered relations.

There exists an unfortunate tendency within the literature (e.g., Soames 1983, 577–78) to think that what Wittgenstein provides at 5.501 is a strict protocol governing the sorts of symbols to which the N operator may be prefixed in certain cases. This is incorrect. Wittgenstein is not trying to draw a contrast between different kinds of *symbols* to place under the operator, but only between three distinct, and purely illustrative *methods* of constructing lists of elementary propositions, or of describing such lists in lieu of writing them down. The propositions on these lists would then (in theory) be assigned semantically atomic sentence letters, in turn placed under the operator. The contrast between methods 1 and 2, then, is thus *not* between putting propositions versus propositional functions under the scope of N. Instead, the contrast between each of methods 1, 2, and 3 is supposed to be between different methods of creating lists of *elementary propositions*, not containing variables, which are subsequently placed within the brackets under the scope of the N operator. This is done prior to the application of the operator, for the purposes of deploying a single, uniform method of constructing all molecular as well as general propositions out of all elementary propositions. Wittgenstein makes these contrasts in order to illustrate and help us to understand how the N operator can be used to express analogues of different sorts of propositions within the propositional and predicate calculus, including both singly- and multiply-general propositions expressed in Russell's notation with quantifiers.

That the methods indicated at 5.501 are supposed to be merely illustrative, as opposed to providing a strict symbolic protocol, is shown by the fact that none of the methods tells us how to describe the range of elementary propositions needed to use N to express multiple quantification, in case those propositions do not constitute an ordered series. This is not a lacuna in Wittgenstein's presentation so much as it is simply an invitation, much like that provided at the outset of Wittgenstein's *Philosophical Investigations* (1953/2001, x), to the reader to think and work out the solution for themselves. Likewise, the fact that Wittgenstein never directly considers the case of universal quantification, let alone mixed multiply-general quantification, constitutes an invitation for the reader to think out the required method for themselves. As we shall see in Section 3, misunderstandings of the intended method, have led Fogelin and others to consider and reflect upon the prospect that Wittgenstein's N operator might not be expressively complete.

In order to better appreciate the nature of these misunderstandings, however, it will be helpful to first talk about the methods described at TLP 5.501 in connection with Wittgenstein's remark at TLP 5.32 that "[a]ll truth-functions are the result of successive applications to elementary propositions of a finite number of truth-operations." In particular, it will be important to gain clarity about what Wittgenstein precisely means by the term "finite" in this context, because it has been the source of much controversy and confusion in the literature, and because a correct understanding of it is crucial to a proper reading of Wittgenstein's proposal. Above we noted that an important advantage of defining the N operator such that it may take an indefinite number of arguments, is that doing so allows us to use it to articulate propositions which, if expressed in Russell's notation would require recourse to additional logical operators, specifically quantifiers. It thereby allows us to extend the expressive capacity of the N operator from the propositional to the predicate calculus. But how precisely does it do this? Given that general propositions may range over an infinite domain of objects (e.g., a, b, c, etc.) and so indicate an infinite number of facts (e.g., fa, fb, fc, etc.), it would seem problematic to suppose that they could be expressed *via* any finite procedure (cf. Morris 2008, 218). In addition to raising some interpretive challenges, consideration of 5.32 thus promises to provide us with important insights with regards to how this extension is supposed to work.

First, as Wittgenstein makes clear at 5.32, the construction always begins with elementary propositions. These are the propositions, selections or lists of which 5.501 provides instructions for how to create, or describe in lieu of writing down. Different propositions will result from applying the N operator to different selections of these propositions in different ways. So for instance, $\mathrm{N}(fa, fb)$ will express a proposition which differs from $\mathrm{N}(Rab, Rbc)$, since the list of elementary propositions upon which N operates ((fa, fb) versus (Rab, Rbc)) differs in each case. On the other hand $\mathrm{N}(\mathrm{N}(fa), \mathrm{N}(fb))$ will express a distinct proposition from $\mathrm{N}(fa, fb)$, despite the fact that they each operate on the same list of elementary propositions, because the operations specified by each N-expression are distinct. Indeed $\mathrm{N}(\mathrm{N}(fa), \mathrm{N}(fb))$ is equvalent to fa & fb, whereas $\mathrm{N}(fa, fb)$ is equivalent to $\sim fa$ & $\sim fb$.

Second, 5.32 also tells us that in applying these operations to selections of elementary propositions in order to express all truth-functions, we will always be iterating the N operator a limited number of times. For example, a limited number of iterations of the N-operator will be used to express the terminal, truth-functional expansions which correspond to existential and universal quantifications. So, for instance, the universally quantified proposition $(\forall x)\, fx$, according to Wittgenstein, is equivalent to a terminal, truth-functional expansion, specifically a conjunction of each of its substitution instances fa, fb, fc, and so on. The list of these substitution instances, which Wittgenstein calls the "values" of the function fx at 5.501, can be created, or described, according to the second of the three methods. This conjunction (fa & fb & fc ... & ... and so on) may then be expressed in N notation, by applying N to each of the "values" (or substitution instances) on this list, and then in turn placing each of the resulting N-expressions under the scope of one, additional iteration of N, like so: $\mathrm{N}(\mathrm{N}(fa), \mathrm{N}(fb), \mathrm{N}(fc), \ldots)$. (As we shall see in Section 5, an additional step, which takes place prior to placing the substitution instances under an iteration of N, will be to assign a semantically atomic sentence letter to each of the elementary propositions which, in this example, appear under the scope of an iteration of N.)

As Fogelin notes (1982, 125–6, 1987, 81) and as we will explore in more detail in Section 3, unless the list of N expressions within the brackets under the scope of the main N operator comes to an end, we can never "move out" so as to apply the N which is the main logical operator (because it has the widest scope) of this sentence, and thus it would be in principle impossible (even for God) to complete the construction in a successive series of steps, without recourse to additional logical operators, such as class forming operators. There are several reasons to be critical of the introduction of class forming operators under the scope of N, which will be explored in more detail in Section 3. But for now it is enough to note that Wittgenstein refers to the general form of a proposition as "the one and only general primitive sign in logic" (TLP 5.472). Since Wittgenstein's symbol for the general form of a proposition (TLP 6) contains no logical operators besides N, it stands to reason that the construction which characterizes the general propositional form is likewise intended to involve no logical operators other than N. Indeed, N, as we shall see, is supposed to facilitate the *elimination* of logical constants, not to require their multiplication. And after all the question at hand is whether N itself, conceived exclusively of other logical operators, is expressively complete. If we are allowed to use other logical operators in the construction, why not just use Russell's quantifiers to express generality?

In order for N to express something equivalent to quantification, however, without recourse to other logical operators, the number of applications of the N operator both under the scope of the main operator and in the sentence as a whole must be limited, or the construction can never be completed. This is what Wittgenstein means and implies when he refers at TLP 5.32 to the fact that all propositions are the result of "a finite number" of truth operations. In this context, by "finite" he simply means "limited" or "restricted." Of course, if N is to be used to express something equivalent to universal quantification, this implies that the number of elementary propositions, as well as the number of objects in the domain of discourse must also be limited. Hence Wittgenstein insists at TLP 5.524 that each of both objects, and elementary propositions, form a totality. Insofar as Wittgenstein intends his operator to be used to express quantification over infinite domains, it must therefore be that he thinks such domains form limited totalities (though not totalities of "finite" cardinality).

It is sometimes instead supposed that Wittgenstein is ultimately ambivalent on the question of whether the domain of quantification is limited, or unlimited, finite or infinite (e.g., McGray 2006, 144, 168; Rogers and Wehmeier 2012, 539–40). However, Wittgenstein simply cannot be ambivalent on these questions. He *can* be ambivalent on the question of whether the empirical universe contains a finite or infinite number of complexes. In the final analysis, however, variables of quantification range neither over complexes, nor over "contingent particulars" (cf. Landini 2007, 136), but rather over metaphysically simple objects. Tractarian simple objects, however, are not empirical but are rather logical objects.[4] They constitute the substance of the world and the scaffolding of logical space. They thus cannot be finite in cardinality, since logical space is a space of possibility. In the space of possibility, infinite and possibly infinite coincide. In the framework of the *Tractatus*, possible objects must be modally real. If a language containing an infinite number of names with different meanings is possible, then there must subsist an actual infinite number of objects to ensure that possibility (even though we cannot meaningfully say so—cf. TLP 5.535). Whether a finite or infinite number of complexes *exist*, it is thus certain that an infinite number of simple, logical objects must *subsist*.

Within the semantic framework of the *Tractatus*, this feature of logical space is not "unknown" (McGray 2006, 168) so much as it is "unsayable." Hence Wittgenstein writes that it is "senseless to speak" (TLP 4.1272) of the number of objects, not that it is "impossible to know" that number. By analogy, that one occurrence of "*a*" refers to the same object as another occurrence of "*a*" is displayed, but cannot literally be said, in a logically adequate notation. But for all that, we can know perfectly well that within the Tractarian logical system, two distinct occurrences of "*a*" refer to one and the same object.

In any case, even supposing that Wittgenstein *is* ambivalent on the question of whether the domain of quantification is finite or infinite, his logical system would in that case have to be designed to accommodate both sorts of domains. That is, unless we (implausibly) assume that Wittgenstein is simply indifferent to concerns of whether his system is expressively complete or not, then, granted he is ambivalent on the question of domain size, he would still have to design that system to accommodate domains both of finite cardinality, as well as those of infinite cardinality (in case it turns out the domain of quantification is in fact infinite in size). But then each sort of domain must ultimately be limited in size, however, since otherwise using N to express universal quantification, for example, would be impossible without recourse to other logical operators, such as class forming operators (again, see Fogelin 1982, 126, 1987, 81). So regardless of whether, as I maintain, Wittgenstein is committed to an infinite domain of quantification, or, as others maintain, he is ambivalent on the issue of domain size, his N operator must be designed two allow for application to limited domains, at least some of which are infinite in size.

Rogers and Wehmeier (2012, 561–62) insist, by contrast with the above reading, that when Wittgenstein asserts at 5.32 that "all truth-functions are results of successive applications to elementary propositions of a finite number of truth operations," he alludes not to a limited or finite number of *applications* of the operator, but rather to a finite number of *operations*. They write:

> 5.32 does *not* say that all truth-functions are results of a finite number of applications of truth-operations, as Fogelin claims, but rather, merely a finite number of truth-operations. Indeed, Wittgenstein argues that they are the result of just one operation, namely N. (Rogers and Wehmeier 2012, 562)

However, it is obvious that if Wittgenstein meant to specify the number of primitive *operators* required to generate all truth-functions when he referred here to a "finite number of truth-operations," and if he were identifying the N operator as that single primitive operator, he would have referred to a *single oper-*

[4]This explains why, when Wittgenstein denies the existence of "logical objects" at TLP 5.4, he adds the caveat "in Frege's and Russell's sense." In other words, according to Wittgenstein there are no objects corresponding to logical constants such as disjunction or existential quantification, for example. This is not, however, to say that Tractarian objects are not "logical" in the distinct sense that they are the timeless and indecomposable referents of simple names such as "*a*" and "*b*".

ator, not to a "finite number" of operators. After all, *one million* is a finite number, but that is quite a few more than the one, single operator Wittgenstein actually and precisely thinks is required. So unless we wish to convict Wittgenstein of being deliberately and hopelessly vague when he speaks of a "finite number of operations" in 5.32, we must concede that he means to identify the number of *applications* of an operator required to express all truth-functions, not to the number of *operators* required to express all truth-functions.

The relevant contrast he wishes to draw at 5.32 is not between *sets of operations* (i.e., applications of the operator) of finite versus infinite cardinality, but rather between *series of successive operations* which are either limited, or unlimited. What he is telling us is that all propositions may be expressed *via* a limited series of successive applications of the N operator. Some of these propositions may correspond to quantification over a domain of infinite cardinality, while others may correspond to quantification over a domain of finite cardinality. The word "finite" can mean something different depending upon whether we are talking about the cardinality of a domain, or asking about whether a procedure is either limited or open ended (just as "Green" sometimes refers to a color and at other times to a man; TLP 3.323). In the context of TLP 5.32, "finite" is used to indicate the fact that the series of successive operations required to express all propositions, is not (because it cannot be) open ended.

In a conversation with Desmond Lee which occurred sometime in 1930–31, Wittgenstein claimed that extensionalism as espoused in the *Tractatus*, rests on two (implausible, as it turned out) assumptions: the claim that all elementary propositions are logically independent, and that infinity is a number.

> For the totality of facts determines both what is the case, and also all that is not the case. This is connected with the idea that there are elementary propositions, each describing an atomic fact, into which all propositions can be analysed. This is an erroneous idea. It arises from two sources. (1) Treating infinity as a number, and supposing that there can be an infinite number of propositions. (2) Statements expressing degrees of quality. This is red contradicts This is white. But the theory of elementary propositions would have to say that if p contradicts q, then p and q can be further analysed, to give e.g. r, s, t, and v, w, and $\sim t$. The fact is self-sufficient and autonomous. (King and Lee 1980, 119)

Wittgenstein's characterization of infinity "as a number" would seem to indicate that, at the time of authoring *Tractatus*, he thought there could be domains, such as that of propositions, which were countable in an actually infinite, and so limited number of steps. In *Tractatus* itself, moreover, Wittgenstein characterizes logical space both as an "infinite whole" (e.g., TLP 4.463) but also as a "limited whole" (e.g., TLP 6.45).[5] Reading Wittgenstein charitably, then, he maintains that all propositions can be constructed out of a limited, but nevertheless infinite number of truth-operations, performed upon a limited, but infinite number of elementary propositions containing a limited, but actually infinite number of objects.

Granted, this raises perplexing questions about how there can, for example, be infinitely many elementary propositions, many of which contain more (or several more) than one name (cf. Anscombe 1959, 137). However, this is why Russell says in his introduction, that the *Tractatus* stands in need of further tech-

[5]Similar conclusions are drawn from these and others of Wittgenstein's remarks, by Marion (1998, 34–35). However, Marion claims that Wittgenstein's treatment of generality "meant that even in cases where no enumeration is possible (an infinite domain would be such a case), it is still possible to consider the universal quantifier as a logical product." By contrast, my claim is that Wittgenstein thought that enumeration was always possible, in theory if not in practice, even in the case of infinite domains (Wittgenstein 2005, 249). This is what enabled it to be understood as a logical product. The fact that such enumeration could not be undertaken in practice was an inconsequential, purely psychological, contingency, which was in no way essential to the correct logical characterization of generality. This point pertains to the second of the four underappreciated features of Wittgenstein's logical philosophy identified above and is discussed in more detail in Section 4.

nical development, specifically with regards to transfinite arithmetic (TLP 1961, xxiii). (Notice that Russell does not specifically identify this as being the case with regards to the N operator.) It would not be surprising at all if Wittgenstein simply derived from Cantor the assumption, *via* his study of Russell, that there were several, distinct infinite totalities[6] and that some of these totalities were equivalent in size, while others differed in size. It could seem plausible to say that "infinity" was the number of each of these totalities, in the absence of a more thoroughly developed transfinite arithmetic.

In the following passage from his 1919 *Introduction to Mathematical Philosophy*, Russell makes it clear he believes that two series can each be infinite, and equivalent in size, and yet one can be placed after the end of the other to create a new series of infinite size. He writes:

> [T]he total number of inductive numbers is the same as the number of even inductive numbers . . .
>
> . . .
>
> Let us . . . consider various different kinds of series which can be made out of the inductive numbers arranged in various plans. We start with the series
>
> $$1, 2, 3, 4, \ldots, n, \ldots, \ldots$$
>
> (and) proceed to thin out this series by repeatedly performing the operation of removing to the end the first even number that occurs . . . If we imagine this process carried on as long as possible, we finally reach the series
>
> $$1, 3, 5, 7, \ldots, 2n + 1, \ldots 2, 4, 6, 8, \ldots 2n, \ldots$$
>
> in which we have first all the odd numbers and then all the even numbers. (Russell 1919, 80, 89–90)

How can one take 2 out of the inductive numbers, and put it at the end, if the inductive numbers do not have an end? As perplexing as it is, I think Wittgenstein got from Russell the (confused) idea that infinity is the number at the end of various infinitely long series. Because Cantor had proved that some of these series must be longer than others, Russell thought that there must be additional, transfinite numbers. This is why he chastises Wittgenstein for failing to properly develop transfinite arithmetic: in the scheme of the *Tractatus*, infinity is simply one number, and no mention is made of other, supplemental transfinite numbers. The *Tractatus* is in this and other respects programmatic, and it is plausible that Wittgenstein meant simply to delegate the requisite, supplemental technical development to others. Given the programmatic idea that propositions and objects should each be conceived as infinite totalities so understood, however, it is easy to see how mixed, multiply-general propositions of the predicate calculus, can be equivalent and so reducible to infinite (but nevertheless terminal) conjunctions of disjunctions, or disjunctions of conjunctions, within the propositional calculus. (The significance of this fact will become clearer when we examine the controversy over the expressive completeness of N in Section 3.) Much as an infinite series of even numbers comes after an infinite series of odd numbers in Russell's example, in the case of $(\forall x)(\exists x)\, Rxy$, for instance, the fully analyzed molecular proposition would consist of a series of infinitely long disjunctions, each of which came after one another and was conjoined to one another. The proposition would come to an end after an infinite number of such iterations.

In his lecture notes from November 1932, G. E. Moore records Wittgenstein as saying that: "In my book I supposed that [in] $(\exists x) fx = fa \vee fb \vee fc$ & *so on* [the '& so on'] was [the '& so on'] of laziness, when it wasn't" (quoted in Proops 2001, 92; cf. Stern, Rogers and and Citron 2016, 215–17).[7] Wittgenstein

[6]Such "totalities" cannot be identical to classes, however, since Wittgenstein repeatedly commits himself to the former (e.g., TLP 1.1) while he unequivocally repudiates the latter (TLP 6.031).

[7]Interestingly, in explicating Wittgenstein's intended application of the N

writes in the *Tractatus*, moreover, that "The concept of successive applications of an operation is equivalent to the concept 'and so on' " (TLP 1961, 5.2523). These two remarks together indicate that Wittgenstein thought quantifiers could be eliminated, and the propositions containing them reduced, to truth-functional expansions which could in principle (aside from laziness), be written out completely. As Wittgenstein himself later explained in the *Big Typescript*:

> My understanding of the general proposition was that $(\exists x).fx$ is a logical sum, and that although its terms weren't enumerated there, they could be enumerated (from the dictionary and the grammar of language). (Wittgenstein 2005, 249)

Wittgenstein's view in the *Tractatus*, in other words, was that existential and universal quantifiers should ultimately be understood, merely, as shorthand for truth-functional expansions, the disjuncts and conjuncts of which, though possibly infinite, were in any case limited in number and thus enumerable at least in theory. And given that the N operator may be used to express any truth-functional expansion within the propositional calculus provided that its values are enumerable and thus that the truth-functional expansion is terminal, the N operator may thus be used to express the truth-functional expansion corresponding to any quantified proposition.

Ramsey confirms this reading when, in "Mathematical Logic," he describes how on Wittgenstein's view the enumerated conjunction corresponding to universal quantification would come to an end: "if we could enumerate the values of x as $a, b \ldots z$, 'For all x, x is red' would be equivalent to the proposition 'a is red and b is red and ... and z is red' " (Ramsey 1931, 74). Here "z" goes proxy for the infinitieth yet final name used to generate a substitution instance for the purposes of constructing the truth-functional expansion corresponding to universal quantification. In Section 5 we will see in more detail how to translate from this truth-functional expansion, and others like it, into Wittgenstein's N notation.

In support of this reading, finally, it is also worthwhile to reflect on Wittgenstein's characterization of logic, at TLP 5.4541, as the study of a "closed regular structure" (or "closed regular shape" (shape = *Gebild*)) in which the answers to logical and philosophical questions are "symmetrically combined":

> The solutions of the problems of logic must be simple, since they set the standard of simplicity.
>
> Men have always had an intuition that there must be a sphere in which the answers to questions are symmetrically united—a priori—into a closed regular structure.
>
> A sphere in which the proposition, *simplex sigillum veri*, is valid. (TLP 1922, 5.4541)

For Wittgenstein, it must be recalled, logic is not ultimately the study, simply, of symbols in a book. It instead concerns the *a priori* logical form, or "logical scaffolding" (TLP 1961, 3.42, 4.023) of reality: "the propositions of logic describe the scaffolding of the world" (TLP 1961, 6.124). And an important feature of that *a priori* logical scaffolding, as Wittgenstein insists at TLP 5.4541, is that it embodies a "closed regular structure" or "closed regular shape," akin to a sphere.

Given Wittgenstein's idea that logical space should be conceived both as an "infinite whole" (TLP 1961, 4.463) but also as a "limited whole" (TLP 1961, 6.45), and given the idea that both the domain of quantification, and the number of elementary propositions constitute an actually infinite, and so limited, total-

operator, McGray insists that dots corresponding to "and so on," are "not dots of laziness" (2006, 152). But in this quotation, Wittgenstein clearly implies, in stark contrast, that they are dots of laziness. In other words, contrary to McGray, the size of the domain of Tractarian objects is not "unknown." We know, from this and other remarks, that Wittgenstein thought the domain of objects constituted an infinite totality, and thus that each member of a series of substitution instances constituting the truth-functional expansion corresponding to a quantification over an infinite domain, could in principle, if not in practice, be completely written down.

ity, it is easy to see how existential and universal quantifications can be equivalent and so reducible to infinite (but nevertheless terminal) conjunctions or disjunctions. It is then easy to see how equivalent truth-functional expansions could be constructed in N notation (we will demonstrate this formally in Section 5).

If, for instance, Wittgenstein conceived of logical space as a "completed infinity" or "infinite totality" akin to a non-Euclidean geometrical space which, though boundless, is limited, this would make it possible to describe all the positive and negative facts there are, including those expressed by the infinitely long (but nevertheless terminal) conjunctions or disjunctions of the propositional calculus, which correspond to quantification over infinite domains within the predicate calculus. In this case, no facts would be left over, and thus there would be nothing left over to say: "The world is determined by the facts, and by these being all the facts" (TLP 1922, 1.11).

3. The Controversy over Operator N

Now that we have a better appreciation of its true nature and function, we can move on to explicate the interpretive controversy over Wittgenstein's N operator, and in particular the controversy initiated by Fogelin over the expressive completeness of N relative to mixed, multiply-general propositions.

"Mixed multiply-general" propositions are simply propositions of the predicate calculus containing at least one of each of both a universal and an existential quantifier. It has been claimed by Fogelin that these propositions cannot be expressed through successive applications of the N operator, whether in any of the ways indicated by Wittgenstein at TLP 5.501, or in any way which is consistent with fundamental philosophical commitments of the *Tractatus*.

An assumption made by Fogelin in motivating this alleged problem, however, is that Wittgenstein intended the N operator to apply first to open sentences, and then subsequently to complex, quantificational structures. According to Fogelin, the following procedure would then be applied in order to construct mixed, multiply-general propositions: first, the N operator would be applied to open sentences such as Fxy, (like so: $\mathrm{N}(fxy)$) which would result in negated, doubly existentially quantified sentences (i.e., $\sim(\exists x)(\exists y)\, fxy$). One would then, in turn, apply the N operator to these doubly existentially quantified sentences, in the hopes of constructing mixed multiply-general propositions (Fogelin 1987, 78–79; Soames 1983, 575–76). Such a procedure will not, however, and as Fogelin correctly notes, result in mixed, multiply-general propositions. Repeated applications of the N operator will instead simply lead from doubly existentially quantified sentences (i.e., $(\exists x)(\exists y)\, fxy$) to their negations, and *vice versa*.

Interestingly, in the context of originally attempting to motivate these concerns about the N operator, in relation to his discussion of Wittgenstein's treatment of generality, Fogelin asks with regards to the expression $\mathrm{N}(Fx \;\&\; Gx)$ (with which the general proposition $\sim(\exists x)\,(Fx \;\&\; Gx)$ is allegedly to be expressed *via* Wittgenstein's N operator): "What shall we say about the sudden appearance of a functional sign under the operation N?" (1987, 64). What we should say about it is that although such things appear in Robert Fogelin's book about Wittgenstein, they simply do not appear at any point in the book of Wittgenstein's which he is writing about. Aside from the fact that functional signs never appear under the scope of the N operator within the pages of *Tractatus*, it is also notable that within Fogelin's version of the construction as described above, by the time we get to our second iteration of N, we are now considering how N would be applied to something which *does* contain quantifiers, but that *does not* contain N. This does not seem to be congruous with Wittgenstein's claim that all truth-functions are to be constructed *via* successive applications of N as opposed to other operators. Indeed, part of the point of using N as a sole, primitive operator, as we shall see momentarily, is to facilitate the elimination of quantifiers and other logical constants (cf. TLP 4.0312).

In his remarks on the alleged "fundamental error in the logic of the *Tractatus*," Fogelin explains that:

> It is easy enough to diagnose the . . . difficulty. When we apply the operator N to the propositions that are the values of the function fxy, both argument places under the function are handled at once in the same way, i.e., both variables are captured. So whatever kind of quantifier emerges governing one of the variables, that same kind of quantifier must emerge governing the other. (Fogelin 1987, 79)

Here Fogelin is saying, essentially, that Wittgenstein either simply does not understand, or does not appreciate, that if two distinct objectual variables occur within one formula, then it is possible for each of those variables to be bound by distinct quantifiers. The idea that Wittgenstein is guilty of such a basic misunderstanding of logical syntax, however, is exceptionally improbable. If Wittgenstein did not recognize or understand this potentiality, then why would he have designed a system of exclusive quantifiers specifically designed to avoid ambiguities associated with it? Fogelin's attempted construction seems to have gotten off track in presuming that N could be meaningfully applied either to open sentences, or quantified sentences. In Section 4 we will examine two implicit, and problematic assumptions which likely motivated this mistaken presumption.

In any case, as a means of circumventing Fogelin's concerns about the expressive completeness of Wittgenstein's N operator, Geach (1981) proposed an "enhanced" treatment of N, which involves taking N to apply to class forming operators (such as $[\ddot{x} : fx]$):

> The N operator yields joint denial of an arbitrary number of propositional arguments, from 1 upwards. Applied to one argument it yields the same result as ordinary negation; applied to two arguments it yields binary joint denial; but it may be applied to more than two arguments, and then these need not be listed, but may be given by specifying the class containing just such-and-such propositions. (Geach 1981, 168)

According to this procedure, for instance, $(\exists x)(\forall y)\, fxy$ would be expressed (1981, 169) as

$$\mathrm{N}(\mathrm{N}(\ddot{x} : (\mathrm{N}(\ddot{y} : \mathrm{N}(fxy)))))$$

Here the two N's on the left correspond to existential quantification and apply to x, while the two N's on the right correspond to universal quantification and apply to y:

> '$(\mathrm{N}(\ddot{y} : \mathrm{N}(fay)))$' says that we are to deny every such proposition as '$\mathrm{N}(fab)$', i.e., that 'fab' is always to be affirmed whatever 'b' stands for; i.e., it is equivalent to $(y)\, fay$'. And then if we replace 'a' by the variable 'x' and embed the result in '$\mathrm{N}(\mathrm{N}(\ddot{x} : (\ldots)))$', we shall be saying that the joint denial of the class of propositions we get from '$(y)\,(fay)$' by varying 'a' is itself to be denied, i.e., that one or the other member of this class is to be affirmed, i.e., that '$(\exists x)(y)\, fxy$' is to be affirmed. (Geach 1981, 169)

Through this ingenious method, we are able to use N in consort with class forming operators, to express something equivalent to the conjunction of disjunctions which corresponds to the relevant mixed, multiply-general quantification.

However, while Geach is correct to think that the N operator may be applied, simultaneously, to more than two arguments, it is *not* strictly correct, as he suggests, to think that one might apply N, ultimately, to a specified *class* of propositions. As Wittgenstein makes clear at TLP 5.52, neither an open sentence, nor a class forming operator will ultimately occur under the scope of the N operator. Instead, what ultimately occurs under the scope of the N operator are the values of $\bar{\xi}$ themselves. These values, as we have seen, are the elementary propositions which result from substituting individual constants for variables within the relevant open sentence. Subsequently, N-expressed truth-functions of those elementary propositions may occur under the scope of the operator. Since, in the basis case, these values are logically independent, elementary propositions, they can in turn be symbolized by semantically atomic sentence letters, a distinct one assigned to each elementary proposition within the brackets following the N operator. Though this procedure has already been discussed to some degree in Section 2, it shall in any case be explicated formally, and in greater detail in Section 5.

As Landini notes, Geach's appeal to a class forming operation, runs counter to Wittgenstein's "unequivocal rejection of the view that a theory of classes is part of logic" (2007, 138). Though not with any specific reference to Geach, Frascolla elaborates on the reasons for this "unequivocal rejection" as follows:

> We can expound the objection more explicitly. The reductionist programme of logicism turns out to be a translation of arithmetical axioms into propositions of the language of the theory of classes (in the type theoretical version), and to be a proof of these translations from the axioms of the latter theory. This translation is considered by Wittgenstein not only superfluous, but also harmful since it changes propositions whose general validity is essential into propositions that, if valid, are endowed merely with accidental general validity. As known, the opposition between essential and accidental general validity of logical propositions is elaborated in the *Tractatus* in overt polemic with Russell's conception of logic. (Frascolla 1994, 38)

In other words, the validity of logical and mathematical propositions cannot derive from the truth of axioms within the theory of classes, because in that case their validity would depend upon the contingent truth of those axioms, and so be "accidental" (as opposed to "essential"). A more specific example of this problem, would be that concerning Russell's reducibility axiom, about which Wittgenstein writes:

> Propositions like Russell's "axiom of reducibility" are not logical propositions, and this explains our feeling that, even if they were true, their truth could only be the result of a fortunate accident.
>
> It is possible to imagine a world in which the axiom of reducibility is not valid. It is clear, however, that logic has nothing to do with the question whether our world really is like that or not. (TLP 6.1232–6.1233)

Since the reducibility axiom is not true in all possible worlds, it can only be accidentally but not necessarily valid. Arithmetical truths thus cannot be logically derived from it, except by bartering away the necessity of such propositions.

As an alternative to the problematic, type-theoretic foundation for mathematics promoted by Russell, therefore, Wittgenstein instead accounts for mathematics and number in terms of what he calls the "general form of an operation" (TLP 6.01). The details of this proposal are not especially important for our present purposes.[8] What is important from our present perspective is that operator N is supposed, by Wittgenstein, to be an instance of the general form of an operation (TLP 6.01; Frascolla 1994, 2). Given, however, that Wittgenstein's introduction of the general form of an operation is supposed to obviate the need to appeal to philosophically problematic class-theoretical notions, it would be surprising if he then intended to reintroduce these problematic notions in consort with the deployment of N in his characterization of the general form of a truth-function. More plausibly, Wittgenstein's introduction of operations is supposed to facilitate the elimination of such appeals within the constructions both of number, as well as truth-functions. This explains why the two topics are so closely linked thematically, and discussed in such close proximity within the text (TLP 6–6.031).

Geach in fact acknowledges that his notation is *prima facie* incongruous with Wittgenstein's intentions, and is thus forced to explain this away by claiming that Wittgenstein "exaggerated"

[8] For a more detailed, systematic exposition, see Frascolla (1994, 8–23).

(1981, 179) when he said (TLP 6.031) that the theory of classes was superfluous. On my view, however, there is no exaggeration, and so no need to explain this remark away. On my view, the whole purpose behind introducing the N operator notation, is to allow Wittgenstein to execute the radical, logical eliminativism (see Landini 2007), announced as his *Grundgedanke*, or "fundamental thought" in TLP 4.0312. The N operator notation is supposed to show that all meaningful language can be built up out of repeated iterations of a single operator, which is nothing other than a truth-function of elementary propositions that are facts and also model them. This allows Wittgenstein to get by with nothing over and above an ontology of facts consisting of structured combinations of objects.

Yet it is hard to see how classes can fit in to this ontology. It is hard, in particular, to see how classes can either be identified with facts, or how facts can have classes as constituents (especially since Wittgenstein stipulates that facts only have objects as constituents; TLP 2.01). It might then seem to defeat the whole purpose of Wittgenstein's eliminativism, if his theory *required* the N operator to be applied to class forming operators. Of course, there is nothing to prevent us from applying Geach's clever notation as shorthand, just as we might apply Russell's quantifier notation, or Rogers and Wehmeier's ingenious circumflex and bar notation.[9] The point, however, is that none of these notations, any more than conjunction or disjunction, can be supposed to provide the *ultimate analysis* of the general propositional form, because they each contain supplemental operators and/or logical constants, which Wittgenstein desires, ultimately, to eliminate for the purposes of his logical philosophy.

This eliminativist agenda explains why, when Wittgenstein depicts truth-tables at 4.442, 5.5 and elsewhere, they do not have logical constants in them. The logical constants are (truth-functional) operations (5.2341), and operations are mere symbols (cf. Soames 1983, 582), which, unlike functions proper, play no essential role in characterizing the sense of a proposition (TLP 5.25). The whole *point* of characterizing the logical constants *as* operations is so that they may be ontologically eliminated, and the N operator facilitates that process by reducing all other truth-operations to one.

Now, while Fogelin concedes that mixed multiply-general propositions can indeed be constructed using Geach's enhanced notation, he nevertheless insists that this notation is incongruous with other fundamental tenets of the *Tractatus*. In particular, Fogelin focuses on an incongruity with Tractarian commitments to the finiteness and successiveness of truth-operations (1982, 126). These concerns of Fogelin's are connected in important ways to Geach's use of class forming operators, and to the question, discussed above, of whether such operators should be understood merely as convenient shorthand, or instead as ineliminable elements of a Wittgensteinian notation.

By "successiveness" (as we shall see in more detail in Section 5), Wittgenstein means simply that within the N-expression of any proposition, N-expressions will typically be nested within one another, and those with the narrowest scope will apply to elementary propositions. In constructing the full N-expression we are interested in, we will thus proceed, successively, from those applications of the N operator which have the narrowest scope, to those which have the widest, until we are done N-expressing the proposition we want.

[9] Rogers and Wehmeier note that, if we were to construe Geach's notation as "shorthand," or as deploying "abbreviations" (2012, 562) for enumerated lists, "then the first and second methods of description of 5.501 would actually be identical." This is quite true, and explains why, in his introduction, Russell characterized Wittgenstein's method of constructing general propositions, described at 5.501, as being "exactly" the same as that used in case N is applied to enumerated lists (TLP 1961, xvi). It also explains why Wittgenstein insists that the particular method used to describe the bracketed propositions, is inessential (TLP 5.501). Ultimately, despite inessential differences of detail, the very same procedure is used to construct all propositions, and that is why this procedure provides the general propositional form.

Fogelin seems not to have any issue with *this* aspect of "successiveness," however. On his view, the problems for successiveness only come about in connection with another alleged commitment of Wittgenstein's identified at 5.32, which is that of "finiteness." As Fogelin explains:

> If the set of base propositions is infinite, then nothing will count as the immediate predecessor of the final application of the operation N in the construction of a universally quantified proposition. (Fogelin 1987, 81)

In other words, if we try to construct an N-expression according to the above procedure, it may be that we will never be able to successively "move out" to the iterations of the N operator which have wider scope in the expression, because series of required, narrower scope iterations of the N operator may be infinite and so unending. And this, according to Fogelin, explains why at TLP 5.32 Wittgenstein insists that all truth-functions must be expressible *via* a finite number of operations.

Fogelin claims that this same basic problem applies to Geach's notation. This is because Geach's notation must ultimately be understood merely as shorthand for a much lengthier, and potentially unlimited procedure of successive applications of N. With regards to Geach's expression for universal quantification ($\mathrm{N}(\ddot{x} : \mathrm{N}(fx))$), for example, Fogelin explains:

> The surface grammar of this expression may suggest only two successive applications of the operator N to an initial set of propositions, but that, of course, is false. For Geach the expression '$(\ddot{x} : \mathrm{N}(fx))$' specifies a set of propositions that is the result of possibly infinitely many applications of the operator to a set of propositions. (Fogelin 1982, 125)

So we seem to have an arrived at an impasse. We can escape Fogelin's concerns about the expressive completeness of N by deploying Geach's method of applying N to class forming operators. However, this method would seem to be incongruous with important aspects of Tractarian doctrine, such as Wittgenstein's repudiation of classes at TLP 6.031. On the other hand, if we interpret Geach's notation as shorthand for a procedure which is consistent with Wittgenstein's repudiation of classes, we run into the problem of "successiveness" identified by Fogelin.[10] That is, if N is supposed to operate upon an infinite list of propositions, where infinity is understood as potential and so unlimited, we will be unable to construct the proposition we want in a limited number of successive steps. Thankfully, in Section 5 I will provide an alternative method of construction, which both evades Fogelin's concerns about expressive completeness, but is also completely consistent with Tractarian doctrine. First, however, it will be helpful to more closely examine as well as correct several problematic assumptions, and associated expository errors, that led us to this apparent deadlock.

4. Problematic Assumptions

Despite their apparent disagreement, both Fogelin and Geach seem at least implicitly to share four problematic and interrelated assumptions. By rejecting these four assumptions, we can avoid being led into the impasse, and dilemma, which their debate might seem to have created for us and which we arrived at in Section 3.

The first assumption we must reject is to think that at the time of authoring *Tractatus*, Wittgenstein construed infinity as poten-

[10] It is important to note that Fogelin himself seems not to endorse Wittgenstein's repudiation of classes: "I have no personal reservations about forming sets . . . in just the way that Geach proposes" (1982, 126). However, it is unclear how his critique of Geach will have any force unless he at the very least accepts that Wittgenstein would repudiate a set theory of the sort that Geach describes. For if it is permissible to form infinite classes in the way that Geach intends, then Geach's notation for mixed, multiply-general propositions, for example, need not be conceived as a mere shorthand, and consequently, the demands of finiteness and successiveness are met.

tial, rather than actual (cf. Marion 1998, 34). Under that assumption, it would seem obvious, as it apparently did to both Fogelin and Geach, that Wittgenstein would need to deploy some device, for instance a class forming operator, or an open sentence, so as to quantify over an infinite domain. This is because, given the assumption that infinity is potential, it would be impossible, even in theory, to write out an entire, truth-functional expansion over such a domain. At the time of authoring *Tractatus*, however, Wittgenstein thought it was *not* in principle impossible to write out the entire truth-functional expansion corresponding to quantification over an infinite domain. Instead he conceived of logical space as an infinite whole (TLP 4.463), comprising a *totality* of (negative and positive) facts (TLP 1.11–1.12), any list of which must therefore come to an *end*. Consequently, and as Methven notes

> Wittgenstein does not seem to have thought that there was any essential difference between the finite and the infinite case of conjunction beyond the fact that, in the infinite case, the propositions to which his N-operator is to be applied will be specified by a propositional function rather than by enumeration. (Methven 2015, 210)

> Wittgenstein is quite clear that how one specifies the elementary propositions upon which such operations are to take place is not essential. (Methven 2015, 209)

Wittgenstein did not think that there was any essential difference between the finite and infinite case, because he thought that infinity was actual, as opposed to potential. He thus thought that the terms of an infinitely long series were just as enumerable, in theory, as were those of any finitely long series. Of course, infinitely long series of propositions would be practically impossible to enumerate; but then the same thing would be true of many very large, but finitely long series of propositions. Thus there would be an obvious temptation, in both cases, to simply use an open sentence, or propositional function, to describe the propositions we are talking about. But, from a logical point of view, it would not be absolutely essential that we do.

This observation leads directly to the second integral, if underappreciated feature of Wittgenstein's system, which is that he *only* intends to recommend the N operator on the grounds of its potential to yield philosophical insight, and not in virtue of any practical, technical advantages it enjoys over other notations. Absent that assumption, it might again seem to be required that N be used in consort with a set forming operator, open sentence, or some other quantificational device, in order to render its deployment practicable. But the N operator is simply not intended as any sort of practical, notational recommendation, and that explains why Wittgenstein never introduced any sort of thoroughly worked out technical proposal according to which it might be so deployed.[11] After all, Wittgenstein offers his book as a *treatise on logical philosophy*, not a *logic textbook* (1961, 3). It is thus enough to say that the list of elementary propositions required to express a general proposition *via* successive applications of N, may simply be stipulated to be the values of

[11]It may be uncommon in the literature on this topic, for commentators to *explicitly* characterize the N operator as if it were adopted and advanced by Wittgenstein in virtue of its practical, technical, advantages over other notations. But this assumption nevertheless seems to be *implicit* in the lengths to which commentators have gone to flesh out the technical details of Wittgenstein's proposal, in an effort to determine whether it could practically be implemented to advantage within a logical system. Fogelin goes so far as to insist that: "I do not think we grasp the full intent of the *Tractatus* unless we see that one of its pretensions—perhaps its chief pretension—was to serve as a replacement for *Principia Mathematica*" (1987, 72). If the reading offered in this paper is correct, such pretensions were far from Wittgenstein's mind when devising the N operator notation. If anything, the *Tractatus* is supposed to obviate the need for a book like *Principia*, not to replace *Principia*. The N operator is thus not designed to figure prominently within a practicable logical system, and Wittgenstein is not a logicist, keen on deploying the system to prove mathematical or other theorems. On Wittgenstein's view, mathematics is a "method of logic" (TLP 6.234), not derived from logic. The N operator, moreover, is introduced not to facilitate a logicist program, but instead to display the essence of meaning and representation, for the purposes of a philosophy of language and logic.

a particular propositional function. Moreover, Wittgenstein is clear that this is only one possible method of stipulating the relevant propositions and is not in any way essential. The advantage of this method is simply that, as opposed to being required to list each of the relevant elementary propositions (which would be possible in principle but not in practice), we may instead use the propositional function, or open sentence, as a means to *describe* the list in lieu of writing it down. We do not then need to actually *place* the open sentence corresponding to the propositional function under the scope of the operator, and, moreover, to do so would violate logical grammar. And this explains why Wittgenstein never does place an open sentence under the scope of N within the pages of TLP.

The N operator is simply not designed to be used for practical purposes, but instead *only* for the purposes of logical philosophy. For Wittgenstein's purposes, it is enough that a sufficiently powerful God, or supercomputer could use the N operator to express all meaningful propositions; it does not even have to be humanly possible. After all, an expressively complete notation cannot be bound by the psychological limitations and physical frailties of human beings, for these are purely contingent, and imperfect features of the world. To think that it is so bound, would be to confuse the psychological, with the logical (see TLP 4.1121). As Ramsey explains:

> Mr. Wittgenstein holds that all propositions express agreement and disagreement with truth-possibilities of atomic propositions . . . although often the atomic propositions in question are not enumerated, but determined as all values of a certain propositional function . . . It is clear, of course, that the state of mind of a man using the one expression differs in several respects from that of a man using the other, but what might be called the logical meaning of the statement, the fact which is asserted to be, is the same in the two cases. (Ramsey 1931, 74)

The "logical meaning" of a general statement is the same whether we use quantification in consort with a propositional function, or enumeration of the sort which characterizes a truth-functional expansion. The use of a propositional function is thus inessential, from a logical point of view (if not from a psychological point of view). Given the possibility, in theory if not in practice, of deploying the N operator to express all meaningful propositions, it is supposed to display the general and essential form both of truth-functions and, given extensionality, of propositions. We do not actually have to be able to *use* the N operator in practice, in order to deploy it to display these philosophically interesting features of logic and language.

There exists an analogy, here, with Wittgenstein's use of exclusive quantifiers so as to eliminate the identity predicate. The point of that analysis is *not*, ultimately, to recommend replacing the identity sign with exclusive quantifiers due to any demonstrable "technical advantages" (Hintikka 1956, 228) thereof (cf. Ramsey 1931, 31–32). The point is instead simply to show that the identity sign "is not an essential constituent of conceptual notation" (TLP 1961, 5.533). In other words, Wittgenstein's point is that the identity sign *can* be eliminated from conceptual notation, not that, for practical purposes, it *should*. From the point of view of logical philosophy, the relevant lesson which follows from the eliminability of the identity sign is not that we should not use it in logical practice, but simply that it cannot be part of the *ultimate* philosophical analysis of that practice. Russell explains this point quite nicely in his introduction:

> Mr. Wittgenstein accordingly banishes identity and adopts the convention that different letters are to mean different things. In practice, identity is needed as between a name and a description or between two descriptions. It is needed for such propositions as "Socrates is the philosopher who drank the hemlock", or "The even prime is the next number after 1." For such uses of identity it is easy to provide on Wittgenstein's system. (TLP 1961, xix)

In other words, use of the identity sign for practical purposes is perfectly compatible with Wittgenstein's *logical system*. But only on the condition that it can ultimately be eliminated for the purposes of a *logical philosophy*, the "fundamental thought" of which is that logical constants, like the identity sign, do not stand for or represent anything (TLP 1961, 4.0312).

The third, and related feature of Wittgenstein's logical philosophy that goes underappreciated by each of both Fogelin and Geach is simply the fact that according to it, elementary propositions have structure but are nevertheless logically independent. This means that once an open sentence Fxy, for example, is used to create (or describe) a list of elementary propositions with a certain structure, that structure subsequently becomes irrelevant within the truth-functional expansion of any mixed or unmixed, multiply- or singly-general proposition. Since elementary propositions are logically independent, they can safely be assigned semantically atomic sentence letters which lack any internal structure. Regardless of any internal structure or content, no other elementary proposition can possibly be inconsistent with any elementary proposition, and no elementary proposition can follow from any elementary proposition. This is why Wittgenstein's symbol for the general form of a proposition, $[\bar{p}, \bar{\xi}, \mathrm{N}(\bar{\xi})]$, contains the sentence letter meta-variable "p" to range over all elementary propositions at the base level of the operation, and not for instance a variable set forming operator, or an "open sentence" variable.

Once the intended list of structured, elementary propositions is stipulated (or described) *via* the deployment of an open sentence in accordance, for example, with the second or third of the methods specified in TLP 5.501,[12] and once semantically atomic sentence letters are subsequently assigned to these propositions, we can essentially forget about the internal structure of the elementary propositions. The construction of the truth-functional expansion corresponding to any mixed or unmixed, multiply- or singly-general proposition, then proceeds just as if that proposition were one of the propositional rather than the predicate calculus. In fact, Wittgenstein's view is that general propositions, and molecular propositions of the propositional calculus, are each ultimately reducible to a single, truth-functional calculus deploying N as its lone, truth-functional operator. The second and third methods identified at 5.501, are each designed to specify selections of elementary propositions with structure for the purposes of assigning them sentence letters which lack structure. The N operator can then be applied to these sentence letters as per the general form of a truth-function identified at TLP 6.

We may then express equivalents both of general propositions, but also of molecular propositions of the propositional calculus, *via* a single, truth-functional operator and in accordance with a uniform procedure. This is why Russell insists in his introduction (TLP 1961, xvi), that the construction of general propositions proceeds in *exactly* the same way as it would in the case of operations upon enumerated lists of sentence letters: because it too ultimately reduces to operations upon enumerable (if not in practice enumerated) lists of semantically atomic sentence letters, and subsequently on truth-functions thereof. Hence, in the general form of a proposition, we begin with all sentence letters (symbolized by $\bar{p}$), take a selection of those (symbolized by $\bar{\xi}$), and then apply N to them (like so: $\mathrm{N}(\bar{\xi})$), to arrive at N-expressed truth-functions of the elementary propositions symbolized by the selected sentence letters.

[12] Soames calls the third of the three methods identified at TLP 5.501 "mysterious" (Soames 1983, 578). However, there is nothing especially mysteriously about it; like method 2 it is simply a device deployed to describe collections of elementary propositions to which one can apply the N operator so as to express targeted propositions. It differs from the second method by focusing on series of forms, rather than simply on monadic forms. So method two would collect sets of elementary propositions along the lines of, e.g., Fa, Fb, Fc, etc., whereas method three would collect elementary propositions along the lines of, e.g., aRb, bRc, cRd, etc., The third method could be useful in formulating general propositions about geometrically ordered relations, for instance.

These observations lead directly to a fourth and final, interrelated and underappreciated feature of the Tractarian logical system, which is the sentential character of the N operator, and more specifically Wittgenstein's insistence that N applies only to elementary propositions and their (N-expressed) truth-functions. It is common within the scholarly literature, by contrast, to treat the N operator as being akin to a quantifier (in addition to Fogelin and Geach, see Rogers and Wehmeier 2012, 564; Soames 1983, 576) and thus as interacting with the internal structure of atomic propositions, represented *via* objectual variables. Such a treatment is presumably thought to be required so as to extend the applicability of N from that of the propositional to the predicate calculus. A problematic consequence of so treating the N operator, as we have seen, is that N might then seem to be susceptible to expressive limitations in case multiple, and distinct objectual variables occur within its scope (Fogelin 1987, 79; Soames 1983, 575–76). Failing to recognize these expressive limitations is the blunder Wittgenstein is alleged, by Fogelin, to have committed.

On my reading, by contrast, the N operator neither binds objectual variables, and nor do objectual variables ever occur within its scope (see Jacquette 2001, 194, 196). This enables Wittgenstein to evade concerns of the sort identified by Fogelin, wherein the expressive completeness of N is undermined in cases of multiple variation. On my reading, N should be understood as a sentential operator, which operates successively first on selections of elementary propositions, and then, in turn, on their truth-functions. Constructing a proposition in N notation (or "N-expressing" a proposition) is thus a matter of first applying the N operator to the relevant selection of elementary propositions, and in turn applying the N operator to these "N-expressions," and so on, until the desired expression is achieved. The "successiveness" of the N operator identified by Wittgenstein at TLP 5.32, is thus a matter, simply, of undertaking repeated iterations of the N operator from those with the narrowest to the widest scope in the relevant expression. The N operator thus paradigmatically operates upon semantically atomic sentence letters of the sort ranged over by sentence letter meta-variables "p" and "q," for instance, as well as (N-expressed) truth-functions thereof.

Having cleared up this and various other interrelated misunderstandings, we can now move on to demonstrate formally, in Section 5, the correct method by which N was intended by Wittgenstein to be deployed so as to construct various truth-functions, including those subsequently controversial truth-functions corresponding to mixed multiply-general quantification.

5. Constructing Mixed Multiply-General (and Other) Propositions

With the Fogelin-Geach controversy behind us, it is time to articulate the method by which N may be used to construct mixed multiply-general propositions, and other supposedly problematical propositions. Crucially, this method both evades Fogelin's concerns about expressive completeness, and is also perfectly consistent with all fundamental philosophical tenets of the *Tractatus*. In light of these features, and in virtue of the issues discussed above, it is likely that Wittgenstein envisioned the construction of N-expressions as proceeding more or less along the lines of the following proposal. (Though there seems to be no reason why he would have rejected Geach's clever class notation, or the ingenious circumflex and bar notation due to Rogers and Wehmeier, as convenient shorthand for the procedure described in this section.)

Integral to this proposal is Wittgenstein's characterization of quantified sentences, as containing "logical prototypes" (TLP 1961, 3.24, 5.522) which can be deployed as a means of describing the lists of elementary propositions to which one would apply the N operator, in order to construct the truth-function corresponding to a particular, quantified sentence. For instance, and as Wittgenstein notes at 5.52, "$\sim(\exists x)\,Fx$" would be expressed by placing all of the elementary propositions which are values of the propositional function Fx under the scope of the N operator. (To say that these are "values of the propositional function Fx," is to *describe* the relevant list of propositions, in lieu of writing the whole list down.) Because these are elementary propositions, moreover, they can be replaced by sentence letters, a different one for each distinct substitution instance of the propositional function.

For example, where we have:

$$\begin{aligned} p &: Fa \\ q &: Fb \\ r &: Fc \\ &\vdots \\ p^i &: Fi \end{aligned}$$

Then "$\sim(\exists x)\,Fx$" would be expressed using N as:

$$\mathrm{N}(p, q, r, \ldots, p^i)$$

Given the assignment of simple sentence letters to each of the infinitely many substitution instances of the logical prototype "Fx" contained within "$\sim(\exists x)\,Fx$", the above N-expression simply negates each of these substitution instances, and in turn conjoins each of these negations. The above N-expression is thus equivalent to the following, infinitely long, truth-functional expansion of "$\sim(\exists x)\,Fx$":

$$\sim Fa \;\&\; \sim Fb \;\&\; \sim Fc \;\&\; \ldots \;\&\; \sim Fi$$

Within the above construction and subsequent constructions, the letter "i" is supposed to stand for the "infinitieth" (and thus final) iteration of whatever it applies to, be it the infinitieth object in the domain, or the infinitieth proposition in a series. Greater clarity could perhaps be achieved *via* the use of transfinite numerals. However, as discussed above, Wittgenstein himself did not deploy such numerals, nor the distinctions that they embody, within his characterization of the infinite in the *Tractatus*. To be faithful to the text, therefore, I have simply supposed the existence of a number, "infinity," which lies at the terminus of various infinitely long series, and have abstracted from the possibility that those series may differ in length or cardinality.

Turning, now, to the construction of a mixed multiply-general proposition, where we have:

$$\begin{aligned} p &: Faa \\ q &: Fab \\ r &: Fac \\ &\vdots \\ p^i &: Fai \\ p^1 &: Fba \\ q^1 &: Fbb \\ r^1 &: Fbc \\ &\vdots \\ p^{1i} &: Fbi \\ p^2 &: Fca \\ q^2 &: Fcb \\ r^2 &: Fcc \\ &\vdots \end{aligned}$$

$$p^{2i} : Fci$$

$$\vdots$$

$$p^{i1} : Fia$$

$$\vdots$$

$$p^{i,i} : Fii$$

Then "$(\exists x)(\forall y)\, Fxy$" may be expressed using the N operator as:

$$\mathrm{N}(\mathrm{N}(\mathrm{N}(\mathrm{N}p, \mathrm{N}q, \mathrm{N}r, \ldots, \mathrm{N}p^{i}), \mathrm{N}(\mathrm{N}p^{1}, \mathrm{N}q^{1}, \mathrm{N}r^{1}, \ldots, \mathrm{N}p^{1i}),$$
$$\mathrm{N}(\mathrm{N}p^{2}, \mathrm{N}q^{2}, \mathrm{N}r^{2}, \ldots, \mathrm{N}p^{2i}), \ldots, \mathrm{N}(\mathrm{N}p^{i1}, \ldots, \mathrm{N}p^{i,i})))$$

"$(\exists x)(\forall y)\, Fxy$" is thus equivalent to an infinite disjunction of infinite conjunctions of elementary propositions, each of which are substitution instances of the "prototype" contained in the original, quantified sentence, and to each of which in virtue of logical independence we may assign a semantically atomic sentence letter. The above N expression is thus equivalent to the following, infinite, truth-functional expansion:

$$(Faa \,\&\, Fab \,\&\, Fac \,\&\, \ldots \,\&\, Fai) \vee (Fba \,\&\, Fbb \,\&\, Fbc \,\&\, \ldots \,\&\, Fbi)$$
$$\vee\, (Fca \,\&\, Fcb \,\&\, Fcc \,\&\, \ldots \,\&\, Fci) \vee \ldots \vee (Fia \,\&\, \ldots \,\&\, Fii)$$

Of course, nothing would prevent one from leaving a given substitution instance in the *form Fab*, for example, but that could potentially obscure how it subsequently *functions*, which is to say, precisely as a semantically atomic sentence letter does: "In order to recognize a symbol by its sign we must observe how it is used with a sense" (TLP 1961, 3.326).

The method of specification integral to the construction of mixed multiply-general propositions such as $(\exists x)(\forall y)\, Fxy$, is thus akin to the second of the three identified at TLP 5.501, adapted, simply, so as to describe a list of substitution instances of a two place propositional function. This point is, actually, nicely made by Russell in the summary of Wittgenstein's procedure given in his introduction (TLP 1961, xiv–xvii), and this makes it somewhat ironic that Fogelin (1987, 63) quotes Russell with approval when the latter writes:

> Wittgenstein's method of dealing with general propositions [i.e., $(x).\, fx$ and $(\exists x).\, fx$] differs from previous methods by the fact that the generality comes only in specifying the sets of propositions concerned, and when this has been done the building up of truth-functions proceeds exactly as it would in the case of a finite number of enumerated arguments $p, q, r, \ldots$. (TLP 1961, xvi)

What Russell is saying here, in stark opposition to Fogelin, is that the intended construction of the predicate calculus is in effect the very same, indeed *exactly* the same construction as that which characterizes the propositional calculus. After open sentences are used merely *to describe* the relevant list of elementary propositions, the construction proceeds *exactly* as it would in the case of dealing with a finite number of enumerated sentence letters, because *it too would operate on terminal lists of sentence letters* such as p, q, and r, each assigned to a unique elementary proposition (e.g., Fab, Fac, Fad, etc.) (Note that, as we saw above, Ramsey also characterized Wittgenstein's truth-functional expansions as involving terminal lists of propositions; see Ramsey 1931, 74.) This explains why Russell goes on in the text that immediately follows, to explicate Wittgenstein's symbol for the general form of a proposition as being one which contains a *sentence letter meta-variable p* which "stands for all atomic propositions" (TLP 1961, xvii).

The fact that the N operator applies only to semantically atomic, unstructured sentence letters, and subsequently to their truth-functions, also explains why Russell (and Wittgenstein) are so careful in setting up their explication of the relevant construction, to note the integral feature of Tractarian semantics that "[a]ll atomic propositions are logically independent of each other" (TLP 1961, xv). The independence thesis accounts for, among other things, why we can essentially forget about the internal structure of elementary propositions when proceeding with the construction of (analogues of) general propositions, including mixed multiply-general propositions. Such constructions proceed in exactly the same way as those involving a finite list of enumerated sentence letters, because they too operate on terminal lists of sentence letters and their truth-functions, not on open sentences, class forming operators, or quantified propositions. The sentence letter meta-variable "p" contained in the general form of a proposition, cannot range over open sentences, or set forming operators, because neither are propositions; and it cannot range over quantified propositions because they are not yet fully analyzed and thus contain logical complexity which would interfere with the operation (as Fogelin's failed attempt at a construction perhaps shows).

Now that we have a clearer appreciation of how the construction is supposed to work, it is easy to see how other supposedly problematical propositions would be constructed using Wittgenstein's N operator. Soames (1983, 577), for example, sees the concerns about expressive completeness identified by Fogelin as considerably more general, and as applying to the following two cases among others:

(1) $(\exists x) {\sim} Px$
(2) $(\exists x)(Px \;\&\; Qx)$

The alleged problem with constructing such propositions emerges on the assumption that the construction would proceed by applying N to the open sentences "Px" and "$Px \;\&\; Qx$". Given that assumption, as Soames insists, the N operator may be used to construct only those propositions "that can be formulated in the predicate calculus by sentences whose only quantifiers are blocks of (one or more) existential quantifiers prefixed to atomic formulas" (1983, 577).

Given the procedure outlined above, however, it becomes very straightforward to see how the N operator could be used to construct the propositions (1) and (2), without our problematically being required to apply the N operator to the specified open sentences (i.e., "atomic formulas"). We are thereby easily able to evade the (apparent) roadblock identified by Soames. (1) is equivalent to a disjunction of negations. Thus, where we have:

$$\begin{gathered} p : Pa \\ q : Pb \\ r : Pc \\ \vdots \\ p^i : Pi \end{gathered}$$

(1) may be expressed in N notation as follows:

$$\mathrm{N}(\mathrm{N}(\mathrm{N}(p), \mathrm{N}(q), \mathrm{N}(r), \ldots, \mathrm{N}(p^i)))$$

To N-express this proposition, we start by applying the N's which have the smallest scope, those which apply to the elementary propositions p, q, r, and so on; these give us the negations of Pa, Pb, Pc, etc. Then we apply, successively, two iterations of the N operator, the second with wider scope than the first, in order to disjoin these negations. The above N expression is thus equivalent to the following, infinite, truth-functional expansion:

$${\sim}Pa \vee {\sim}Pb \vee {\sim}Pc \vee \ldots \vee {\sim}Pi$$

By contrast, (2) is equivalent to a disjunction of conjunctions. Thus, where we have:

$$\begin{gathered} p : Pa \\ q : Pb \\ r : Pc \\ \vdots \end{gathered}$$

$$p^i : Pi$$
$$p^1 : Qa$$
$$q^1 : Qb$$
$$r^1 : Qc$$
$$\vdots$$
$$p^{1i} : Qi$$

(2) may be expressed using the N operator as follows:

$$\mathrm{N}(\mathrm{N}(\mathrm{N}(\mathrm{N}(p), \mathrm{N}(p^1)), \mathrm{N}(\mathrm{N}(q), \mathrm{N}(q^1)), \mathrm{N}(\mathrm{N}(r), \mathrm{N}(r^1)), \ldots, \mathrm{N}(\mathrm{N}(p^i), \mathrm{N}(p^{1i}))))$$

Here, again, we start by applying the N's with the smallest scope to the relevant elementary propositions, in this case in pairs $(p, p^1), (q, q^1)$, and so on, and then apply an additional "N" with wider scope, to each pair. This gives us a series of conjunctions. To the whole series, we then apply two iterations of the N operator, each with increasing scope, in order to disjoin these conjunctions. The above N expression is thus equivalent to the following, infinite, truth-functional expansion:

$$(Pa \;\&\; Qa) \vee (Pb \;\&\; Qb) \vee \ldots \vee (Pi \;\&\; Qi)$$

While thus far we have been assuming inclusive quantification, it is easy to see how to adapt the above procedure so as to construct equivalents of propositions expressed using Wittgenstein's exclusive quantifiers. Take the following example (TLP 5.532):

$$(\exists x)(\exists y)\, Fxy$$

As per Wittgenstein, this Russellian expression deploying inclusive quantifiers may then be translated into Wittgenstein's exclusive quantifier notation as follows:

$$(\exists x, y).F(x, y) \,.\vee.\, (\exists x).F(x, x)$$

This proposition is a disjunction of disjunctions (and is thus equivalent to one, lengthy disjunction). Where we have:

$$p : Fab$$
$$q : Fac$$
$$r : Fad$$
$$\vdots$$
$$p^i : Fai$$
$$\vdots$$
$$p^1 : Faa$$
$$q^1 : Fbb$$
$$r^1 : Fcc$$
$$\vdots$$
$$p^{1i} : Fii$$

Then this proposition may be expressed in N notation as follows:

$$\mathrm{N}(\mathrm{N}(p, q, r, \ldots, p^i, \ldots, p^1, q^1, r^1, \ldots, p^{1i}))$$

The above N expression is equivalent to the following, infinite, truth-functional expansion:

$$Fab \vee Fac \vee Fad \vee \ldots \vee Fba \vee Fbc \vee Fbd \vee \ldots \vee Faa \vee Fbb \vee Fcc \vee \ldots \vee Fii$$

Wittgenstein did not bother to spell all of this out, because as Russell notes in his introduction (TLP xv–xvii), he relied on Scheffer's proof "that all truth-functions can be obtained out of simultaneous negations" (xvii). The extension of this method to the predicate calculus is achieved, not by applying N to open sentences, but by allowing it to operate upon lists of arguments, containing an indefinite number of arguments, and possibly even an infinite number of arguments (i.e., sentence letters) occurring under its scope. This method of construction works perfectly well, in part, because it only ever takes as arguments elementary propositions, and N-expressed truth-functions of those elementary propositions. So we never encounter the roadblocks identified by Fogelin and Soames which emerge from applying the N operator to open sentences (i.e., "atomic formulas"). The role played by open sentences is simply that they are *one means* of stipulating the entries on a list of propositions required to N-express the truth-functional expansions which correspond to existential and universal quanitifiation. They are especially useful as a means of *describing* the entries on the list, in lieu of having to write the whole list down, which would be practically impossible.

Notice that, as described by Russell in the quotation from his introduction above, the "specifying" takes place prior to the "building up," or "construction," of truth-functions. Note, moreover, that as described by Russell in the quotation above, the construction involves applying N to sentence letters, not to open sentences, or quantificational structures. By the time we get to the point of using the N operator to express propositions, open sentences and/or quantifiers have already been eliminated in favour of a list of elementary propositions corresponding to the prototype they contain. These elementary propositions have subsequently been assigned semantically atomic sentence letters, and are now ready to have the N operator (straightforwardly) applied to them. Remember, Wittgenstein's is a "logical atomist" project. In that context, we use analysis to show what the basic constituents of logical space are, and then we show how facts and propositions can be constructed out of them. We do not start the construction until after we have completed the analysis.

The above procedure can be made even more perspicuous and compelling, finally, if we consider James McGray's attempt (2006, 154–55) to demonstrate the equivalence of Geach's N notation to Russell's quantifier notation for mixed, multiply-general propositions. In twelve steps McGray claims to show the equivalence between Geach's $\mathrm{N}(\mathrm{N}(\ddot{x} : \mathrm{N}(\ddot{y} : \mathrm{N}(Lxy))))$ and Russell's $(\exists x)(\forall y)\, Lxy$. McGray's presentation moves from the former to the latter, but really what we are interested in, and what McGray's presentation also shows, is how to get from Russell's quantifier notation to an equivalent expression in N notation. Taken in reverse order, we have:

12) $(\exists x)(\forall y)\, Lxy$	Russell's mixed multiply-general proposition
11) $(\forall y)\, Lay \vee (\forall y)\, Lby \vee (\forall y)\, Lcy \ldots$	existential expansion of the logical sum
10) $(Laa \;\&\; Lab \;\&\; Lac \;\&\; \ldots) \vee (Lba \;\&\; Lbb \;\&\; Lbc \;\&\; \ldots) \vee (Lca \;\&\; Lcb \;\&\; Lcc \;\&\; \ldots) \vee \ldots$	universal expansion of each disjunct

By the time we reach our third step (which is McGray's step 10), we already have something expressed in the propositional calculus. Assuming Laa, Lab, Lbc, Lcc, and so on, are elementary propositions, we can simply assign them each a distinct sentence letter, and N can easily be applied to them, and to N-expressed truth-functions of them, so as to complete the transition from Russell's to Wittgenstein's notation. We showed how to do that for the analogous expression, $(\exists x)(\forall y)\, Fxy$, above. The point is, we do not require Geach's notation, which would be arrived at by reversing the nine additional steps of McGray's proof, in order to move from something expressed in Russell's quantifier notation, to Wittgenstein's N notation. Geach's notation may be useful for illustrative purposes, or for shorthand. But granted the idea that infinity is actual, rather than merely potential, it is in principle eliminable in favour of a notation and procedure of the sort outlined above.

In summary, in consort with Sheffer's proof that all truth-functions of propositions of the propositional calculus can be obtained *via* joint negation, and with Wittgenstein's claim that all general propositions can be reduced to truth-functional expansions of the propositional calculus (a reduction which Russell refers to as Wittgenstein's "derivation of general propositions from conjunctions and disjunctions"—TLP 1961, xvii), the "uniform method of construction" (TLP 1961, xvii), embodied in the general form of a truth-function entails that quantified propositions in general, and mixed multiply-general propositions in particular, are adequately expressible *via* successive applications of Wittgenstein's N operator to selections of semantically atomic sentence letters assigned to elementary propositions.

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