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The Propositional Logic of Frege's *Grundgesetze*: Semantics and Expressiveness

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In this paper we compare the propositional logic of Frege's *Grundgesetze der Arithmetik* to modern propositional systems, and show that Frege does not have a separable propositional logic, definable in terms of primitives of *Grundgesetze*, that corresponds to modern formulations of the logic of "not", "and", "or", and "if . . . then . . . ". Along the way we prove a number of novel results about the system of propositional logic found in *Grundgesetze*, and the broader system obtained by including identity. In particular, we show that the propositional connectives that are definable in terms of Frege's horizontal, negation, and conditional are exactly the connectives that fuse with the horizontal, and we show that the logical operators that are definable in terms of the horizontal, negation, the conditional, and identity are exactly the operators that are invariant with respect to permutations on the domain that leave the truth-values fixed. We conclude with some general observations regarding how Frege understood his logic, and how this understanding differs from modern views.

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1. Introduction

Gottlob Frege is often, and quite rightly, identified as the father and founder of modern formal logic. While there is no doubt that *Begriffsschrift* and *Grundgesetze* set the stage for the revolution in logic that occurred over the next half-century (the ramifications of which are still being investigated and assessed today), there remain many unanswered questions regarding the exact nature of Frege's logic, and how it influenced the development of the formal approaches, techniques, and constructions with which we are familiar today. The primary reason for this unfortunate gap in our understanding of this important aspect of the history of logic is not hard to identify, at least with regard to work on Frege in the English-speaking world: the lack—until very recently (Frege 2013)—of a complete translation of *Grundgesetze*, the work where Frege presents his final, mature account of formal logic.

The present essay is an attempt to fill in some of these missing details. The topic we shall focus on, however, might strike some readers as odd, unexpected, or not worthy of the sort of extended examination given here. The present essay focuses on Frege's propositional logic. Furthermore, for the most part we will ignore that aspect of Frege's logic that typically receives the most attention: Frege's deductive system. Instead, we shall focus on a topic often thought to be of minimal interest within the study of formal logic: the semantics of Frege's propositional logic, and in particular, whether or not the connectives introduced by Frege in *Grundgesetze* are *expressively complete*.[1]

Although most of the technical arguments below will focus on results regarding the expressive power of various sub-systems of *Grundgesetze*, the larger philosophical/historical issues with which we are primarily concerned can be expressed in terms of a very simple question—one that might be thought to have a simple and obvious answer:

Question: What, exactly, *is* Frege's propositional logic?

As we shall see, answering this question adequately and completely turns out to be much more complicated than one might initially suspect.

Of course, this question has a trivial and affirmative answer on one reading. After all, *Grundgesetze* contains operators (the horizontal, negation, and the conditional) that, although functioning differently from modern propositional connectives in various ways (e.g. being total functions on the entire domain), are nevertheless clear Fregean *analogues* of the modern notion of propositional connective. Hence, on this simple reading the propositional logic of *Grundgesetze* is the logic obtained via restricting our attention to the explicit propositional operators contained in *Grundgesetze*—that is, the horizontal, the negation stroke, and the conditional stroke.

Things in the *Grundgesetze* are not so simple, however. While the point of the previous paragraph is correct as far as it goes, it misses some deep and important aspects of the particular, and sometimes peculiar, way that the logic of *Grundgesetze* operates. Frege does, of course, introduce operators that are clearly the

[1] Note that this focus on Frege's semantics, and our explicit construction of a formal semantics inspired by the informal semantics described by Frege in *Grundgesetze*, is counter to accounts, such as that found in Landini (2012), that understand Frege's logic as defined and understood primarily (or even solely) in terms of deduction. For recent accounts that place more importance on the role of semantics in Frege's logic, however, see Heck (2012) and Cook (2013).

analogues of (and are the historical precedents of) our modern propositional connectives. But, as we shall see, he also introduces additional operators that are not analogues of modern propositional connectives, but which allow him to *define* new propositional connectives not definable in terms of the explicit propositional connectives he does introduce. Hence the emphasis, in what follows, on the expressive power of various collections of logical operators from *Grundgesetze*.

These complications stem in great part from the dual role that the identity function plays in Frege's logic. The identity function (that is, the dyadic function that maps α, β to the True if and only if α is identical to β, and maps any pair of distinct objects to the False) is often used as if it were a propositional connective—in particular, it is often used in lieu of a biconditional when both arguments are guaranteed to be truth-values. But there are good reasons for thinking that the identity function is *not* a propositional connective (or an analogue of one)—at least, not in the contemporary sense of "propositional connective"—and we should recognize that, in using the identity function *as if* it were the biconditional, Frege is, strictly speaking, going beyond the bounds of propositional logic.

As a result, there are logical operations that are (in a sense to be made precise in §3) propositional operators, but which can only be defined, within the language of *Grundgesetze*, in terms of identity. What about these? Should these be counted as part of Frege's propositional logic? Arguing for one or another answer here is probably pointless, of course. Rather, the more fruitful task we undertake in this paper is to carefully characterize and compare the various subsystems of *Grundgesetze* that might be thought to correspond, more or less loosely, to the contemporary understanding of propositional logic.

Before moving on, it is worth making an important terminological clarification. We have, and will continue, to use the expression "propositional connective" for the horizontal, the negation stroke, and the conditional stroke, as well as for other, relevantly similar functions that can be defined within *Grundgesetze* (again, see §3 for a precise definition of what is meant by "relevantly similar" here). This use of terminology should not be taken to imply that Frege's logical operators are, strictly speaking, connectives in the modern sense. On the contrary, his horizontal, negation, and conditional stroke are total functions from the domain of discourse to the True ($\top$) and the False ($\bot$). They are not connectives in the contemporary sense, where a connective is an operator that applies to well-formed formulas and produces more complex well-formed formulas. Nevertheless, the negation stroke in *Grundgesetze* is, roughly speaking, clearly meant to do much of the same work as does "$\neg$" in contemporary presentations of propositional logic—in particular, they are both formalizations of "not" or "it is not the case that" in English[2] and hence it seems natural to use uniform terminology here to emphasize the conceptual and historical connections between Frege's horizontal, negation, and conditional strokes and our contemporary connectives, even as we examine in detail the significant technical differences between the two approaches. These observations, of course, further emphasize the peculiar nature of Frege's use of identity, since it is, again roughly speaking, meant to do much of the same work as both the biconditional "$\leftrightarrow$" (a propositional connective) and the identity predicate "$=$" (not a propositional connective) in modern formal logic.

As a result, there are a number of (consistent) "sub-systems" of the full (and of course inconsistent) logic of *Grundgesetze* that are of interest when investigating whether, and in what sense, Frege's *Grundgesetze* contains a clear development of something resembling modern propositional logic:

1. The class of propositional connectives definable in terms

[2]And natural language negation in German, etc.

of Frege's primitive propositional connectives (i.e. the horizontal, negation, and the conditional).

2. The class of propositional connectives definable in terms of Frege's primitive propositional connectives plus the identity function.
3. The class of logical operators (propositional connective or not) definable in terms of Frege's primitive propositional connectives plus the identity function.

One of the main results of this paper is showing that these three collections are distinct: the first is a proper subset of the second, which is a proper subset of the third. Further, there is no list of primitive operators of *Grundgesetze* such that the second collection—the collection of all propositional connectives definable in *Grundgesetze*—is exactly the collection of logical operators definable in terms of the members of that list. In addition, as a sort of corollary we will also show that there is no subsystem of *Grundgesetze*, identifiable in terms of some subcollection of Frege's primitive logical operators, that captures *exactly* the second class of logical operators listed above (and nothing else). In short, the class of propositional connectives of *Grundgesetze* is not *separable* from the logic of *Grundgesetze* as a whole (or, at the very least, not separable from the logic of the Fregean identity function). The remainder of this paper is devoted to developing and discussing these and related results, although we will of course point out interesting corollaries of the results needed for this argument and explore interesting digressions as they arise. After all, the real purpose of this paper is not merely to make a rather picky observation about the expressive limitations of the language of *Grundgesetze*, but to use the examination of this issue to develop a much deeper understanding of the precise way that the logic of *Grundgesetze* functions.

The paper proceeds as follows: In the next section we will present a brief overview of that portion of the language and logic of *Grundgesetze* that we need for this project—in particular, the primitive propositional connectives (horizontal, negation, and the conditional) and the identity function. This section will also include a brief discussion of Frege's use of Roman letters as a device for expressing generality (for reasons that will become apparent). Then, in §3, we investigate the class of propositional connectives that are definable in terms of the primitive propositional connectives of *Grundgesetze* (again, the horizontal, negation, and the conditional), and show that this class can be described in two other ways that would have been of central importance to Frege given the manner in which he (informally) sets up the semantics of *Grundgesetze*. In §4 we then introduce a novel set of propositional connectives—what we shall call the *alternative horizontal*, the *alternative negation*, and the *alternative conditional*, and show that these connectives are not expressible in terms of Frege's primitive propositional connectives. In short, in this section we show that the first class in the list above is not identical to the class of all propositional connectives. We then consider the system that would result from combining the "official" Fregean connectives and the alternative versions of these same connectives in §5, and examine which sub-collections of these six connectives are and are not expressively complete with respect to the class of all Fregean propositional connectives. In §6 we then show that, if we add Frege's identity function to Frege's official set of connectives (the horizontal, negation, and conditional strokes), we are able to express every Fregean propositional connective (although we get "too much", in a sense, since this collection of operators also allows us to define functions that are not propositional connectives at all). This, of course, amounts to showing that the second class is distinct from the third. Finally, after a brief digression on the various readings of the biconditional that are possible within *Grundgesetze* in §7, we precisely characterize the class of logical operators definable in terms of the horizontal, negation, the conditional, and identity in §8. In this section we shall also identify a striking and heretofore unnoticed connection between this subsystem of the logic of *Grundgesetze* and the role that permutation invariance plays in current debates regarding the nature of logic and of logical operators generally. Finally, we draw some general conclusions regarding the nature of Frege's propositional logic (and the lack of a *separable* propositional logic) in the conclusion.

2. Frege's *Grundgesetze* Connectives

At the outset, it is important to remember that Frege requires that every function be defined on *every* argument—that is, for Frege any function whatsoever is a total function on the entire domain.[3] As a result, Frege's propositional connectives behave differently from modern connectives, since we usually do not assume that the latter are defined on arguments other than $\{\top, \bot\}$.

The first propositional connective (or analogue thereof) that Frege introduces in *Grundgesetze* is the *horizontal*.[4] The horizontal is a unary function that maps every object to either the True or the False—that is, in Frege's terminology is a 1[st]-level concept. Thus, if α names an object, then:

$$— \alpha$$

names a truth-value. Frege provides the following informal description of the semantic behavior of the horizontal:

> I regard it as a function-name such that
>
> $$— \Delta$$
>
> is the True when Δ is the True, and is the False when Δ is not the True. (Frege 2013, I: §5)

In short, if α names an object, then "$— \alpha$" can be understood, loosely, as denoting the truth-value of:

$$\alpha \text{ is (i.e. is identical to) the True.}$$

In what follows we shall represent the horizontal as "$f_{\mathsf{HOR}}(\xi)$" (or simply f_{HOR}) rather than "$— \xi$", since this will make the proofs that follow a bit more perspicuous and also provides a nice template for introducing additional functions not present as primitive logical operators in *Grundgesetze*.[5] The logical prop-

[3]The discussion of the nature of Frege's logical operators in this section borrows heavily from Cook (2013), which is also highly recommended to the reader desiring more details regarding Frege's logical notation.

For a detailed discussion of what, exactly, Frege's requirement that functions be total amounts to, and in particular whether the requirement should be understood as requiring that functions be total on all objects whatsoever, or only on those objects that happen to be in the current (not necessarily universal) domain, see the exchange in the symposium: Cook (2015), Rossberg (2015), Wehmeier (2015), Blanchette (2015).

[4]It is worth noting that, in *Grundgesetze*,, the terms "horizontal", "negation", and "conditional" are names of linguistic items, not ("officially") names of the functions these linguistic items name. Since nothing hinges on this in the present context, however, we shall use "horizontal", "negation", and "conditional" to refer to both the functions in question and the function name used to refer to them—thus, "horizontal" is systematically ambiguous in what follows, denoting either the horizontal stroke or the horizontal function that the stroke denotes, depending on context (and similarly for the other primitive or defined logical operators).

[5]The claim that Frege's two-dimensional notation is difficult to read or reason with is nothing more than a pernicious myth, with its roots in uncharitable reviews of Frege's work by his contemporaries—in particular Venn (1880) and Schröder (1880). Frege himself refutes such slanders in "On the Aim of 'Conceptual Notation' ":

> I pause here to answer some objections to Schröder. He compares my representation of the exclusive "*A* or *B*" with his mode of writing:
>
> $$ab_1 + a_1b = 1$$
>
> and finds here, as elsewhere in my "conceptual notation", a monstrous waste of space. In fact, I cannot deny that my expression takes up more room than Schröder's, for which its part is again more spread-out than Boole's original:
>
> $$a + b = 1$$
>
> But this criticism is based upon the view that my "conceptual notation" is supposed to be a presentation of abstract logic. These formulas [of my "conceptual notation"] are actually only empty schemata; and in their

erties of the horizontal function are thus summed up in the following table:[6]

α	$f_{\mathsf{HOR}}(\alpha)$
$\top$	$\top$
$\bot$	$\bot$
other	$\bot$

It should be emphasized that the table above is not a truth-table in the modern sense: Frege's logical operators are not truth-functions, since they are not functions from n-tuples of truth-values to truth-values. Instead, Frege's horizonal is a function from *any* object to a truth-value (similar comments apply to the negation-stroke and conditional-stroke, discussed below).

Next up is *negation*. Negation is also a 1^{st}-level concept: if "α" names an object, then:

$$\top\, \alpha$$

names a truth-value. Frege summarizes the semantics for negation as follows:

> The value of the function:
>
> $$\top\, \xi$$
>
> is to be the False for every argument for which the value of the function:
>
> $$-\!\!-\, \xi$$
>
> is the True, and it is to be the True for all other arguments. (Frege 2013, I: §6)

Frege's gloss on negation is given in terms of the semantics for the horizontal, but we can easily reformulate it along the same lines as the informal semantic clause for the horizontal along the following lines:

> Negation is a function-name such that:
>
> $$\top\, \Delta$$
>
> is the True when Δ is not the True, and is the False when Δ is the True. (Frege 2013, I: §5)

Hence, if α names an object, then "$\top\, \alpha$" can be understood, loosely, as denoting the truth-value of:

α fails to be (i.e. fails to be identical to) the True.

In what follows we shall represent Frege's negation function as "$f_{\mathsf{NEG}}(\xi)$" (or simply f_{NEG}), and we can sum up the logical properties of negation in the following table:

α	$f_{\mathsf{NEG}}(\alpha)$
$\top$	$\bot$
$\bot$	$\top$
other	$\top$

> application, one must think of whole formulas in the places of A and B—perhaps extended equations, congruences, projections. Then the matter appears quite different. The disadvantage of the waste of space of the "conceptual notation" is converted into the advantage of perspicuity; the advantage of terseness for Boole is transformed into the disadvantage of unintelligibility. (Frege 1882–83, 97)

Nevertheless, it is unfortunately true that Frege's notation can become somewhat unwieldy in metaproofs about the logical system of *Grundgesetze*—that is, when we *are* treating his logical system as an *abstract logic*—hence our introduction of f_{HOR}, f_{NEG}, and so on.

[6]It is perhaps worth noting that the fact that Frege does not distinguish between the False and non-truth-values in his definition of the horizontal stroke (and similarly for his definitions of the negation stroke and the conditional stroke, see below) suggests that a table with only two rows, corresponding to (i) the true, and (ii) everything else, might be more natural. While this observation is correct, distinguishing between the False and non-truth-values at this stage will help to motivate the notion of non-Truth-non-distinguishing connectives examined in §4 as well as the introduction of the alternative Fregean connectives, and the corresponding notion of non-Falsity-non-distinguishing connectives, examined in §5. At any rate, although he did not do so in his definitions of the connectives, and hence cannot do so solely in terms of these connectives, the logical resources of *Grundgesetze* (in particular, identity) do allow for Frege to distinguish between the False and other objects that are not identical to the True (see, e.g., the constructions in *Grundgesetze* §10!)

The third, and final, primitive propositional connective in *Grundgesetze* is the *conditional stroke*. Frege's conditional stroke is a binary function symbol that attaches to names of objects—in Frege's terms it names a 1st-level relation: if α and β name objects, then:

$$\begin{array}{l} \top\!\!-\, \beta \\ \llcorner\, \alpha \end{array}$$

is the name of a truth-value. Frege describes the logical behavior of the conditional as follows:

> I introduce the function with two arguments:
>
> $$\begin{array}{l} \top\!\!-\, \xi \\ \llcorner\, \zeta \end{array}$$
>
> by means of the specification that its value shall be the False if the True is taken as the ζ argument, while any object that is not the True is taken as the ξ-argument; that in all other cases the value of the function shall be the True. (Frege 2013, I: §12)

Hence, if α and β name objects, then $\begin{array}{l} \top\!\!-\, \beta \\ \llcorner\, \alpha \end{array}$ can be understood, loosely, as denoting the truth-value of:

> Either α fails to be (i.e fails to be identical to) the True, or β is (i.e is identical to) the True.

We will use "$f_{\text{CON}}(\xi_1, \xi_2)$" (or simply f_{CON}) to symbolize this binary function below. The logical properties of f_{CON} are summed up in the following table:

f_{CON}	$\top$	$\bot$	other
$\top$	$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\top$
other	$\top$	$\top$	$\top$

(The left-most vertical column represents the value of "α", the antecedent or *subcomponent* of the conditional statement; the top-most horizontal row the value of "β", the consequent or *supercomponent*.)

These are the only propositional connectives that occur primitively in *Grundgesetze*, if by "propositional connective" we mean an n-ary 1st-level function $g(\xi_1, \xi_2, \ldots, \xi_n)$ that maps every object to a truth-value, and which does not distinguish between any two non-truth-values as argument (we shall tighten up this criterion for being a propositional connective in §3 below). Before moving on to the identity function, however, it is worth examining two features of Frege's propositional connectives that will play a central role in the results below.

First, Frege suggests that the negation-stroke and the conditional-stroke can be understood as consisting merely of the actual vertical strokes or lines involved in their formalization, with the attached horizontal portions of their notation understood as separate occurrences of the horizontal. For example, when discussing negation he writes that:

> ... '$\top\, \Delta$' always refers to the same as '$\top\, (-\, \Delta)$', as '$-\, (\top\, \Delta)$' and as '$-\, (\top\, (-\, \Delta))$'. We therefore regard '$\top$' as composed of the small vertical stroke, the *negation stroke*, and the two parts of the horizontal stroke each of which can be regarded as a *horizontal* in our sense. The transition from '$\top\, (-\, \Delta)$' or from '$-\, (\top\, \Delta)$' to '$\top\, \Delta$', as well as that from '$-\, (-\, \Delta)$' to '$-\, \Delta$', I will call the *fusion of horizontals*. (Frege 2013, I: §6)

and when discussing the conditional he writes that:

> It follows that:
>
> $$\begin{array}{l} \top\!\!-\, \Gamma \\ \llcorner\, \Delta \end{array}$$
>
> is the same as:
>
> $$-\left(\begin{array}{l} \top\!\!-\, (-\, \Gamma) \\ \llcorner\, (-\, \Delta) \end{array}\right)$$
>
> and therefore that in:
>
> $$`\begin{array}{l} \top\!\!-\, \Gamma \\ \llcorner\, \Delta' \end{array}$$

> we can regard the horizontal stroke before 'Δ', as well as each of the two parts of the upper horizontal stroke partitioned by the vertical, as *horizontals* in our particular sense. We speak here, just as previously, of the *fusion of horizontals*. (Frege 2013, I: §12)

Frege's suggestion that negation and the conditional have occurrences of horizontals as parts should not be taken too literally, since it is not clear that the result of prefixing the vertical portion of the negation symbol "$\top$" (without its horizontal "parts") to an argument α—that is, something like "$_{\mid}\,\alpha$"—is a well-formed concept-script expression in *Grundgesetze*.[7] Nevertheless, what is clear is that Frege is suggesting that the horizontal, negation, and the conditional can be treated *as if* they contain occurrences of the horizontal.[8] In short, Frege is pointing out that, for any α:[9]

$$\top\,\alpha = \top\,(—\,\alpha), = —\,(\top\,\alpha) = —\,(\top\,(—\,\alpha))$$

or, in the notation introduced above:

$$\begin{aligned} f_{\mathsf{NEG}}(\alpha) &= f_{\mathsf{NEG}}(f_{\mathsf{HOR}}(\alpha)) \\ &= f_{\mathsf{HOR}}(f_{\mathsf{NEG}}(\alpha)) \\ &= f_{\mathsf{HOR}}(f_{\mathsf{NEG}}(f_{\mathsf{HOR}}(\alpha))) \end{aligned}$$

Along similar lines, it must be the case, for any α and β, that:

$$\begin{array}{l}\!\top\!\!—\,\beta\\ \llcorner\,\alpha\end{array} = —\left(\begin{array}{l}\!\top\!\!—\,\beta\\ \llcorner\,\alpha\end{array}\right) = \begin{array}{l}\!\top\!\!—\,\beta\\ \llcorner\,(—\,\alpha)\end{array}$$
$$= \begin{array}{l}\!\top\!\!—\,(—\,\beta)\\ \llcorner\,\alpha\end{array} = \begin{array}{l}\!\top\!\!—\,(—\,\beta)\\ \llcorner\,(—\,\alpha)\end{array} = —\left(\begin{array}{l}\!\top\!\!—\,(—\,\beta)\\ \llcorner\,\alpha\end{array}\right)$$
$$= —\left(\begin{array}{l}\!\top\!\!—\,\beta\\ \llcorner\,(—\,\alpha)\end{array}\right) = —\left(\begin{array}{l}\!\top\!\!—\,(—\,\beta)\\ \llcorner\,(—\,\alpha)\end{array}\right)$$

or:

$$\begin{aligned} f_{\mathsf{CON}}(\alpha,\beta) &= f_{\mathsf{HOR}}(f_{\mathsf{CON}}(\alpha,\beta)) \\ &= f_{\mathsf{CON}}(f_{\mathsf{HOR}}(\alpha),\beta) \\ &= f_{\mathsf{CON}}(\alpha, f_{\mathsf{HOR}}(\beta)) \\ &= f_{\mathsf{CON}}(f_{\mathsf{HOR}}(\alpha), f_{\mathsf{HOR}}(\beta)) \\ &= f_{\mathsf{HOR}}(f_{\mathsf{CON}}(\alpha, f_{\mathsf{HOR}}(\beta))) \\ &= f_{\mathsf{HOR}}(f_{\mathsf{CON}}(f_{\mathsf{HOR}}(\alpha),\beta)) \\ &= f_{\mathsf{HOR}}(f_{\mathsf{CON}}(f_{\mathsf{HOR}}(\alpha), f_{\mathsf{HOR}}(\beta))) \end{aligned}$$

Frege calls these equivalences, and the transformations that result from replacing one of the concept-script expressions listed above with another, equivalent formulation, the *fusion of horizontals*.

The role that the fusion of horizontals plays in the logic of *Grundgesetze* helps to explain what might otherwise appear to be an unnecessary redundancy in Frege's notation. As we have seen, Frege explicitly introduces both the horizontal and negation. But once we have negation, we no longer need the horizontal as an additional primitive notation, since the horizontal

[7]Further, the "metatheoretic" argument that every concept-script expression in *Grundgesetze* has a reference, found in Frege (2013, I: §§10, 30–31), do not cover such cases, rendering these arguments incomplete (and obviously so) if Frege genuinely intended us to understand "$\top\,\xi$" and "$\begin{array}{l}\!\top\!\!—\,\xi_2\\ \llcorner\,\xi_1\end{array}$" as complex symbols. It is worth noting that Frege talks here of "regarding" the negation and the conditional as if they had a certain complex structure, not that they in fact *have* that structure. Frege might be understood as claiming that the negation stroke $\top\,\xi$ and the various function names obtained by fusing negation with one or more horizontals—that is:

$$—\,(\top\,\xi),\ \ \top\,(—\,\xi),\ \ —\,(\top\,(—\,\xi))$$

have the same reference (i.e. name the same function), despite having distinct senses. Defending this nuanced reading is beyond the scope of the present work. (Thanks to a referee for suggesting this idea.)

[8]These comments also apply to the concavity stroke—that is, to Frege's analogue of the universal quantifier (see, e.g. Frege 2013, I: §8). This fact, while obviously important, is orthogonal to the present project.

[9]Note that the occurences of "=" here are metatheoretic: these are not identity claims formulated within the language of *Grundgesetze* itself.

is explicitly definable in terms of negation.[10] In other words, if α names an object, then:

$$— \alpha = \neg\neg\, \alpha$$

or, equivalently:

$$f_{\mathsf{HOR}}(\alpha) = f_{\mathsf{NEG}}(f_{\mathsf{NEG}}(\alpha))$$

Frege explicitly proves this in *Grundgesetze*—the following, which is labelled IVb, occurs in §49:[11]

$$\vdash (— a) = (\neg\neg\, a)$$

[10]The horizontal is also definable in terms of the conditional. For any α and β:

$$— \alpha = \begin{array}{l} \text{———}\ \alpha \\ \quad \llcorner\text{—}\ \beta \\ \qquad \llcorner\ \beta \end{array}$$

[11]Interestingly, Frege claims in the introduction to *Begriffschift* that he could have combined theorem (31) (Frege 2002, 156):

$$\begin{array}{l} \vdash\text{———}\ a \\ \quad \llcorner\neg\neg\ a \end{array}$$

and theorem (41) (Frege 2002: 158):

$$\begin{array}{l} \vdash\neg\neg\ a \\ \quad \llcorner\text{——}\ a \end{array}$$

into (Frege (2002): 107):

$$\vdash (\neg\neg\, a = a)$$

This is correct, in *Begriffschrift*, since Frege understands the horizontal, negation, and the conditional as something akin to our contemporary conception of sentential operators, rather than as functions on the domain, in the earlier work. In particular, there is no indication that connectives need to be total functions in *Begriffsschrift*. Note that the formula above is *not* a theorem of *Grundgesetze* since $\neg\neg\, a = a$ is the False when a is not a truth-value.

Thus, Frege does not need the horizontal as an additional primitive in his system—at least, he does not need it because of any additional expressive power it affords us once negation is already in play. But the comments above regarding the fusion of horizontals suggest another role that the horizontal plays in the presentation of *Grundgesetze*: The horizontal is required in order to provide an exact description of the connectives that are definable in terms of the primitive propositional connectives of *Grundgesetze*. As we shall see below, it is not merely the case that both negation and the conditional fuse with the horizontal. In addition, the propositional connectives that fuse with the horizontal are *exactly* those definable in terms of Frege's negation and conditional. In short, the horizontal, redundant or not, denotes *exactly* the notion required to characterize the propositional connectives definable in terms of Frege's primitive propositional connectives. Although there does not seem to be any textual evidence that Frege was aware of this fact or was in a position to prove it, we find the fact that Frege retained the horizontal in spite of its obvious expressive redundancy to be rather suggestive that he might have at least suspected something like this result.

There is a second observation regarding the nature of Frege's propositional connectives that is worth making at this point. In providing the informal "semantic clauses" for the horizontal, negation, and the conditional, Frege divides arguments into two kinds: the True, and any object that is not the True (including but obviously not limited to the False). Hence the horizontal maps the True to the True, and any other object to the False; negation maps the True to the False, and any other object to the True, and the semantics for the conditional can be summarized in a similar simplified form via the following table:

f_{CON}	$\top$	not : $\top$
$\top$	$\top$	$\bot$
not : $\top$	$\top$	$\top$

Thus, none of Frege's primitive propositional connectives can distinguish between the False as input and any non-truth-value as input. Importantly, as we shall show below, the propositional connectives introduced by Frege—that is, the horizontal, negation, and the conditional—do not merely have this property, which we shall call *non-Truth non-distinguishing*, as a matter of happenstance. Instead, the propositional connectives that do not distinguish between the False and non-truth-values are *exactly* those that fuse with the horizontal, and hence are *exactly* those that are definable in terms of Frege's primitive propositional connectives.

As we have already noted, however (although the proof of this fact will have to wait until §5), this class of connectives is not identical to the class of all propositional connectives. In order to obtain resources sufficient for defining all of the possible propositional connectives, we will have to move beyond the propositional connectives Frege explicitly introduces in *Grundgesetze*. In particular, we will need the dyadic identity function "=". Frege provides the following informal description of the semantic behavior of the identity function:

> We have already used the equality-sign rather casually to form examples but it is necessary to stipulate something more precise regarding it:
>
> $$\text{‘}\Delta = \Gamma\text{’}$$
>
> refers to the True, if Γ is the same as Δ; in all other cases it is to refer to the False. (Frege 2013, I: §7)

Thus, identity is a binary function, and:

$$\alpha = \beta$$

is the True if and only if α and β are identical, and is the False otherwise. Other than its slightly different syntactic form (i.e. it is a two place function from objects to truth-values, rather than a relation in the contemporary sense of "relation") Frege's identity function works very similarly to the modern notion of identity mobilized in, for example, contemporary first-order logic. In the sections to follow we shall use $f_{=}(\xi_1, \xi_2)$ (or simply $f_{=}$) to refer to this binary function, and we shall be careful to use "=" only when making metatheoretic claims about the logical system within which $f_{=}$ occurs.

As we have already emphasized, the identity function $f_{=}(\xi_1, \xi_2)$ is not a propositional connective—even on the Fregean way of understanding what it is for a total function to be a propositional connective—since it can distinguish between non-truth-values. To make this point clearer, consider what happens if we attempt to present Frege's dyadic identity function in terms of a 3×3 chart similar to the manner in which we were able to represent the logical behavior of the conditional:

$f_{=}$	$\top$	$\bot$	other
$\top$	$\top$	$\bot$	$\bot$
$\bot$	$\bot$	$\top$	$\bot$
other	$\bot$	$\bot$	???

Everything goes swimmingly until we get to the bottom-right entry. Then the problem becomes apparent: unlike the propositional connectives (i.e. the horizontal, negation, and the conditional) discussed above, knowing that two arguments are both non-truth-values is not enough to know the result of applying the identity function to them. In addition, we need to know whether or not they are the *same* non-truth-value. Since a propositional connective should, in some sense, only care about truth-values as inputs, and should, as a result, output some default value when all inputs are non-truth-values (and, further, should output the *same* default value in all such cases, since otherwise the connective would in fact be distinguishing between different sequences of inputs based on distinctions other than what truth-value, if any, the inputs are), identity is not a propositional connective.

This completes our survey of those primitive notions from *Grundgesetze* that will be needed in what follows. There are two additional issues that we need to deal with, however. First, Frege's logic does not contain any primitive names—either of truth-values or of anything else—and hence if we consider the subsystem of *Grundgesetze* that contains only the horizontal, negation, the conditional, and the identity function, we will find ourselves unable to proceed very far, since the subsystem in question will contain no sentences.[12] There are two ways to rectify this: The first, and arguably simplest, would be to supplement the system with a stock of primitive names (where these names should be understood as having both truth-values and non-truth-values as their potential referents). From a technical perspective this approach would be perfectly adequate, but it does involve adding something genuinely new to *Grundgesetze*.[13] Instead, we will take ourselves to be working within a slightly wider sub-system of *Grundgesetze* that includes, in addition to the notions surveyed above, the Roman letter generality device.[14] Thus, we will take formulas with object-level variables occurring in them to be understood as (or as equivalent to) instances of (prenex-only) universal quantification. There is, of course, plenty of controversy regarding how, exactly, we are to understand Frege's Roman letters. Fortunately for us, none of these subtle issues in any way affects the rather unsubtle use we make of the device—as a way to make claims about all objects in a particular domain, and nothing more.[15]

Second, we need to flag a methodological assumption adopted in the constructions below. We assume throughout that functions are individuated extensionally in the logic of *Grundgesetze*—that is, that two functions are identical if and only if they map the same arguments to the same outputs—and we construct our formal version of Frege's informal semantics accordingly.[16] Not all Frege scholars agree with this assumption (see, e.g. Benis-Sinaceur, Panza, and Sandu 2015). Nevertheless, this simplifying assumption should not prevent such scholars from accepting the results proven below, since the vast majority of results in this paper require proving that a function of such-and-such kind exists (or something similar), but do not depend on there being a *unique* such function. In short, all of the results below can be easily (but tediously) re-cast as holding of a non-extensional interpretation of *Grundgesetze*.[17]

[12]Since *Grundgesetze* contains no primitive names, one must first form complex names via applying 2^{nd}-level functions to 1^{st}-level functions to obtain complex names of objects before one can form names via the application of 1^{st}-level functions to names of objects.

[13]Note that our use of f_{HOR}, f_{NEG}, f_{CON}, and $f_{=}$ does not involve adding anything new to the logic of *Grundgesetze*. Instead, we have merely introduced new, more typographically convenient names for the very same functions present in Frege's original work. In other words, introducing these new notations no more implies that we have introduced new functions, or new understandings of these functions, than does the fact that the authors of this paper refer to these notions using "horizontal", "negation", "conditional", and "identity", while Frege referred to them as "Wagerechten", "Verneinungsstrich", "Bedingungsstrich", and "Gleichheit".

[14]Thus, we will interpret formulas of the languages containing f_{HOR}, f_{NEG}, f_{CON}, and $f_{=}$, as well as the extended languages obtained by adding additional functions, as bound by implicit prenex first-order quantifiers.

[15]It is worth mentioning, however that we are sympathetic to the reading of the Roman letter generality device provided in Heck (2012), especially §3.2.

[16]Frege, of course, denied that we can express the claim that two functions are identical (or not) in the formal language of *Grundgesetze*—the best we can do in the formal context is assert that two functions agree on every argument (and, equivalently, that their value-ranges are identical). This has no bearing on whether or not Frege (or his commentators) could take a meaningful stand on the identity, or not, of functions from the perspective of the metatheory within which the semantics and deductive system of *Grundgesetze* is detailed.

[17]Of course, both authors of this essay believe that Frege did, in fact, have an extensional understanding of functions. Hence the implicit assumption of such in the formalism is not only a simpifying device, but also reflects our own views regarding how best to understand the logic of *Grundgesetze*. The point, however, is that those scholars who disagree can easily modify the constructions and proofs given below so that they are appropriate to a non-extensional reading.

3. Negation, Conditional, and Horizontal

Having identified the primitive propositional connectives of *Grundgesetze* (and introduced the new notation—"f_{HOR}", "f_{NEG}", and "f_{CON}"—for these notions) we are now in a position to make some technical observations regarding the logical "behavior" of the system that results from restricting attention to the corresponding subsystem of *Grundgesetze*.

Before proving such results, we first need to construct a bit of formal machinery that corresponds to Frege's informal discussion of the semantics of the propositional connectives. As the discussion of the previous section makes clear, Frege understood propositional connectives to be functions mapping n-tuples from the domain of all objects whatsoever (which includes the truth-values $\top$ and $\bot$) to the truth-values. Thus, in order to study Frege's connectives, we need the following notion of *Fregean domain*:[18]

Definition 3.1. *A* Fregean domain *is any set Δ where:*

$$\top, \bot \in \Delta$$

[18]It is perhaps worth commenting on our mobilization of Fregean domains, since Frege is widely (but not universally) interpreted as admitting only a single, universal domain of quantification (again, see the exchange: Cook 2015, Rossberg 2015, Wehmeier 2015, Blanchette 2015). Presumably, a contemporary theorist who believes that there is only one legitimate domain of quantification (for whatever reasons) will nevertheless gain much from a study of arbitrary models, so long as that theorist is unsure of the exact content of the one true legitimate domain, for the simple reason that anything that is true of all models will also be true of the one true legitimate domain. Presumably a formal reconstruction of Frege's informal semantics, such as the one mobilized here, can adopt the same approach. There are questions regarding how such an approach relates to Frege's notorious dispute with Hilbert regarding the role of models in consistency proofs, of course, but this in no way affects the usefulness of modern model-theoretic methods as tools for understanding the complexities of the formal system of Frege's *Grundgesetze*. We plan to examine how such methods relate to Frege's anti-model-theory attitudes in future work.

It is now very natural to define a *Fregean propositional connective* as a specific kind of function that maps n-tuples of objects from a Fregean domain to the truth-values:[19]

Definition 3.2. *An n-ary function g is a* Fregean propositional connective *if and only if, for any Fregean domain Δ where $|\Delta| \geq 3$, g is total on Δ^n, the range of g is $\{\top, \bot\}$:*

$$g : \Delta^n \rightarrow \{\top, \bot\}$$

and, for any n-tuple:

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in \Delta^n$$

any:

$$B \subseteq \{m \in \mathbb{N} : 1 \leq m \leq n\}$$

and any $\delta \in \Delta$ where $\delta \notin \{\top, \bot\}$, we have:

$$g(\alpha_1, \alpha_2, \ldots, \alpha_n) = g(\beta_1, \beta_2, \ldots, \beta_n)$$

where:

$$\beta_i = \begin{cases} \delta, & \text{if } \alpha_i \notin \{\top, \bot\} \text{ and } i \in B; \\ \alpha_i, & \text{if } \alpha_i \in \{\top, \bot\} \text{ or } i \notin B. \end{cases}$$

FPC *is the class of Fregean propositional connectives.*

Loosely speaking, a Fregean propositional connective is any total function that maps n-tuples from a Fregean domain Δ to $\{\top, \bot\}$ that does not distinguish between any two non-truth-values. As our discussion of Frege's horizontal, negation, and conditional

[19]We do not claim that this is Frege's notion (since, of course, Frege never explicitly identified a subsystem of *Grundgesetze* as the propositional subsystem), nor do we claim it is in some sense the *right* understanding of "propositional connective". All that we require is that it is a relatively *natural* and *independently interesting* understanding of what it is to be a propositional connective in the Fregean setting.

makes clear, Fregean propositional connectives can be naturally represented in ways that resemble (but are not identical to) familiar treatments of the connectives in three-valued logic, since there are three relevant kinds of input to consider: $\top$, $\bot$, and non-truth-values. For example, binary Fregean propositional connectives (such as the conditional stroke, but also binary connectives definable in terms of other notions) can be represented as 3×3 matrices of outputs of the form:[20]

con	$\top$	$\bot$	other
$\top$	V_1	V_2	V_3
$\bot$	V_4	V_5	V_6
other	V_7	V_8	V_9

where $V_i \in \{\top, \bot\}$ for $1 \leq i \leq 9$. Similar comments apply to n-ary Fregean propositional connectives and 3^n-ary matrices for $n \geq 0$.

The three propositional connectives explicitly given in *Grundgesetze* are, of course, Fregean propositional connectives:

Proposition 3.3. $f_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f_{\mathsf{CON}} \in \mathsf{FPC}$.

Note, however, that we make no assumption at this point that *Grundgesetze* contains all of the Fregean propositional connectives. In fact, as we shall see, although all Fregean propositional connectives are definable in the language of *Grundgesetze*, some of these connectives can only be constructed within *Grundgesetze* via the use of functions (in particular, identity) that are not Fregean propositional connectives. With this in mind, it is worth noting explicitly that $f_=$ is *not* a Fregean propositional connective, since it is able to distinguish between different non-truth-values:

$$f_=(\alpha, \beta) = \begin{cases} \bot, & \text{if } \alpha, \beta \notin \{\top, \bot\} \text{ and } \alpha \neq \beta; \\ \top, & \text{if } \alpha, \beta \notin \{\top, \bot\} \text{ and } \alpha = \beta. \end{cases}$$

We shall now identify three sub-classes of Fregean propositional connectives that are of interest in the present context. The first, and most obvious, perhaps, is the class of *Fregean simply definable connectives*—those Fregean propositional connectives that are definable in terms of the explicit propositional connectives contained in *Grundgesetze*:[21]

Definition 3.4. *An n-ary Fregean propositional connective g is a* Fregean simply definable connective *if and only if g is definable in terms of:*[22]

$$f_{\mathsf{HOR}}, f_{\mathsf{NEG}}, \textit{ and } f_{\mathsf{CON}}$$

DEF *is the class of Fregean simply definable connectives.*

The following is obvious:

Proposition 3.5. $f_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f_{\mathsf{CON}} \in \mathsf{DEF}$.

In the previous section we identified two interesting properties shared by the connectives we are representing as f_{HOR}, f_{NEG}, and f_{CON}. The first of these was the fact that each of these connectives fuses with the horizontal. We can provide a precise characterization of this property as follows:

Definition 3.6. *An n-ary Fregean propositional connective g is a* horizontal fusing connective *if and only if, for any Fregean domain Δ, n-tuple:*

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in \Delta^n$$

[20] It should be emphasized once again, however, that these are *not* truth tables, strictly speaking, and Frege's logic is not a three-valued logic, since the third type of inputs are, by definition, not truth-values!

[21] The purpose of the modifier "simply" will become apparent in §4 and §6 below, where we introduce distinct classes of propositional connectives which we shall call the *Fregean alternative definable connectives* and the *Fregean identity definable connectives* respectively.

[22] By "definable" here, and below, we mean definable in terms of the functions listed via composition.

and:

$$B \subseteq \{m \in \mathbb{N} : 1 \leq m \leq n\}$$

we have:

$$\begin{aligned} g(\alpha_1, \alpha_2, \ldots, \alpha_n) &= f_{\mathsf{HOR}}(g(\alpha_1, \alpha_2, \ldots, \alpha_n)) \\ &= g(\beta_1, \beta_2, \ldots, \beta_n) \\ &= f_{\mathsf{HOR}}(g(\beta_1, \beta_2, \ldots, \beta_n)) \end{aligned}$$

where:

$$\beta_i = \begin{cases} f_{\mathsf{HOR}}(\alpha_i), & \text{if } i \in B \\ \alpha_i, & \text{if } i \notin B. \end{cases}$$

FUSE *is the class of horizontal fusing connectives.*

As discussed above, Frege himself demonstrates the following:

Proposition 3.7. $f_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f_{\mathsf{CON}} \in$ FUSE (Frege 2013, I: §§5, 6, 12).

The second interesting property of the primitive functions f_{HOR}, f_{NEG}, and f_{CON} is that they are unable to distinguish between taking the False as an argument, and taking a non-truth-value as argument. We thus obtain the third sub-category of Fregean propositional connectives:

Definition 3.8. *An n-ary Fregean propositional function g is* non-Truth non-distinguishing *if and only if, for any Fregean domain Δ where $|\Delta| \geq 3$, n-tuple:*

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in \Delta^n$$

and:

$$B \subseteq \{m \in \mathbb{N} : 1 \leq m \leq n\}$$

and $\delta \in \Delta$ where $\delta \notin \{\top, \bot\}$, we have:

$$g(\alpha_1, \alpha_2, \ldots, \alpha_n) = g(\beta_1, \beta_2, \ldots, \beta_n)$$

where:

$$\beta_i = \begin{cases} \alpha_i, & \text{if } \alpha_i = \top \text{ or } i \notin B; \\ \delta, & \text{if } \alpha_i = \bot \text{ and } i \in B; \\ \bot, & \text{if } \alpha_i \neq \top \text{ and } \alpha_1 \neq \bot \text{ and } i \in B. \end{cases}$$

NTND *is the class of non-Truth non-distinguishing functions.*

A binary non-Truth non-distinguishing propositional function will correspond to a 3×3 matrix[23] of the following form:

con	$\top$	$\bot$	other
$\top$	V_1	V_2	V_2
$\bot$	V_3	V_4	V_4
other	V_3	V_4	V_4

where $V_i \in \{\top, \bot\}$ for $1 \leq i \leq 4$. Similar comments apply to n-ary Fregean propositional connectives and 3^n-ary matrices.

Again, Frege's primitive connectives are also in this class of Fregean propositional connectives and, as already discussed above, Frege's informal semantic clauses for his primitive propositional connectives immediately entail the following:

Proposition 3.9. $f_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f_{\mathsf{CON}} \in$ NTND. (Frege 2013 I: §§5, 6, 12).

The remainder of this section will be devoted to proving that these three classes of connectives are in fact, identical—that is:

$$\mathsf{DEF} = \mathsf{FUSE} = \mathsf{NTND}$$

We shall prove this theorem via three lemmas, which explicitly provide the following equivalent claim:

$$\mathsf{DEF} \subseteq \mathsf{FUSE} \subseteq \mathsf{NTND} \subseteq \mathsf{DEF}$$

[23] Which, we will emphasize one final time, is *not* a truth table!

Before proving any of these "containment" claims, however, we prove the following lemma, which, intuitively, provides an alternative formulation of the non-Truth non-distinguishing condition—one that will be extremely useful in what follows:

Lemma 3.10. *Given any Fregean propositional functions* $g_1, g_2 \in \mathsf{NTND}$, *if for any:*

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in \{\top, \bot\}^n$$

we have:

$$g_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = g_2(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

then, given any Fregean domain Δ *and any:*

$$\langle \beta_1, \beta_2, \ldots, \beta_n \rangle \in \Delta^n$$

we have:

$$g_1(\beta_1, \beta_2, \ldots, \beta_n) = g_2(\beta_1, \beta_2, \ldots, \beta_n)$$

Proof. Assume g_1, g_2 are such that, that for any:

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in \{\top, \bot\}^n$$

we have:

$$g_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = g_2(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

Let Δ be any Fregean domain, and:

$$\langle \beta_1, \beta_2, \ldots, \beta_n \rangle \in \Delta^n$$

For $1 \leq i \leq n$, let:

$$\delta_i = \begin{cases} \beta_i, & \text{if } \beta_1 = \top \text{ or } \beta_i = \bot; \\ \bot, & \text{otherwise.} \end{cases}$$

Then, since g_1 and g_2 are non-truth non-distinguishing, we have:

$$\begin{aligned} g_1(\beta_1, \beta_2, \ldots, \beta_n) &= g_1(\delta_1, \delta_2, \ldots, \delta_n) \\ &= g_2(\delta_1, \delta_2, \ldots, \delta_n) \\ &= g_2(\beta_1, \beta_2, \ldots, \beta_n). \end{aligned}$$

□

We can now prove the first of the three ingredients for our theorem:

Lemma 3.11. $\mathsf{DEF} \subseteq \mathsf{FUSE}$

Proof. Assume g is an n-ary Fregean propositional function such that $g \in \mathsf{DEF}$. Let g^* be the n-ary function obtained by replacing each instance of $f_{\mathsf{NEG}}(\beta)$ with:

$$f_{\mathsf{HOR}}(f_{\mathsf{NEG}}(f_{\mathsf{HOR}}(\beta)))$$

and each occurrence of $f_{\mathsf{CON}}(\beta, \delta)$ with:

$$f_{\mathsf{HOR}}(f_{\mathsf{CON}}(f_{\mathsf{HOR}}(\beta), f_{\mathsf{HOR}}(\delta)))$$

in the definition of g. Then, by **Proposition 3.7**, we have, for any Fregean domain Δ and $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in \Delta^n$:

$$g(\alpha_1, \alpha_2, \ldots, \alpha_n) = g^*(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

The construction of g^* provides an n-ary function h such that:

$$f_{\mathsf{HOR}}(h(f_{\mathsf{HOR}}(\alpha_1), f_{\mathsf{HOR}}(\alpha_2), \ldots, f_{\mathsf{HOR}}(\alpha_n))) = g^*(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

Given any Fregean domain Δ, let:

$$\begin{aligned} &\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in \Delta^n \\ &B \subseteq \{m \in \mathbb{N} : 1 \leq m \leq n\} \end{aligned}$$

and ($1 \leq i \leq n$):

$$\beta_i = \begin{cases} f_{\mathsf{HOR}}(\alpha_i), & \text{if } i \in B; \\ \alpha_i, & \text{if } i \notin B. \end{cases}$$

Then, by **Proposition 3.7**:

$$\begin{aligned}
g(\alpha_1, \alpha_2, \ldots, \alpha_n) &= g^*(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= f_{\mathsf{HOR}}(h(f_{\mathsf{HOR}}(\alpha_1), f_{\mathsf{HOR}}(\alpha_2), \ldots, f_{\mathsf{HOR}}(\alpha_n))) \\
&= f_{\mathsf{HOR}}(f_{\mathsf{HOR}}(h(f_{\mathsf{HOR}}(\alpha_1), f_{\mathsf{HOR}}(\alpha_2), \ldots, f_{\mathsf{HOR}}(\alpha_n)))) \\
&= f_{\mathsf{HOR}}(g^*(\alpha_1, \alpha_2, \ldots, \alpha_n)) \\
&= f_{\mathsf{HOR}}(g(\alpha_1, \alpha_2, \ldots, \alpha_n))
\end{aligned}$$

$$\begin{aligned}
g(\alpha_1, \alpha_2, \ldots, \alpha_n) &= g^*(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= f_{\mathsf{HOR}}(h(f_{\mathsf{HOR}}(\alpha_1), f_{\mathsf{HOR}}(\alpha_2), \ldots, f_{\mathsf{HOR}}(\alpha_n))) \\
&= f_{\mathsf{HOR}}(h(f_{\mathsf{HOR}}(\beta_1), f_{\mathsf{HOR}}(\beta_2), \ldots, f_{\mathsf{HOR}}(\beta_n))) \\
&= g^*(\beta_1, \beta_2, \ldots, \beta_n) \\
&= g(\beta_1, \beta_2, \ldots, \beta_n)
\end{aligned}$$

$$\begin{aligned}
g(\alpha_1, \alpha_2, \ldots, \alpha_n) &= g^*(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= f_{\mathsf{HOR}}(h(f_{\mathsf{HOR}}(\alpha_1), f_{\mathsf{HOR}}(\alpha_2), \ldots, f_{\mathsf{HOR}}(\alpha_n))) \\
&= f_{\mathsf{HOR}}(f_{\mathsf{HOR}}(h(f_{\mathsf{HOR}}(\alpha_1), f_{\mathsf{HOR}}(\alpha_2), \ldots, f_{\mathsf{HOR}}(\alpha_n)))) \\
&= f_{\mathsf{HOR}}(f_{\mathsf{HOR}}(h(f_{\mathsf{HOR}}(\beta_1), f_{\mathsf{HOR}}(\beta_2), \ldots, f_{\mathsf{HOR}}(\beta_n)))) \\
&= f_{\mathsf{HOR}}(g^*(\beta_1, \beta_2, \ldots, \beta_n)) \\
&= f_{\mathsf{HOR}}(g(\beta_1, \beta_2, \ldots, \beta_n))
\end{aligned}$$

Hence, $g \in \mathsf{FUSE}$. □

The second ingredient is obtained as follows:

Lemma 3.12. $\mathsf{FUSE} \subseteq \mathsf{NTND}$

Proof. Assume g is an n-ary Fregean propositional function such that $g \in \mathsf{FUSE}$. Let Δ be any Fregean domain where $|\Delta| \geq 3$, n-tuple:

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in \Delta^n$$

and:

$$B \subseteq \{m \in \mathbb{N} : 1 \leq m \leq n\}$$

and $\delta \in \Delta$ such that $\delta \neq \top$ and $\delta \neq \bot$, where:

$$\beta_i = \begin{cases} \alpha_i, & \text{if } \alpha_i = \top \text{ or } i \notin B; \\ \delta, & \text{if } \alpha_i = \bot \text{ and } i \in B; \\ \bot, & \text{if } \alpha_i \neq \top \text{ and } \alpha_1 \neq \bot \text{ and } i \in B. \end{cases}$$

Let:

$$\delta_i^* = \begin{cases} f_{\mathsf{HOR}}(\alpha_i) = \bot, & \text{if } \alpha_i \neq \top \text{ and } \alpha_1 \neq \bot \text{ and } i \in B; \\ \alpha_i, & \text{otherwise.} \end{cases}$$

Since $g \in \mathsf{FUSE}$, we have:

$$g(\alpha_1, \alpha_2, \ldots, \alpha_n) = g(\delta_1^*, \delta_2^*, \ldots, \delta_n^*)$$

Let:

$$\delta_i^{**} = \begin{cases} f_{\mathsf{HOR}}(\beta_i) = \bot, & \text{if } \beta_i = \delta \text{ and } i \in B; \\ \alpha_i, & \text{otherwise.} \end{cases}$$

Since $g \in \mathsf{FUSE}$, we have:

$$g(\beta_1, \beta_2, \ldots, \beta_n) = g(\delta_1^{**}, \delta_2^{**}, \ldots, \delta_n^{**})$$

But $(1 \leq i \leq n)$:

$$\delta_i^* = \delta_i^{**}$$

So:

$$g(\alpha_1, \alpha_2, \ldots, \alpha_n) = g(\beta_1, \beta_2, \ldots, \beta_n)$$

Thus $g \in \mathsf{NTND}$. □

The following provides the third and final ingredient:

Lemma 3.13. $\mathsf{NTND} \subseteq \mathsf{DEF}$

Proof. Assume g is an n-ary Fregean propositional function such that $g \in \mathsf{NTND}$. The fact that the contemporary classical pair of connectives $\{\rightarrow, \neg\}$ is expressively complete for the classical propositional connectives is folklore. It follows that there is a function $h \in \mathsf{DEF}$ such that g and h agree on $\{\top, \bot\}^n$. By **Lemma 3.11** and **Lemma 3.12**, we know that $\mathsf{DEF} \subseteq \mathsf{NTND}$, so $h \in \mathsf{NTND}$. By **Lemma 3.10**, we know that h is the unique[24] function in NTND that agrees with g on $\{\top, \bot\}^n$—hence for any Fregean domain Δ and $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in \Delta^n$:

$$g(\alpha_1, \alpha_2, \ldots, \alpha_n) = h(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

Thus, $g \in \mathsf{DEF}$. □

This completes the circle:

Theorem 3.14. DEF = FUSE = NTND

Proof. Immediate from **Lemmas 3.11, 3.12** and **3.13**. □

Thus, the primitive propositional connectives of the *Grundgesetze* provide a system of propositional logic that can be characterized in three ways. We have not yet determined, however, whether DEF (= FUSE = NTND) is identical to the class of all Fregean propositional connectives FPC. A negative answer to this question will be obtained as a corollary to the results of the next section.

4. An Alternative "Fregean" System

In the previous sections we made much of the fact that, when defining the primitive propositional connectives of *Grundgesetze*, Frege divides possible arguments into two kinds: the True, and those objects that are not identical to the True. This is not the only way that Frege could have proceeded, however. Instead, he could have defined versions of the horizontal, negation, and the conditional based, not on the distinction between the True and the non-True, but on the distinction between the non-False and the False.

There is a simple way to see how such an account would proceed. We need merely reformulate Frege's informal descriptions of the semantics of each of the connectives, uniformly replacing "the True" with "not the False" and "not the True" with "the False" in the description of the arguments to be plugged into the connectives. Note that we make no alterations to the description of the outputs that result, since, in order to be a propositional connective at all, the output of each such function must be a truth-value.

If we apply this transformation to the description of the horizontal provided by Frege in §5 of *Grundgesetze*, we obtain the following:

> I regard it as a function-name such that:
>
> $$— \Delta$$
>
> is the True when Δ is not the False, and is the False when Δ is the False.

The following table summarizes the logical behavior of this alternative horizontal, which we represent as "$f^{\mathsf{Alt}}_{\mathsf{HOR}}$":

α	$f^{\mathsf{Alt}}_{\mathsf{HOR}}(\alpha)$
$\top$	$\top$
$\bot$	$\bot$
other	$\top$

Applying the transformation to (our paraphrase of) Frege's explication of negation provides:

[24]For the reader interested in proving these results in a non-extensional context: Here h would not necessarily be unique, but would be one of a class of extensionally equivalent such functions. The proof would proceed along the same lines otherwise.

> Negation is a function-name such that:
>
> $$\top\, \Delta$$
>
> is the True when Δ is the False, and is the False when Δ is not the False.

We shall use "$f^{\mathsf{Alt}}_{\mathsf{NEG}}$" to represent this propositional function. The table for this alternative version of negation is:

α	$f^{\mathsf{Alt}}_{\mathsf{NEG}}(\alpha)$
$\top$	$\bot$
$\bot$	$\top$
other	$\bot$

Finally, applying the transformation to Frege's informal description of the semantics of the conditional gives:

> I introduce the function with two arguments:
>
> $$\begin{array}{l} -\!\!\!\ulcorner\ \xi \\ \ \ \llcorner\ \zeta \end{array}$$
>
> by means of the specification that its value shall be the False if any object that is not the False is taken as the ζ argument, while the False is taken as the ξ-argument; that in all other cases the value of the function shall be the True.

This propositional connective that results, which we shall write as "$f^{\mathsf{Alt}}_{\mathsf{CON}}$", has the following table:

$f^{\mathsf{Alt}}_{\mathsf{CON}}$	$\top$	$\bot$	other
$\top$	$\top$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\top$
other	$\top$	$\bot$	$\top$

We note the (obvious) fact that these new connectives are, in fact, Fregean propositional connectives:

Proposition 4.1. $f^{\mathsf{Alt}}_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{CON}} \in \mathsf{FPC}$.

Before moving on to investigate these alternate versions of Frege's connnectives, it is worth noting that Frege's choice of f_{HOR}, f_{NEG} and f_{CON}, rather than $f^{\mathsf{Alt}}_{\mathsf{HOR}}$, $f^{\mathsf{Alt}}_{\mathsf{NEG}}$ and $f^{\mathsf{Alt}}_{\mathsf{CON}}$ (or anything else), is far from arbitrary. Given his primary purpose in formulating the logic of *Grundgesetze*—to provide gapless proofs of the *truth* of the axioms of arithmetic and real (and complex?) analysis within a formal logic—connectives that distinguish between the True and everything else make sense. After all, within a logic based on the alternative connectives just introduced, the provability of:

$$f^{\mathsf{Alt}}_{\mathsf{HOR}}(\alpha)$$

would not imply that α is the True (i.e., intuitively, that α "expressed" something true), but merely that α is not the False. Such a result would be relatively useless for Frege's purposes. Thus, the reader should not take the attention lavished on these alternative connectives as meant to suggest that they were an actual *alternative* to the primitive propositional connectives used by Frege in constructing the theorems of *Grundgesetze*.

Nevertheless, the fact that these alternative connectives are not of much use when attempting to demonstrate that certain claims are true (i.e. that particular names are names of the True) does not entail that the alternative connectives do not play *any* explicit role in *Grundgesetze*. On the contrary, in §10 of *Grundgesetze*, immediately after the notorious permutation argument, and as part of the preliminary results leading to his flawed proof (in §§30–31) that every expression of the language of *Grundgesetze* has a reference, Frege identifies the True and the False with particular value-ranges:

> Thus, without contradicting our equating '$\grave{\varepsilon}\Phi(\varepsilon) = \grave{\varepsilon}\Psi(\varepsilon)$' with '$-\!\overset{\mathfrak{a}}{\smile}\!-\ \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', it is always possible to determine that an arbitrary value-range be the True and another arbitrary value-range be the False. Let us therefore stipulate that $\grave{\varepsilon}(-\!\!-\ \varepsilon)$ be the True and

> that $\dot{\varepsilon}(\varepsilon = (\top\!\overset{\mathfrak{a}}{\smile}\!-\ \mathfrak{a} = \mathfrak{a}))$ be the False. $\dot{\varepsilon}(-\!\!-\ \varepsilon)$ is the value-range of the function $-\!\!-\ \xi$, whose value is the True only if the argument is the True, and whose value is the False for all other arguments. (Frege 2013, I: §10)

Put a bit more formally, Frege is stipulating that the following identities are to hold (even if these identities are never officially codified as Basic Laws):

$$\top = \dot{\varepsilon}(-\!\!-\ \varepsilon)$$
$$\bot = \dot{\varepsilon}(\varepsilon = (\top\!\overset{\mathfrak{a}}{\smile}\!-\ \mathfrak{a} = \mathfrak{a}))$$

Note that "$\top\!\overset{\mathfrak{a}}{\smile}\!-\ \mathfrak{a} = \mathfrak{a}$" is just a convenient name of the False. Thus, Frege identifies the True with the value-range of the concept that holds solely of the True (loosely speaking, with the singleton of the True), and identifies the False with the value-range of the concept that holds solely of the False (again, loosely speaking, with the singleton of the False). Now, the value-range identified with the False is not identical with the value range of Frege's primitive negation—that is, with $\dot{\varepsilon}(\top\ \varepsilon)$ or, in our notation, $\dot{\varepsilon}(f_{\mathsf{NEG}}(\varepsilon))$—since Frege's primitive negation outputs the True not only for the False as input, but also for any non-truth-value as input. Interestingly, however, this value-range *is* identical to the value-range of our alternative negation—that is:[25]

$$\dot{\varepsilon}(f^{\mathsf{Alt}}_{\mathsf{NEG}}(\varepsilon)) = \dot{\varepsilon}(\varepsilon = (\top\!\overset{\mathfrak{a}}{\smile}\!-\ \mathfrak{a} = \mathfrak{a}))$$

and we can reformulate Frege's identities as:[26]

$$\top = \dot{\varepsilon}(f_{\mathsf{HOR}}(\varepsilon))$$
$$\bot = \dot{\varepsilon}(f^{\mathsf{Alt}}_{\mathsf{NEG}}(\varepsilon))$$

Thus, although it does not play the role of negation in the propositional reasoning found within *Grundgesetze*—this role is reserved for f_{NEG}—our alternative negation $f^{\mathsf{Alt}}_{\mathsf{NEG}}$ does play a rather striking role within Frege's metatheoretic justification of his logic. Although we take the alternative connectives to be of inherent and independent interest in the context of Frege's logic, for reasons that will shortly be apparent, we take this center-stage appearance of the alternative negation $f^{\mathsf{Alt}}_{\mathsf{NEG}}$ to provide further justification for exploring these additional connectives.

As we shall see, these alternative connectives *are* definable within *Grundgesetze*, although they are not definable in terms of Frege's primitive propositional connectives f_{HOR}, f_{NEG} and f_{CON}. In addition, examining the alternative connectives $f^{\mathsf{Alt}}_{\mathsf{HOR}}$, $f^{\mathsf{Alt}}_{\mathsf{NEG}}$ and $f^{\mathsf{Alt}}_{\mathsf{CON}}$ in some detail will provide us with a rather elegant means to present the expressive completeness results in §5 and §6 below. In short, although there are reasons why Frege did not, and should not, have used these alternative connectives as primitive connectives used for the primary reasoning within *Grundgesetze*, he does use them (at least one of them), and, in addition, a careful examination of these alternative notions will provide a number of insights into the logical properties of the primitive propositional connectives that do appear in *Grundgesetze*.

Now that we have these alternative Fregean propositional connectives to hand, we can formulate three classes of connectives that are related to these connectives in the same way that DEF, NTND, and FUSE are related to f_{HOR}, f_{NEG} and f_{CON}. We begin with the class of connectives definable in terms of $f^{\mathsf{Alt}}_{\mathsf{HOR}}$, $f^{\mathsf{Alt}}_{\mathsf{NEG}}$ and $f^{\mathsf{Alt}}_{\mathsf{CON}}$:

Definition 4.2. *An n-ary Fregean propositional function g is an* alternative Fregean definable function *if and only if g is definable*

[25] Frege's definition, in effect, of $f^{\mathsf{Alt}}_{\mathsf{NEG}}(\xi)$ as $\xi = (\top\!\overset{\mathfrak{a}}{\smile}\!-\ \mathfrak{a} = \mathfrak{a})$, with its explicit usage of negation, foreshadows **Proposition 4.13** and **Theorem 6.9**, which state that $f^{\mathsf{Alt}}_{\mathsf{HOR}}$, $f^{\mathsf{Alt}}_{\mathsf{NEG}}$, and $f^{\mathsf{Alt}}_{\mathsf{CON}}$ are not definable in terms of f_{HOR}, f_{NEG}, and f_{CON}, but are definable in terms of f_{HOR}, f_{NEG}, f_{CON}, *and* $f_{=}$.

[26] Note that even if, on a non-extensional reading of Frege's functions, $\xi = (\top\!\overset{\mathfrak{a}}{\smile}\!-\ \mathfrak{a} = \mathfrak{a})$ and $f^{\mathsf{Alt}}_{\mathsf{NEG}}(\xi)$ are not the same function, they nevertheless have the same value-range since:

$$-\!\overset{\mathfrak{e}}{\smile}\!-\ ((\mathfrak{e} = (\top\!\overset{\mathfrak{a}}{\smile}\!-\ \mathfrak{a} = \mathfrak{a})) = f^{\mathsf{Alt}}_{\mathsf{NEG}}(\mathfrak{e}))$$

in terms of:

$$f^{\mathsf{Alt}}_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{NEG}}, \text{ and } f^{\mathsf{Alt}}_{\mathsf{CON}}$$

$\mathsf{DEF}^{\mathsf{Alt}}$ *is the class of alternative Fregean definable functions.*

Obviously:

Proposition 4.3. $f^{\mathsf{Alt}}_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{CON}} \in \mathsf{DEF}^{\mathsf{Alt}}$.

Likewise, we have an alternative notion of horizontal fusion, obtained by considering those Fregean propositional connectives that fuse with our new alternative horizontal $f^{\mathsf{Alt}}_{\mathsf{HOR}}$:

Definition 4.4. *An n-ary Fregean propositional function g is an* alternative horizontal fusing function *if and only if, for any Fregean domain* Δ, *n-tuple:*

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in \Delta^n$$

and:

$$B \subseteq \{m \in \mathbb{N} : 1 \leq m \leq n\}$$

we have:

$$\begin{aligned} g(\alpha_1, \alpha_2, \ldots, \alpha_n) &= f^{\mathsf{Alt}}_{\mathsf{HOR}}(g(\alpha_1, \alpha_2, \ldots, \alpha_n)) \\ &= g(\beta_1, \beta_2, \ldots, \beta_n) \\ &= f^{\mathsf{Alt}}_{\mathsf{HOR}}(g(\beta_1, \beta_2, \ldots, \beta_n)) \end{aligned}$$

where:

$$\beta_i = \begin{cases} f^{\mathsf{Alt}}_{\mathsf{HOR}}(\alpha_i), & \text{if } i \in B; \\ \alpha_i, & \text{if } i \notin B. \end{cases}$$

$\mathsf{FUSE}^{\mathsf{Alt}}$ *is the class of alternative horizontal fusing functions.*

The following observation is easily verifiable, and parallels **Proposition 3.7** above:

Proposition 4.5. $f^{\mathsf{Alt}}_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{CON}} \in \mathsf{FUSE}^{\mathsf{Alt}}$.

Finally, and utterly unsurprisingly given how we arrived at $f^{\mathsf{Alt}}_{\mathsf{HOR}}$, $f^{\mathsf{Alt}}_{\mathsf{NEG}}$ and $f^{\mathsf{Alt}}_{\mathsf{CON}}$ in the first place, we will consider the class of Fregean propositional functions that do not distinguish between the True and any non-truth-values:

Definition 4.6. *An n-ary Fregean propositional function g is* non-Falsity non-distinguishing *if and only if, for any Fregean domain* Δ *where* $|\Delta| \geq 3$, *n-tuple:*

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in \Delta^n$$

and:

$$B \subseteq \{m \in \mathbb{N} : 1 \leq m \leq n\}$$

and $\delta \in \Delta$ *such that* $\delta \notin \{\top, \bot\}$ *we have:*

$$g(\alpha_1, \alpha_2, \ldots, \alpha_n) = g(\beta_1, \beta_2, \ldots, \beta_n)$$

where:

$$\beta_i = \begin{cases} \alpha_i, & \text{if } \alpha_i = \bot \text{ or } i \notin B; \\ \delta, & \text{if } \alpha_i = \top \text{ and } i \in B; \\ \top, & \text{if } \alpha_i \neq \top \text{ and } \alpha_1 \neq \bot \text{ and } i \in B. \end{cases}$$

NFND *is the class of non-Falsity non-distinguishing functions.*

A binary non-Falsity non-distinguishing propositional function will have a corresponding table of the following form:

con	$\top$	$\bot$	other
$\top$	V_1	V_2	V_1
$\bot$	V_3	V_4	V_3
other	V_1	V_2	V_1

where $V_i \in \{\top, \bot\}$ for i such that $1 \leq i \leq 4$.[27] Although we will not work through the proofs of results analogous to those in §3 in detail (since they are extremely similar to the corresponding proofs found in §3), we nevertheless mention (but do not prove) a result corresponding to **Lemma 3.10** for the convenience of the reader interested in explicitly reconstructing the proofs of results in this section:

Lemma 4.7. *Given any Fregean propositional functions* $g_1, g_2 \in \mathsf{NFND}$*, if for any:*

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in \{\top, \bot\}^n$$

we have:

$$g_1(\alpha_1, \alpha_2, \ldots, \alpha_n) = g_2(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

then, given any Fregean domain Δ *and any:*

$$\langle \beta_1, \beta_2, \ldots, \beta_n \rangle \in \Delta^n$$

we have:

$$g_1(\beta_1, \beta_2, \ldots, \beta_n) = g_2(\beta_1, \beta_2, \ldots, \beta_n)$$

We now gesture at the proof that:

$$\mathsf{DEF}^{\mathsf{Alt}} = \mathsf{FUSE}^{\mathsf{Alt}} = \mathsf{NFND}$$

which, similar to the proof in §3, proceeds by proving:

$$\mathsf{DEF}^{\mathsf{Alt}} \subseteq \mathsf{FUSE}^{\mathsf{Alt}} \subseteq \mathsf{NFND} \subseteq \mathsf{DEF}^{\mathsf{Alt}}$$

Lemma 4.8. $\mathsf{DEF}^{\mathsf{Alt}} \subseteq \mathsf{FUSE}^{\mathsf{Alt}}$

Proof. Mirror image of the proof of **Lemma 3.11**. Left to the reader. □

Lemma 4.9. $\mathsf{FUSE}^{\mathsf{Alt}} \subseteq \mathsf{NFND}$

Proof. Mirror image of the proof of **Lemma 3.12**. Left to the reader. □

Lemma 4.10. $\mathsf{NFND} \subseteq \mathsf{DEF}^{\mathsf{Alt}}$

Proof. Mirror image of the proof of **Lemma 3.13**. Left to the reader. □

This completes the circle:

Theorem 4.11. $\mathsf{DEF}^{\mathsf{Alt}} = \mathsf{FUSE}^{\mathsf{Alt}} = \mathsf{NFND}$

Proof. Immediate from **Lemmas 4.8**, **4.9** and **4.10**. □

Finally, we make some observations about the connections between the $f^{\mathsf{Alt}}_{\mathsf{HOR}}$, $f^{\mathsf{Alt}}_{\mathsf{NEG}}$ and $f^{\mathsf{Alt}}_{\mathsf{CON}}$ and Frege's f_{HOR}, f_{NEG} and f_{CON}. First, we note that:

Proposition 4.12. $f_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f_{\mathsf{CON}} \notin \mathsf{NFND}$ $(= \mathsf{FUSE}^{\mathsf{Alt}} = \mathsf{DEF}^{\mathsf{Alt}})$

Proposition 4.13. $f^{\mathsf{Alt}}_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{CON}} \notin \mathsf{NTND}$ $(= \mathsf{FUSE} = \mathsf{DEF})$

[27] A question of purely technical interest: What do we get if we restrict attention within this Fregean framework, to the class of *truth-value non-distinguishing* Fregean propositional connectives, where an n-ary Fregean propositional function g is *truth-value non-distinguishing* if and only if, for any Fregean domain Δ, n-tuple:

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in \Delta^n$$

and:

$$B \subseteq \{m \in \mathbb{N} : 1 \leq m \leq n\}$$

we have:

$$g(\alpha_1, \alpha_2, \ldots, \alpha_n) = g(\beta_1, \beta_2, \ldots, \beta_n)$$

where:

$$\beta_i = \begin{cases} \top, & \text{if } \alpha_i = \bot \text{ and } i \in B; \\ \bot, & \text{if } \alpha_i = \top \text{ and } i \in B; \\ \alpha_i, & \text{otherwise.} \end{cases}$$

f_{OTH}, which we shall introduce in the next section, is truth-value non-distinguishing connective.

An immediate consequence is that neither DEF nor $\mathsf{DEF}^{\mathsf{Alt}}$ is expressively complete—that is, neither allows for the construction of all Fregean propositional connectives in FPC, or, more succinctly:

$$\mathsf{DEF} \neq \mathsf{FPC}$$
$$\mathsf{DEF}^{\mathsf{Alt}} \neq \mathsf{FPC}.$$

In the next section we shall examine various collections of connectives drawn from both DEF and $\mathsf{DEF}^{\mathsf{Alt}}$ that are expressively complete in this sense. Before doing so, however, the following observation emphasizes how minimal the overlap is between DEF and $\mathsf{DEF}^{\mathsf{Alt}}$, and hence how far short of expressive completeness each of these falls:

Theorem 4.14. *The only connectives in* FPC *that are in both* DEF *and* $\mathsf{DEF}^{\mathsf{Alt}}$ *are the constant functions.*

Proof. We will do the case for binary connectives. The generalization is straightforward. Assume $g \in \mathsf{DEF}$ and $g \in \mathsf{DEF}^{\mathsf{Alt}}$. Let:

g	$\top$	$\bot$	other
$\top$	V_1	V_2	V_3
$\bot$	V_4	V_5	V_6
other	V_7	V_8	V_9

be the table corresponding to g. Now, since $g \in \mathsf{DEF}$ (= NTND), we have:

$$V_2 = V_3$$
$$V_4 = V_7$$
$$V_5 = V_6 = V_8 = V_9$$

and since $\in \mathsf{DEF}^{\mathsf{Alt}}$ (= NFND), we have:

$$V_1 = V_3 = V_7 = V_9$$
$$V_2 = V_8$$
$$V_4 = V_6$$

It follows that:

$$V_1 = V_2 = V_3 = V_4 = V_5 = V_6 = V_7 = V_8 = V_9.$$

□

5. Combining the Systems

If we combine Frege's "official" propositional connectives with the alternative set of connectives explored in the previous section, we obtain a language that is capable of expressing all Fregean propositional connectives. In other words, this collection of connectives is expressively complete in the following sense:

Definition 5.1. *A set of Fregean propositional connectives* Σ *is* FPC-expressively complete *if and only if, for any Fregean propositional connective* $g \in \mathsf{FPC}$, g *is definable in terms of functions in* Σ.

In fact, we do not need all six connectives in $\mathsf{DEF} \cup \mathsf{DEF}^{\mathsf{Alt}}$ for FPC-expressive completeness.

To begin, we will first determine which subsets of these six connectives are not expressively complete in the relevant sense, and then show that the remaining subsets are expressively complete in this manner. Our first two results are immediate corollaries of results in the previous section, but are worth noting explicitly:

Theorem 5.2. *No subset of:*

$$\{f_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f_{\mathsf{CON}}\}$$

is FPC*-expressively complete.*

Proof. $f^{\mathsf{Alt}}_{\mathsf{HOR}} \notin \mathsf{NTND}$ (= DEF). □

Theorem 5.3. *No subset of:*

$$\{f^{\mathsf{Alt}}_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{CON}}\}$$

is FPC*-expressively complete.*

Proof. $f_{\mathsf{HOR}} \notin \mathsf{NFND}$ $(= \mathsf{DEF}^{\mathsf{Alt}})$. □

The next result is equally straightforward:

Theorem 5.4. *No subset of:*

$$\{f_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{NEG}}\}$$

is FPC*-expressively complete.*

Proof. No n-ary connective, for $n \geq 2$, is definable in terms of:

$$\{f_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{NEG}}\}.$$

□

One more theorem completes the negative results:

Theorem 5.5. *No subset of:*

$$\{f_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{HOR}}, f_{\mathsf{CON}}, f^{\mathsf{Alt}}_{\mathsf{CON}}\}$$

is FPC*-expressively complete.*

Proof. Given any n-ary Fregean connective g definable in terms of:

$$\{f_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{HOR}}, f_{\mathsf{CON}}, f^{\mathsf{Alt}}_{\mathsf{CON}}\}$$

we have $g(\top, \top, \dots \top) = \top$ (by induction on the complexity of g). □

With non-FPC-expressive completeness out of the way, we can now show that the remaining subsets of:

$$\{f_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{NEG}}, f_{\mathsf{CON}}, f^{\mathsf{Alt}}_{\mathsf{CON}}\}$$

are FPC-expressively complete. First, we note that, for any α:

Proposition 5.6.

$$f_{\mathsf{HOR}}(\alpha) = f^{\mathsf{Alt}}_{\mathsf{NEG}}(f_{\mathsf{CON}}(x, f^{\mathsf{Alt}}_{\mathsf{NEG}}(\alpha)))$$
$$f^{\mathsf{Alt}}_{\mathsf{HOR}}(\alpha) = f_{\mathsf{NEG}}(f^{\mathsf{Alt}}_{\mathsf{CON}}(x, f_{\mathsf{NEG}}(\alpha)))$$

Also, negation is definable in terms of the alternative negation and the horizontal, and the alternative negation is definable in terms of negation and the alternative horizontal, since for any α:

Proposition 5.7.

$$f_{\mathsf{NEG}}(\alpha) = f^{\mathsf{Alt}}_{\mathsf{NEG}}(f_{\mathsf{HOR}}(\alpha))$$
$$f^{\mathsf{Alt}}_{\mathsf{NEG}}(\alpha) = f_{\mathsf{NEG}}(f^{\mathsf{Alt}}_{\mathsf{HOR}}(\alpha))$$

The following connective, which we shall call the *other* connective (or f_{OTH}), will be useful, and is definable in two ways:

$$\begin{aligned} f_{\mathsf{OTH}}(\xi) &= f_{\mathsf{NEG}}(f_{\mathsf{CON}}(f_{\mathsf{NEG}}(\xi), f^{\mathsf{Alt}}_{\mathsf{NEG}}(\xi))) \\ &= f_{\mathsf{NEG}}(f^{\mathsf{Alt}}_{\mathsf{CON}}(f_{\mathsf{NEG}}(\xi), f^{\mathsf{Alt}}_{\mathsf{NEG}}(\xi))) \end{aligned}$$

The behavior of f_{OTH} is summarized by the following table:

α	$f_{\mathsf{OTH}}(\alpha)$
$\top$	$\bot$
$\bot$	$\bot$
other	$\top$

Note that:[28]

$$f_{\mathsf{OTH}} \notin \mathsf{NTND}$$
$$f_{\mathsf{OTH}} \notin \mathsf{NFND}$$

Combining these results and the definition of f_{OTH} provides the following useful lemma:

Lemma 5.8. *Given any set of Fregean connectives Σ, if either:*

f_{NEG} and $f^{\mathsf{Alt}}_{\mathsf{CON}}$ are definable in terms of connectives in Σ.

or:

[28] f_{OTH} is a truth-value non-distinguishing connective—see note 27.

$f^{\mathsf{Alt}}_{\mathsf{NEG}}$ and f_{CON} are definable in terms of connectives in Σ.

then all of:

$$f_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{NEG}}, f_{\mathsf{OTH}}$$

are definable in terms of connectives in Σ.

Proof. Straightforward from **Propositions 5.6** and **5.7** and the definition of f_{OTH}. □

Finally, we need to introduce some defined conjunctions and disjunctions. Frege provides us with a recipe for both disjunction and conjunction defined in terms of (modulo our change in notation) f_{NEG} and f_{CON}. For example, Frege defines disjunction as:

$$f_{\mathsf{OR}}(\xi_1, \xi_2) = f_{\mathsf{CON}}(f_{\mathsf{NEG}}(\xi_1), \xi_2)$$

(translating his treatment into the present notation, see Frege 2013, I: §12), and he defines conjunction as:

$$f_{\mathsf{AND}}(\xi_1, \xi_2) = f_{\mathsf{NEG}}(f_{\mathsf{CON}}(\xi_1, f_{\mathsf{NEG}}(\xi_2)))$$

(again, see Frege 2013, I: §12). The tables for Frege's conjunction and disjunction are:

f_{OR}	$\top$	$\bot$	other
$\top$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\bot$
other	$\top$	$\bot$	$\bot$

f_{AND}	$\top$	$\bot$	other
$\top$	$\top$	$\bot$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$
other	$\bot$	$\bot$	$\bot$

In the present context, however, we will also be considering cases where we need to define a conjunction and disjunction in terms of $f^{\mathsf{Alt}}_{\mathsf{NEG}}$ and $f^{\mathsf{Alt}}_{\mathsf{CON}}$. We can do so straightforwardly by mimicking the structure of Frege's definitions to arrive at:

$$f^{\mathsf{Alt}}_{\mathsf{OR}}(\xi_1, \xi_2) = f^{\mathsf{Alt}}_{\mathsf{CON}}(f^{\mathsf{Alt}}_{\mathsf{NEG}}(\xi_1), \xi_2)$$
$$f^{\mathsf{Alt}}_{\mathsf{AND}}(\xi_1, \xi_2) = f^{\mathsf{Alt}}_{\mathsf{NEG}}(f^{\mathsf{Alt}}_{\mathsf{CON}}(\xi_1, f^{\mathsf{Alt}}_{\mathsf{NEG}}(\xi_2)))$$

The tables for these connectives are:

$f^{\mathsf{Alt}}_{\mathsf{OR}}$	$\top$	$\bot$	other
$\top$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$
other	$\top$	$\top$	$\top$

$f^{\mathsf{Alt}}_{\mathsf{AND}}$	$\top$	$\bot$	other
$\top$	$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
other	$\top$	$\bot$	$\top$

respectively. As we would expect, all of f_{OR}, $f^{\mathsf{Alt}}_{\mathsf{OR}}$, f_{AND}, or $f^{\mathsf{Alt}}_{\mathsf{AND}}$ are associative and commutative, so we may write:[29]

$$f_{\mathsf{OR}}(\xi_1, \xi_2, \dots \xi_{n-1}, \xi_n)$$
$$f^{\mathsf{Alt}}_{\mathsf{OR}}(\xi_1, \xi_2, \dots \xi_{n-1}, \xi_n)$$
$$f_{\mathsf{AND}}(\xi_1, \xi_2, \dots \xi_{n-1}, \xi_n)$$
$$f^{\mathsf{Alt}}_{\mathsf{AND}}(\xi_1, \xi_2, \dots \xi_{n-1}, \xi_n)$$

for:

$$f_{\mathsf{OR}}(\xi_1, (f_{\mathsf{OR}}(\xi_2, \dots f_{\mathsf{OR}}(\xi_{n-1}, \xi_n) \dots)))$$
$$f^{\mathsf{Alt}}_{\mathsf{OR}}(\xi_1, (f^{\mathsf{Alt}}_{\mathsf{OR}}(\xi_2, \dots f^{\mathsf{Alt}}_{\mathsf{OR}}(\xi_{n-1}, \xi_n) \dots)))$$
$$f_{\mathsf{AND}}(\xi_1, (f_{\mathsf{AND}}(\xi_2, \dots f_{\mathsf{AND}}(\xi_{n-1}, \xi_n) \dots)))$$
$$f^{\mathsf{Alt}}_{\mathsf{AND}}(\xi_1, (f^{\mathsf{Alt}}_{\mathsf{AND}}(\xi_2, \dots f^{\mathsf{Alt}}_{\mathsf{AND}}(\xi_{n-1}, \xi_n) \dots)))$$

respectively, and permute the order of arguments in such expressions at will.

We are now ready to prove our main expressive completeness results. Our two main theorems follow immediately:

Theorem 5.9. *Any superset (relative to our six connectives) of:*

$$\{f_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{CON}}\}$$

is FPC*-expressively complete.*

[29]Thanks are owed to a referee for pointing out a very embarrassing technical mistake hereabouts in an earlier version of this paper.

Proof. By **Lemma 5.8** above, $f^{\mathsf{Alt}}_{\mathsf{NEG}}$, f_{HOR} and f_{OTH} are expressible in terms of f_{NEG} and $f^{\mathsf{Alt}}_{\mathsf{CON}}$. Additionally, $f^{\mathsf{Alt}}_{\mathsf{AND}}$, and $f^{\mathsf{Alt}}_{\mathsf{OR}}$ are defined in terms of $f^{\mathsf{Alt}}_{\mathsf{CON}}$ and f_{NEG}, so are definable as well. Assume g is an n-ary function in FPC. Let α be any object not in $\{\top, \bot\}$. Then, for any n-tuple $S = \langle v_1, v_2, \ldots v_n\rangle \in \{\top, \bot, \alpha\}^n$, let:

$$\Phi_S = f^{\mathsf{Alt}}_{\mathsf{AND}}(\beta_1, \beta_2, \ldots \beta_n)$$

Where:

$$b_n = \begin{cases} f_{\mathsf{HOR}}(\xi_n), & \text{if } v_n = \top; \\ f^{\mathsf{Alt}}_{\mathsf{NEG}}(\xi_n), & \text{if } v_n = \bot; \\ f_{\mathsf{OTH}}(\xi_n), & \text{if } v_n = \alpha. \end{cases}$$

Let:

$$h(\xi_1, \xi_2, \ldots \xi_m) = f^{\mathsf{Alt}}_{\mathsf{OR}}(\{\Phi_S : g(S) = \top\})$$

Then $g = h$. □

Unsurprisingly, **Theorem 5.9** has a (not-so-evil) twin:

Theorem 5.10. *Any superset (relative to our six connectives) of:*

$$\{f^{\mathsf{Alt}}_{\mathsf{NEG}}, f_{\mathsf{CON}}\}$$

is FPC*-expressively complete.*

Proof. Proceed as in **Theorem 5.9**, replacing $f^{\mathsf{Alt}}_{\mathsf{OR}}$ and $f^{\mathsf{Alt}}_{\mathsf{AND}}$ with f_{OR} and f_{AND} respectively. □

The following are worth noting:

Corollary 5.11. *Any superset (relative to our six connectives) of:*

$$\{f_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{CON}}\}$$

is FPC*-expressively complete.*

Proof. By **Proposition 5.7**, f_{NEG} is definable in terms of f_{HOR} and $f^{\mathsf{Alt}}_{\mathsf{NEG}}$. We then apply **Theorem 5.9**. □

Corollary 5.12. *Any superset (relative to our six connectives) of:*

$$\{f^{\mathsf{Alt}}_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f_{\mathsf{CON}}\}$$

is FPC*-expressively complete.*

Proof. By **Proposition 5.7**, $f^{\mathsf{Alt}}_{\mathsf{NEG}}$ is definable in terms of $f^{\mathsf{Alt}}_{\mathsf{HOR}}$ and f_{NEG}. We then apply **Theorem 5.10**. □

The previous results regarding FPC-expressive incompleteness imply that:

$$\{f_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{CON}}\}$$
$$\{f^{\mathsf{Alt}}_{\mathsf{NEG}}, f_{\mathsf{CON}}\}$$
$$\{f_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{CON}}\}$$
$$\{f^{\mathsf{Alt}}_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f_{\mathsf{CON}}\}$$

are minimal—that is, that no proper subset of any of these four sets is FPC-expressively complete. The reader is encouraged to verify (tediously) that these results exhaust all possible subsets of:[30]

$$\{f_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{NEG}}, f_{\mathsf{CON}}, f^{\mathsf{Alt}}_{\mathsf{CON}}\}$$

6. Adding Identity

Thus, any Fregean-style propositional logic that contains the resources to define either f_{NEG} and $f^{\mathsf{Alt}}_{\mathsf{CON}}$, or $f^{\mathsf{Alt}}_{\mathsf{NEG}}$ and f_{CON}, is sufficiently expressive to represent *any* Fregean propositional function. But so what? We have also shown that the set of propositional connectives Frege actually introduces—that is, f_{HOR}, f_{NEG}, and f_{CON}—are not expressively complete in this sense.

[30] We leave it as an exercise for the reader to determine which subsets of:

$$\{f_{\mathsf{HOR}}, f^{\mathsf{Alt}}_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f^{\mathsf{Alt}}_{\mathsf{NEG}}, f_{\mathsf{CON}}, f^{\mathsf{Alt}}_{\mathsf{CON}}, f_{\mathsf{OTH}}\}$$

are FPC-expressively complete.

So what does any of this have to do with the actual formal logic of *Grundgesetze*?

The answer to this question is simple: The formal language developed in the *Grundgesetze* is, in fact, able to express every Fregean propositional function. It is just not able to do so in terms of the primitive propositional functions of the *Grundgesetze* itself. Instead, we need to use the identity function.

To begin, let us note the following obvious fact explicitly:

Theorem 6.1. $f_{=} \notin \mathsf{FPC}$

Proof. Let Δ be a Fregean domain where $|\Delta| \geq 4$, and let α and β be any two distinct objects in Δ other than $\top$ and $\bot$. Then $f_{=}(\alpha, \alpha) = \top \neq \bot = f_{=}(\alpha, \beta)$, violating our definition of Fregean propositional connective. □

In other words, Fregean propositional connectives, as we have understood them here (and in accordance with what we take to be the most natural way to understand the concept of propositional connective within Frege's framework) cannot distinguish between different non-truth-values. The identity function can do so, however, so it is not a Fregean propositional connective.

The following pair of definitions make the distinction between the class of Fregean propositional connectives definable in terms of f_{HOR}, f_{NEG}, f_{CON} and $f_{=}$ and the wider class of *all* functions (propositional or not) whose ranges are $\{\top, \bot\}$ definable in terms of this same collection explicit:

Definition 6.2. *An n-ary function g is a* Fregean identity-definable function *if and only if g is a function mapping n-tuples from a Fregean domain* Δ *to* $\{\top, \bot\}$ *and g is definable in terms of:*

$$f_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f_{\mathsf{CON}} \text{ and } f_{=}$$

$\mathsf{DEF}^{\mathsf{Fun}}_{=}$ *is the class of Fregean identity-definable functions.*

Definition 6.3. *An n-ary function g is a* Fregean identity-definable connective *if and only if g is a Fregean propositional connective and g is definable in terms of:*

$$f_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f_{\mathsf{CON}} \text{ and } f_{=}$$

$\mathsf{DEF}^{\mathsf{Con}}_{=}$ *is the class of Fregean identity-definable connectives.*[31]

We note the following now-obvious facts:

Proposition 6.4. $f_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f_{\mathsf{CON}}, f_{=} \in \mathsf{DEF}^{\mathsf{Fun}}_{=}$

Proposition 6.5. $f_{\mathsf{HOR}}, f_{\mathsf{NEG}}, f_{\mathsf{CON}} \in \mathsf{DEF}^{\mathsf{Con}}_{=}$.

Proposition 6.6. $f_{=} \notin \mathsf{DEF}^{\mathsf{Con}}_{=}$.

At first glance, it might look like we haven't gained much, since $f_{=}$ isn't itself a new Fregean propositional connective not definable in terms of f_{HOR}, f_{NEG} and f_{CON}. But even though $f_{=}$ is not a "new" propositional connective, it, in combination with f_{HOR}, f_{NEG} and f_{CON}, allows us to define propositional connectives not definable in terms of f_{HOR}, f_{NEG} and f_{CON} alone. A striking example of just such a function is provided by the "other" connective f_{OTH}, which played a central role in the proofs of **Theorem 5.9** and **5.10** above. f_{OTH} is definable in terms of f_{HOR}, f_{NEG}, f_{CON}, and $f_{=}$:

Proposition 6.7. $f_{\mathsf{OTH}}(\xi) = f_{\mathsf{NEG}}(f_{=}(\xi, f_{\mathsf{HOR}}(\xi)))$

Given the definability of f_{OTH}, however, we can apply the results of the previous section to show that every Fregean propositional function is definable in terms of f_{HOR}, f_{NEG}, f_{CON}, and $f_{=}$. The key is to note that $f^{\mathsf{Alt}}_{\mathsf{NEG}}$ is definable once the identity function is in play:

[31] We could, of course, also investigate the *alternative* class of connectives $f^{\mathsf{Alt}}_{\mathsf{HOR}}$, $f^{\mathsf{Alt}}_{\mathsf{NEG}}$, f_{CON}, and $f_{=}$ (or various combinations of Alt and non-Alt connectives), constructing analogous notions such as the class of connectives $\mathsf{DEF}^{\mathsf{Alt}}_{=}$ definable in terms of the three alternative connectives and $f_{=}$. Results for such systems, analogous to those presented here for Frege's actual notions, are simple to obtain, but are left to the reader.

Proposition 6.8.

$$f^{\mathsf{Alt}}_{\mathsf{NEG}}(\xi) = f_{\mathsf{NEG}}(f_{\mathsf{CON}}(f_{\mathsf{NEG}}(\xi), f_{\mathsf{OTH}}(\xi)))$$

Given the fact that we can define alternative negation $f^{\mathsf{Alt}}_{\mathsf{NEG}}$ in terms of f_{HOR}, f_{NEG}, f_{CON}, and $f_{=}$, we get the following as an immediate consequence:

Theorem 6.9. $\mathsf{FPC} = \mathsf{DEF}^{\mathsf{Con}}_{=} \subsetneq \mathsf{DEF}^{\mathsf{Fun}}_{=}$

Proof. Combine **Theorem 5.10** with **Propositions 6.6, 6.7** and **6.8**. □

Let us now return to our motivating question:

Question: What, exactly, *is* Frege's propositional logic?

The answer, as we suggested it would be in the introduction, is complicated.

On the one hand, Frege clearly has primitive operators that, although they behave somewhat differently from modern propositional connectives (being total functions from the domain to $\{\top, \bot\}$, for example), are clearly best thought of as genuine propositional connectives. His primitive operators f_{HOR}, f_{NEG}, and f_{CON} are the paradigm examples, as are any propositional connectives definable in terms of these three notions (i.e. all propositional connectives in DEF). There is no doubt that Frege's deductive system allows us to manipulate these notions in complex and subtle ways. Further, there can be no doubt that, in formulating the logic of *Grundgesetze*, Frege must have carefully considered the manner in which these propositional operators behave. So in this sense, there is a simple and obvious case to be made for the claim that the propositional logic of *Grundgesetze* is the subsystem of *Grundgesetze* containing the horizontal, negation, and the conditional stroke (and nothing else).[32]

On the other hand, a slightly different perspective provides a somewhat different answer. If we are asking whether the logic of the *Grundgesetze* contains a separable subsystem that encompasses exactly the propositional core of the system, and we understand that core as encompassing all and only those functions that (i) map objects to truth-values and (ii) cannot distinguish between distinct non-truth-values (i.e. exactly the Fregean propositional connectives $\mathsf{FPC} = \mathsf{DEF}^{\mathsf{CON}}_{=}$), then the results developed above show that Frege's propositional logic, on this understanding, outstrips DEF and is not separable in the sense explicated above. Frege's system does allow us to express every Fregean propositional connective, but in order to do so we must make use of notions that go beyond the primitive propositional connectives themselves, using notions ($f_{=}$ in particular) that are not themselves Fregean propositional connectives. In short, on this understanding of propositional logic, Frege does not explicitly identify propositional logic as a significant and separate subsystem of the logic of *Grundgesetze*, and he could not have done so in principle, since propositional logic (on this understanding) is, as a matter of mathematical fact, not a separate or separable subsystem given the primitives he chose for his system. Simply put, on this reading there is no sub-collection of the primitive notions of *Grundgesetze* such that the functions definable in terms of that set are exactly FPC.

The reader might object at this point that this second perspective seems worryingly anachronistic. Why should we care if Frege could isolate something that looks like modern propositional logic within his system? After all, the subsystem of propositional logic obtained by restricting attention to f_{HOR}, f_{NEG}, and f_{CON} obviously plays a central role in Frege's system,

[32] It is also worth noting that Frege provides a "separable", deductively complete proof system for the DEF subsystem consisting of his propositional rules of inference and Basic Law I. See, e.g., Landini (2012) for discussion. We plan to present a detailed examination of, and comparison between, the deductive theories corresponding to DEF and $\mathsf{DEF}^{\mathsf{Con}}_{=}$ in a sequel to the present paper.

since it consists of exactly those connectives that fuse with f_{HOR}, which itself plays a special role in terms of both the formulation of, and the assertion of, axioms and theorems within the logic of *Grundgesetze*. For example, the horizontal plays an essential role in the statement of Basic Law IV—see the discussion in the next section. In addition, Frege claims in §6 of *Grundgesetze* that the judgement stroke $\vdash$ is *composed of* two symbols: a horizontal and the vertical assertion stroke proper.[33] Why worry about whether or not Frege's system allows us to isolate FPC, since it seems unlikely that Frege himself could have separated (or would have been interested in separating) this sub-system, given his own goals and purposes?[34]

Of course, the previous paragraph sets up a bit of a straw man, since there are obvious reasons why philosophers and historians of logic might be interested in exploring technical, philosophical, and historical connections between logic as it is currently understood and the logical systems that lie at the historical origin of modern formal logic.[35] But setting the worry up in this way allows us to raise another kind of question. While it is quite right that Frege would have been more interested in the class of functions that are definable in terms of his primitive notions f_{HOR}, f_{NEG}, and f_{CON} than in the more general (but not separable) class of all Fregean propositional connectives FPC, he would also quite naturally have been interested in all those functions definable in terms of his three propositional connectives plus $f_{=}$, since this is another sub-system of *Grundgesetze* that is easily isolated from Frege's higher-order, value-range-involving logic as a whole. Thus, we will conclude the paper by making some observations regarding what, exactly, one gets when one considers the entire system obtained via f_{HOR}, f_{NEG}, f_{CON}, and $f_{=}$.

Before doing so, however, we shall take a slight detour, making some observations regarding the role of the identity function within the *Grundgesetze* and in particular its use as an analogue for the material biconditional.

7. Identity and Biconditionality

Frege scholars often note that, given Frege's understanding of sentences as names of truth-values, the binary identity function $f_{=}$ plays a dual role: it is used by Frege to express both that two objects are identical and that two sentences are materially equivalent (since on Frege's understanding two sentences will be materially equivalent if and only if they name the same truth-value).[36] For example, in previous work one of the authors described Frege's use of the identity function as follows:

> As a result, within the formalism of *Grundgesetze* the equality-sign plays (to modern eyes) two distinct roles. When attached to proper names generally, it provides the truth-value of the claim that those names are names of the same object. When attached to truth-value-names, however, it provides the truth-value of the

[33]It is interesting to compare this claim to superficially similar claims about the connection between the horizontal and the other connectives (and the concavity), which are phrased in terms of our being permitted to "regard" the connective in a certain way.

[34]A version of this worry was first brought to our attention by David Taylor, and §8 exists in great part because of his pressing this issue and related worries. Thanks are due to him for raising the questions addressed here and in §8, although any dissatisfaction with our discussion should be blamed on the authors, and not on David.

[35]In addition, recall the (non-inferential) role that $f^{\mathsf{Alt}}_{\mathsf{NEG}}$ plays in §10 of *Grundgesetze*.

[36]We should clearly distinguish between a *sentence*—that is, a name of a truth-value—and an *assertion*—that is, an expression that manifests a *judgement*. An assertion (or, in *Grundgesetze*, a *concept-script proposition*, see §26) is the result of prefixing a sentence with the judgement stroke $\vdash$, and assertions express the content of judgements on Frege's account. Some scholars prefer to reserve the term "sentence" for what we are here calling assertions, but this is merely a terminological matter upon which nothing substantial hinges (assuming that we are clear and consistent with respect to how we are using these terms).

claim that those truth-value names denote the same truth-value—that is, it plays something analogous to the role of the material biconditional within modern logical calculi. Of course, on Frege's understanding these are not really two separate tasks. Rather, the biconditional reading of the equality sign is merely just a special case of the more general "identity" reading. (Cook 2013, A12)

This is certainly right as far as it goes: whenever Frege is interested in expressing that two sentences are materially equivalent, he expresses this in terms of identity. A simple example is given by any instance of Basic Law IV. If α and β are names of any object then Basic Law IV provides:[37]

$$\vdash\!\!\!\begin{array}{l} (-\,\alpha) = (-\,\beta) \\ \llcorner\ (-\,\alpha) = (\neg\,\beta) \end{array}$$

But, as we have already emphasized above, the binary identity function is *not* a Fregean propositional connective, since it can distinguish between non-truth-values (something no genuine propositional connective can do). Of course, with respect to Frege's actual constructions, this does not matter, since Frege is careful throughout *Grundgesetze* to use identity as an analogue of the biconditional only in those contexts when both arguments are guaranteed to be truth-value names (i.e. genuine sentences, and not names of non-truth-values). Note, for example, his careful insertion of horizontals in Basic Law IV above. The analogous "law" without the horizontals guaranteeing that the arguments flanking the identity symbols are truth-values—that is:

$$\vdash\!\!\!\begin{array}{l} a = b \\ \llcorner\ a = (\neg\, b) \end{array}$$

is invalid:[38] just let a and b be any two distinct non-truth-values, and the subcomponent is true, while the supercomponent is false.[39] Nevertheless, it is worth examining whether *Grundgesetze* is able to express something like a "true" biconditional, and if so, why Frege did not use such a construction when formulating the laws of the formal system contained therein. The first task involved in answering this question is to determine what, exactly, might count as an analogue of the modern material biconditional within Frege's framework.

If the biconditional is to be a binary Fregean propositional connective (and surely it must be such), then it will be expressible as a 3×3 matrix similar to those given above for the conditional and the various constructed conjunctions and disjunctions. Further, it seems like the following two principles must hold of any binary connective worthy of the label "biconditional":

1. If a binary function g is a *material biconditional*, then, if α is a truth-value, then $g(\alpha, \alpha) = \top$.
2. If a binary function g is a *material biconditional*, then, if α is a truth-value, and β is any object distinct from α, then $g(\alpha, \beta) = g(\beta, \alpha) = \bot$.

These two principles do not, however, isolate a single Fregean propositional connective. Instead, there are two distinct Fregean propositional connectives that satisfy (1) and (2), which we shall

[37]Note that, if α is an object, then $-\!\!-\,\alpha$ is a truth-value, and hence "$-\!\!-\,\alpha$" is a sentence (as is "$\neg\,\alpha$").

[38]Interestingly, the analogue of this formula might be valid within the logic of *Begriffsschrift*, since it seems that in *Begriffsschrift*, unlike *Grundgesetze*, identity (strictly speaking, identity of content $\equiv$) is intersubstitutable with the conjunction of two conditionals. See note 11.

[39]Another way of putting this point is as follows: The identity function "fuses" with external applications of the horizontal—that is, for all x, y:

$$f_{=}(\xi_1, \xi_2) = f_{\mathsf{HOR}}(f_{=}(\xi_1, \xi_2))$$

but does not fuse with "internal" applications of the horizontal—that is, none of:

$$\begin{aligned} f_{=}(\xi_1, \xi_2) &= f_{=}(f_{\mathsf{HOR}}(\xi_1), \xi_2) \\ f_{=}(\xi_1, \xi_2) &= f_{=}(\xi_1, f_{\mathsf{HOR}}(\xi_2)) \\ f_{=}(\xi_1, \xi_2) &= f_{=}(f_{\mathsf{HOR}}(\xi_1), f_{\mathsf{HOR}}(\xi_2)) \end{aligned}$$

are true.

call f_{BIC_1} and f_{BIC_2}:

f_{BIC_1}	$\top$	$\bot$	other
$\top$	$\top$	$\bot$	$\bot$
$\bot$	$\bot$	$\top$	$\bot$
other	$\bot$	$\bot$	$\top$

f_{BIC_2}	$\top$	$\bot$	other
$\top$	$\top$	$\bot$	$\bot$
$\bot$	$\bot$	$\top$	$\bot$
other	$\bot$	$\bot$	$\bot$

The difference between the two lies in how they handle argument pairs where both arguments are non-truth-values. f_{BIC_1} maps such pairs to the True, whereas f_{BIC_2} maps such pairs to the False. In short, the difference between f_{BIC_1} and f_{BIC_2} depends on how far "down" the diagonal of "$\top$"s should extend.

Fortunately, for our purposes we need not settle on one or the other of these as the "correct" understanding of the biconditional (understood as a genuine propositional connective) within the context of *Grundgesetze*. Instead, the observations we wish to make apply to both.

First, it is worth noting that neither of these biconditionals is definable in terms of the horizontal, negation, and the conditional:[40]

Theorem 7.1. *$f_{\mathsf{BIC}_1} \notin \mathsf{DEF}$, $f_{\mathsf{BIC}_2} \notin \mathsf{DEF}$.*

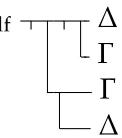
Proof. $f_{\mathsf{BIC}_1} \notin \mathsf{NTND}$ (= DEF), $f_{\mathsf{BIC}_2} \notin \mathsf{NTND}$ (= DEF). □

At first glance this might seem odd, since, as we have already discussed, Frege *does* provide an explicit definition of conjunction in terms of the conditional and negation. As a result, surely we can construct an adequate biconditional merely by following the familiar recipe:

$$\Phi \equiv \Psi =_{\text{df}} (\Phi \rightarrow \Psi) \wedge (\Psi \rightarrow \Phi)$$

[40]Via similar reasoning, neither f_{BIC_1} nor f_{BIC_1} is definable in terms of $f^{\mathsf{Alt}}_{\mathsf{HOR}}$, $f^{\mathsf{Alt}}_{\mathsf{NEG}}$, and $f^{\mathsf{Alt}}_{\mathsf{CON}}$.

That is, we can construct a biconditional within *Grundgesetze* as:[41]

$\Delta \equiv_{\text{Grund}} \Gamma =_{\text{df}}$ 

This connective—which we shall now represent as "$f_{\equiv}$"—corresponds to the following table:

$f_{\equiv}$	$\top$	$\bot$	other
$\top$	$\top$	$\bot$	$\bot$
$\bot$	$\bot$	$\top$	$\top$
other	$\bot$	$\top$	$\top$

and is not equivalent to either f_{BIC_1} or f_{BIC_2}. In particular, it fails to satisfy the second of the two desiderata given above for a genuine biconditional.

Of course, both of f_{BIC_1} and f_{BIC_2} are expressible in terms of negation, the horizontal, the conditional, *and identity*. **Theorem 6.9** above guarantees this.[42] Thus, either of these propositional connectives (f_{BIC_1} and f_{BIC_2}) are expressible in terms of the resources contained in *Grundgesetze*, but neither is expressible solely in terms of the primitive propositional connectives given in *Grundgesetze*. Thus, we have an additional, natural example (actually, two such examples) of the expressive limitations of the propositional connectives that are explicitly contained in *Grundgesetze*—that is, f_{HOR}, f_{NEG}, and f_{CON}.

A final question remains: If Frege had these, in some sense "better", versions of the biconditional at his disposal (even if

[41]The occurrence of "Grund" in the offset formula is intended to emphasize that this version of the biconditional is *not* the conceptual identity relation found in *Begriffsschrift*.

[42]We leave it to the reader to construct explicit definitions of f_{BIC_1} and f_{BIC_2} in terms of f_{NEG}, f_{CON}, and $f_{=}$ via following the recipe implicit in the proof of **Theorem 5.9**.

he had to move beyond the purely propositional resources of *Grundgesetze* in order to formulate them) then why did he not make use of them—that is, why did he not formulate his Basic Laws in terms of f_{BIC_1} or f_{BIC_2}?

The answer to this question is simple, straightforward, and somewhat obvious: Given the primitives that are present in the logic of *Grundgesetze*, the constructions of f_{BIC_1} and f_{BIC_2} are complicated. Furthermore, for Frege's purposes, he did not *need* either of these "improved" biconditionals, since he was primarily concerned, in formulating Basic Laws and theorems of *Grundgesetze*, in expressing biconditionals that hold between sentences—that is, between expressions of *Grundgesetze* that refer to truth-values. And when restricted to expressions referring to truth-values, all of the different understandings of the biconditional considered in this section are equivalent. In short, we do have the following:

Theorem 7.2. *For any* Δ *such that* $|\Delta| \geq 4$*, and any distinct propositional functions:*

$$g, h \in \{f_{=}, f_{\equiv}, f_{\mathsf{BIC}_1}, f_{\mathsf{BIC}_2}\}$$

there are $\alpha, \beta \in \Delta$ *such that:*

$$g(\alpha, \beta) \neq h(\alpha, \beta)$$

Proof. Straightforward application of facts informally discussed above. □

The practical import of **Theorem 7.2** for Frege's project is blunted by the next result, however:

Theorem 7.3. *For any* Δ*, and any propositional function:*

$$g, h \in \{f_{=}, f_{\mathsf{BIC}_1}, f_{\mathsf{BIC}_2}\}$$

and any $\alpha, \beta \in \Delta$*:*

$$g(f_{\mathsf{HOR}}(\alpha), f_{\mathsf{HOR}}(\beta)) = h(f_{\mathsf{HOR}}(\alpha), f_{\mathsf{HOR}}(\beta)) = f_{\equiv}(\alpha, \beta)$$

Proof. Straightforward application of facts informally discussed above. □

As we have already noted in our brief discussion of Basic Law IV above, Frege in effect applies this result when formulating the Basic Laws governing propositional logic via judicious insertion of horizontals. Thus, he had no real need to utilize the "better" biconditionals f_{BIC_1} and f_{BIC_2}.

8. Identity and Logical Permutations

Now that we have looked at Frege's use of the identity function $f_{=}$ in more detail, our final task is to characterize the class of functions that are definable in terms of Frege's primitive propositional connectives and identity. Before proving a bunch of theorems, it is worth noting that the system obtained by considering all functions definable in terms of f_{HOR}, f_{NEG}, f_{CON}, and $f_{=}$ is not only inherently interesting, but would have been interesting to Frege himself. The reason is simple: These four functions are the only primitive first-level functions (i.e. functions mapping objects to objects) that appear in *Grundgesetze* whose behavior is independent of value-ranges.

Frege does have one additional primitive first-level function—the backslash "\". But this function—what one of us has elsewhere (Cook 2013) called the "singletons stripping operator"—treats inputs differently depending on whether or not they are value-ranges.[43] Thus, although the backslash is a first-level function, the intended use of this function requires that one apply it to a complex expression that involves second-level functions (the value-range operator in particular). Hence the use of the backslash \ within *Grundgesetze* is tied to second-level functions in a manner in which f_{HOR}, f_{NEG}, f_{CON}, and $f_{=}$ are not.

[43] In particular, $\backslash(x)$ is y if x is the value range of the concept that holds solely of y, and is x itself otherwise.

Thus, our final task is to provide a characterization of $\mathsf{DEF}^{\mathsf{FUN}}_{=}$. As before, we will freely use contemporary ideas and techniques to characterize this class of functions. What is particularly interesting is that $\mathsf{DEF}_{=}$ is straightforwardly characterizable in terms of a notion that would have been rather alien to Frege, but which plays a central role in debates about the nature of logic and logical operations today—permutation invariance.[44]

A permutation is any function that maps a domain one-to-one onto itself. The idea underlying the role that permutation invariance plays in modern debates regarding logicality is that logic is topic-neutral in some sense—that is, logic does not "care" about the particular identity of the objects to which it is applied—and hence anything that holds as a matter of logic should also hold if one "permutes" the domain. In the results that follow we shall apply a version of this idea, but with one significant modification: if we are working with the logic of Frege's *Grundgesetze*, and hence with Fregean domains as defined above, which contain the *logical* objects $\top$ and $\bot$, then presumably this logic *does* "care" about the identity of at least these two objects (and further, logic "cares" about non-identities between either of these objects and any non-truth-value). As a result, we will not mobilize the full notion of permutations on Fregean domains, but will instead focus on permutations that leave the truth-values fixed. In short, we shall be concerned here with *logical permutations*:

Definition 8.1. *Given a Fregean domain Δ, a unary function π is a* logical permutation *on Δ if and only if π is a permutation on Δ and:*

$$\pi(\top) = \top$$
$$\pi(\bot) = \bot$$

We can now define a "new" class of functions:

[44]For representative examples of discussions of logicality involving permutation invariance, see Tarski (1936), Sher (2008), and Bonnay (2008). For recent examinations of permutation invariance and related notions within the neo-logicist literature, see Fine (2002), Antonelli (2010), and Cook (2017).

Definition 8.2. *An n-ary function g is a* logically invariant logical function *if and only if g is a function mapping n-tuples from a Fregean domain Δ to $\{\top, \bot\}$ and, for any logical permutation π on Δ:*

$$g(\xi_1, \xi_2 \ldots \xi_n) = g(\pi(\xi_1), \pi(\xi_2), \ldots \pi(\xi_n))$$

LINV *is the class of logically invariant logical functions.*

We now prove that the functions definable in terms of f_{HOR}, f_{NEG}, f_{CON}, and $f_{=}$, (i.e., the functions in $\mathsf{DEF}^{\mathsf{Fun}}_{=}$) are exactly the logically invariant semantic functions. One direction of the proof is easy:

Lemma 8.3. $\mathsf{DEF}^{\mathsf{Fun}}_{=} \subseteq \mathsf{LINV}$

Proof. Straightforward induction on the length of formulas, left to the reader. □

Proving the converse, however, takes a bit more work. First, a definition:

Definition 8.4. *Given a Fregean domain Δ and two n-tuples of objects:*

$$\sigma_1 = \langle \alpha_1, \alpha_2 \ldots \alpha_n \rangle \in \Delta^n$$
$$\sigma_2 = \langle \beta_1, \beta_2 \ldots \beta_n \rangle \in \Delta^n$$

σ_1 is invariantly equivalent *to σ_2 (i.e. $\sigma_1 \equiv_{\mathsf{IE}} \sigma_2$) if and only if:*

1. *For any k such that $1 \leq k \leq n$, $\alpha_k = \top$ if and only if $\beta_k = \top$.*
2. *For any k such that $1 \leq k \leq n$, $\alpha_k = \bot$ if and only if $\beta_k = \bot$.*
3. *For any k, j such that $1 \leq k < j \leq n$, $\alpha_k = \alpha_j$ if and only if $\beta_k = \beta_j$.*

$[\sigma_1]_{\mathsf{IE}} = \{\sigma_2 : \sigma_2 \equiv_{\mathsf{IE}} \sigma_1\}$

Simply put, two n-tuples of objects from a Fregean domain Δ are invariantly equivalent just in case they have $\top$s and $\bot$s in the same positions, and one of them contains the same object in two distinct positions if and only if the other contains the same object in those same two positions (but note that the object that occurs in those spots in the first n-tuple need not itself be the same object that inhabits those two spots in the second n-tuple if the objects in question are not truth-values). We leave it to the reader to verify that, given any Fregean domain Δ and any finite n, invariant equivalence is an equivalence relation on n-tuples, and hence the $[\xi]_{\mathsf{IE}}$s are equivalence classes. Next up is a crucial lemma:

Lemma 8.5. *Given any Fregean domain Δ, n-tuples $\sigma_1, \sigma_2 \in \Delta^n$ such that $\sigma_1 \equiv_{\mathsf{IE}} \sigma_2$, and n-ary $g \in \mathsf{LINV}$:*

$$g(\sigma_1) = g(\sigma_2)$$

Proof. Given two n-tuples:

$$\sigma_1 = \langle \alpha_1, \alpha_2 \ldots \alpha_n \rangle$$
$$\sigma_2 = \langle \beta_1, \beta_2 \ldots \beta_n \rangle$$

from a Fregean domain Δ where $\sigma_1 \equiv_{\mathsf{IE}} \sigma_2$, let π be any permutation on Δ such that:

$$\pi(\alpha_k) = \beta_k$$

for any k such that $1 \leq k \leq n$. Note that the fact that $\sigma_1 \equiv_{\mathsf{IE}} \sigma_2$ guarantees the existence of such a permutation. Then, since $g \in \mathsf{LINV}$:

$$\begin{aligned} g(\sigma_1) &= g(\alpha_1, \alpha_2 \ldots \alpha_n) \\ &= g(\pi(\alpha_1), \pi(\alpha_2) \ldots \pi(\alpha_n)) \\ &= g(\beta_1, \beta_2 \ldots \beta_n) \\ &= g(\sigma_2) \end{aligned}$$

□

Lemma 8.6. $\mathsf{LINV} \subseteq \mathsf{DEF}^{\mathsf{Fun}}_{=}$

Proof. Assume $g \in \mathsf{LINV}$ (g n-ary), and assume $\sigma_1 \in \Delta^n$ where:

$$\sigma_1 = \langle \alpha_1, \alpha_2, \ldots \alpha_n \rangle$$

For each k where $1 \leq k \leq n$, let:

$$\Phi_{\langle \sigma_1, k \rangle}(\xi_k) = \begin{cases} f_{=}(\xi_k, (f_{=}(\xi_k, \xi_k))), & \text{if } \alpha_k = \top; \\ f_{=}(\xi_k, f_{\mathsf{NEG}}((f_{=}(\xi_k, \xi_k)))), & \text{if } \alpha_k = \bot; \\ f_{\mathsf{OTH}}(\xi_k), & \text{otherwise.} \end{cases}$$

and for each k, j such that $1 \leq k < j \leq n$, let

$$\Psi_{\langle \sigma_1, k, j \rangle}(\xi_k, \xi_j) = \begin{cases} f_{=}(\xi_k, \xi_j), & \text{if } \alpha_k = \alpha_j; \\ f_{\mathsf{NEG}}(f_{=}(\xi_k, \xi_j)), & \text{if } \alpha_k \neq \alpha_j. \end{cases}$$

Now, let:

$$\begin{aligned} \Theta_{\sigma_1} = f_{\mathsf{AND}}(&\Phi_{\langle \sigma_1, 1 \rangle}, \Phi_{\langle \sigma_1, 2 \rangle}, \ldots \Phi_{\langle \sigma_1, n \rangle}, \Psi_{\langle \sigma_1, 1, 2 \rangle}, \Psi_{\langle \sigma_1, 1, 3 \rangle}, \ldots \\ &\Psi_{\langle \sigma_1, 1, n \rangle}, \Psi_{\langle \sigma_1, 2, 3 \rangle}, \Psi_{\langle \sigma_1, 2, 4 \rangle} \ldots \Psi_{\langle \sigma_1, n\text{-}1, n \rangle}) \end{aligned}$$

Note that, for all $\sigma_2 \in \Delta^n$:

$$\Theta_{\sigma_1}(\sigma_2) = \begin{cases} \top, & \text{if } [\sigma_1]_{\mathsf{IE}} = [\sigma_2]_{\mathsf{IE}}; \\ \bot, & \text{if } [\sigma_1]_{\mathsf{IE}} \neq [\sigma_2]_{\mathsf{IE}}. \end{cases}$$

Finally, let:

$$h(\xi_1, \xi_2, \ldots \xi_n) = f_{\mathsf{AND}}(\{\Theta_\sigma(\xi_1, \xi_2, \ldots \xi_n) : g(\sigma) = \top\})$$

Then, for all $\sigma \in \Delta^n$:

$$h(\sigma) = g(\sigma)$$

and $h(\xi_1, \xi_2, \ldots \xi_n) \in \mathsf{DEF}^{\mathsf{Fun}}_{=}$. □

Combining these gives us the desired characterization:

Theorem 8.7. $\mathsf{DEF}^{\mathsf{Fun}}_{=} = \mathsf{LINV}$

Proof. Immediate consequence of **Lemmas 8.3** and **8.6**. □

Thus, the functions definable in terms of Frege's horizontal, negation, conditional, and identity are exactly those that are invariant under permutations of the domain that keep the True and the False fixed.

9. Conclusion

We can sum up our results as follows: There is no single system of *Grundgesetze* that *exactly* matches up with modern formulations of propositional logic. Instead, there are (at least) three distinct systems, corresponding to three distinct collections of connectives—DEF, FPC (= $\mathsf{DEF}^{\mathsf{Con}}_{=}$), and $\mathsf{DEF}^{\mathsf{Fun}}_{=}$—that each, in their own way, capture something "propositional logic"-like within *Grundgesetze*. There are a number of reasons for this mismatch between (these sub-systems of) Frege's logic and our own, some of which are more well-known than others:

- The fact that Frege's treatment of sentences as names of truth-values allows him to equivocate, in a sense, between identity and biconditionality (very well-known).
- The unique role that Frege's horizontal operator, and the fusion of horizontals, play in the formalism of *Grundgesetze* (reasonably well known).
- The fact that Frege's treatment of (first-level) logical operators as total functions from the domain to $\{\top, \bot\}$ allows him to formulate logical notions (such as the alternative connectives and the f_{OTH} operator) that do not correspond to any standard classical connectives (less well-known).
- The fact that the system obtained via considering the first-level logical operators (including identity) has close connections to modern mobilizations of permutation invariance in the literature on logical constants (unknown until now).

As we have already stressed, Frege would surely have been aware of the subsystems corresponding to DEF and $\mathsf{DEF}^{\mathsf{Fun}}_{=}$ (but not necessarily that corresponding to FPC) as distinct subsystems of the full logic of *Grundgesetze*, and likely would have been interested in the particular characteristics of each of these systems. Thus, the results of the previous sections are important not because they show that Frege didn't have a propositional logic or a precise notion of propositional logic corresponding to one or another of these systems. Rather, they are important because they provide a precise characterization of the two subsystems of *Grundgesetze* that Frege himself was in a position to identify and about which he might have raised questions similar to those asked and answered here: the system corresponding to his primitive horizontal-fusing (equivalently, non-truth non-distinguishing) functions f_{HOR}, f_{NEG}, and f_{CON} and the system corresponding to his primitive first level logical functions f_{HOR}, f_{NEG}, f_{CON}, and $f_{=}$.[45] In short, the real purpose of this paper is not to show that Frege's logic lacks some feature that contemporary formalisms enjoy, but rather to better understand the features of Frege's logic (and various subsystems of it) on its own terms.

Of course, the work is far from done. In particular, the next obvious step, which we plan to carry out in future work, is to study the deductive systems corresponding to these various subsystems of the logic of *Grundgesetze*—of particular interest are completeness proofs for these deductive systems relative to the informal semantics formulated in the present essay. We plan to explore such results in a sequel.

[45] Recall the earlier comments about the fact that \, although a first-level function, involves the notion of extension and hence second-level functions in an essential way in its intended application.

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