Journal for the History of Analytical Philosophy

Volume 3, Number 7

Reply to Cook, Rossberg and Wehmeier

Patricia Blanchette

All contributions included in the present issue were originally presented at an 'Author Meets Critics' session organised by Richard Zach at the Pacific Meeting of the American Philosophical Association in San Diego in the Spring of 2014.

Reply to Cook, Rossberg and Wehmeier

Patricia Blanchette

1. Summary of *Frege's Conception of Logic*

Frege's attempt to reduce arithmetic to logic gave rise not just to a beautiful system of quantified logic, but also to a deep and fruitful understanding of the nature of theories, of theoretical reduction, and of logic itself. While the method Frege chose for effecting his reduction has turned out to be unworkable for multiple reasons, the conception of logic that grounds that attempt is, in my view, an important and very promising approach to this notoriously slippery subject-matter. The goal of *Frege's Conception of Logic* is to give a clear account of Frege's central views about the nature of logic and of a network of closely-related issues, and to assess the plausibility of those views.

A central theme of the book is that an understanding of Frege's fundamental views about logic requires an appreciation of his view of the connection between logic and what we might call 'conceptual analysis.' The heart of Frege's life-work, the attempt to reduce arithmetic to logic, consisted of a two-stage project: first, the careful analysis of central arithmetical notions in terms of what he took to be simpler ones; and secondly, the rigorous proof of the thus-analyzed truths from purely-logical principles. As is evident to even the most casual reader of Frege's *Grundgesetze*, the truths that Frege actually proves do not seem, at first or even at second glance, very similar to ordinary truths of arithmetic: they are instead truths about quite complicated functions, and about odd objects called 'value-ranges.' Nevertheless, Frege's central claim is that the proofs establish the purely logical status of arithmetic. The immediate question facing the Frege-interpreter is that of how the proven truths are related to such ordinary arithmetical claims as, for example, that expressed by 'every number has a successor.' Only in virtue of some relatively straightforward relation between the two kinds of truth, one that preserves logical truth, will the logicist claim (that arithmetic is part of logic) follow from the success of Frege's proof-theoretic project.

The answer defended in *Frege's Conception of Logic* is that the truths Frege proves in *Grundgesetze* are (or would be, if all had gone well) self-evidently logically equivalent with good analyses of the ordinary truths of arithmetic. This is a mouthful, and it's not a thesis that Frege articulates clearly. The claim I defend in Chapter 1 is that a careful look at Frege's extremely-careful procedure, especially in *Grundlagen* and *Grundgesetze*, makes it plain that this is the way he understands the relation between the truths he proves and the ordinary truths of arithmetic.

This raises the question of how Frege understood the relationship between analysandum and analysans, and the further question of whether this relationship, as he understands it, is really sufficient to underwrite the logicist inference, that inference from the purely-logical status of analysans (as demonstrated via proof) to the purely-logical status of analysandum, i.e. of ordinary arithmetical truth. This question is addressed in Chapters 2 – 4. Chapter 2 sharpens the question by examining Frege's view of thoughts (*Gedanken*). Thoughts are the nonlinguistic things expressed by sentences (both of natural and of formal languages); they are the items one knows, doubts, and believes; they are the items that bear to one another such relations as those of logical entailment and inconsistency; and they are the items whose truth is demonstrated via proof. Our central question, then, is that of how the thoughts in which a *Grundgesetze* proof culminates are related to the thoughts expressed by the sentences of ordinary arithmetical discourse. This way of framing the issue makes it clear that there are some important *prima facie* difficulties for the Fregean enterprise. The first is that the large semantic difference between the sentences of ordinary arithmetic and the sentences appearing in *Grundgesetze* would seem to guarantee that the thoughts expressed in each case are strikingly different. Perhaps most vividly: it would appear that the singular terms (e.g. numerals) of Frege's formal work don't even refer to the same objects as do the numerals of ordinary arithmetical sentences, with the result that the connection between Frege's proven thoughts and the thoughts of ordinary arithmetic begins to look very tenuous. The response defended in Chapter 4 is that when we correctly understand Frege's view of the connection between sentences, senses, and references, we can see that the close connection between thoughts (that is, between those proven in *Grundgesetze* and those that make up ordinary arithmetic) needed for Frege's project does not require preservation of singular-term reference, but turns instead on a semantically rich similarity relation between whole thoughts. It is argued here that, if Frege had been right that every universally-quantified biconditional is logically equivalent to an associated identity-statement, then he would have been right that the truths of ordinary arithmetic bear just this relationship to truths demonstrable via pure logic.

Prior to the detailed treatment in Chapter 4 of Frege's reductive analyses, Chapter 3 treats a particular difficulty concerning Frege's use of function-terms. Frege is often held, and with a good deal of textual support, to have thought that every referring one-place function-term of first level must refer to a function that's defined over every object. This bizarre view, if upheld strictly, would have the result either (a) that such ordinary function terms as " + 2" refer to functions that are defined over such objects as the Eiffel Tower, despite the fact that no competent user of the term is in possession of criteria that would determine the result of applying that function to that object, or (b) that such ordinary function-terms fail to refer. The disruptive effects of adopting the latter view include the conclusion that, given Frege's general semantic views, no sentence of ordinary arithmetic has a truth-value. The purpose of Chapter 3 is to argue that the textual passages that are taken to support this account of Frege do not in fact do so, and that he simply doesn't hold the bizarre view in question.

By the end of Chapter 4, I take it to be established that, on Frege's view, the question of whether a given thought is logically entailed by a collection of other thoughts can turn in part on what's expressed by some of the simple terms in the sentences used to express those thoughts. This is why conceptual analysis, which is often for Frege a matter of bringing to light hidden complexity in the contents of such terms, is critical to demonstrations of logical entailment, as in the central project of *Grundgesetze*. This means, amongst other things, that relations of logical entailment between thoughts do not, in general, supervene on the syntactic structure of the sentences used to express those thoughts. Because syntactically simple terms can express logically-complex content, logical entailment (a relation between thoughts) is generally considerably richer than is formal derivability (a relation between sentences).

The recognition of the importance of semantic content to logical relations helps to clarify what's going on in the famous disagreement between Frege and David Hilbert over proofs of consistency and independence in geometry. Chapter 5 is a discussion of this disagreement. It is argued that, contrary to some widespread and dismissive views of Frege's side of the debate, Frege had a good point: that, if one means by 'consistent' and 'independent' what Frege means by these terms, according to which they apply in the first instance to thoughts rather than to sentences or to defining conditions, then Frege was right: Hilbert's demonstrations do not demonstrate consistency or independence. This is not to say that Hilbert was in any sense mistaken: Hilbert's technique works unproblematically for the demonstration of what he (and, for the most part, everyone following him) meant by 'consistent' and 'independent.' The importance of this point is twofold: we gain a clearer understanding of Frege's way of conceiving of the logical relations, and we gain a clearer understanding of just what one needs to suppose about the fundamental logical relations in order to take it that broadly Hilbert-style techniques are effective.

Chapter 6 examines some of the central similarities and differences between Frege's understanding of logical entailment and the more modern view on which entailment is a matter of truth-preservation across structures. While there are some clear and important connections between the two conceptions, the differences are significant: the model-theoretic entailment of a sentence φ by a set Γ of sentences is neither a necessary nor a sufficient condition of logical entailment between the thoughts expressed by those sentences. The differences in each direction are, I think, instructive both for understanding Frege, and for understanding some important features of model-theoretic entailment.

Chapter 7 treats the question of whether Frege's understanding of logic and of formal systems is compatible with metatheory. The central claim defended here is that Frege does not hold the 'universalist' position often attributed to him, on which it is impossible coherently to make sense of metatheoretic investigations. I argue that Frege's view is compatible with metatheory, and that he is invested in a handful of metatheoretic claims, most importantly those that have to do with the adequacy of his formal system to the demands of the logicist project. This chapter also investigates the relationship of some central contemporary meta-theoretic issues to Frege's project. The conclusion here is that while the completeness of a formal system is not of particular interest from a Fregean point of view, the properties of consistency and soundness are of clear interest, as is something very like categoricity. Here again, I take it that the comparison between Frege and his successors is useful not just for a better understanding of Frege, but also for a better understanding of our own use and treatment of formal systems.

Chapter 8, a brief conclusion, sums up what I take to be some of the central lessons to be learned by the failures and successes of various parts of Frege's program. The inconsistency of Frege's formal system infects not just that formal apparatus, but the whole of Frege's way of conceiving of value-ranges. It therefore undermines not just the (attempted) proofs in *Grundgesetze* and the proof-sketches in *Grundlagen*, but also the attempted analyses in both works. The first incompleteness theorem also shows that Frege's axiomatic approach to the demonstration of logicism is unworkable. None of these points, however, affects the broadly Fregean account of the nature of logic and of the connections between logic, language, and justified inference. I take it that Frege is on entirely solid ground when insisting on the relevance of content to logical entailment, and on the consequent importance of notions of independence and consistency that are not demonstratable via the construction of models. And I take it that the view that underwrites this position, on which conceptual analysis is an important tool in the investigation of logical relations, is a view that's here to stay, and one whose significance we have yet to fully understand.

2. Reply to Cook, Rossberg and Wehmeier

Let me begin by saying how much I appreciate the thoughtful and helpful contributions made by Roy Cook, Marcus Rossberg, and

Kai Wehmeier. It is a delightful experience to have one's work subjected to the kind of serious and sympathetic scrutiny to which these three have subjected mine, and I am very grateful.

2.1 General Introduction: Total Functions

I'll begin by focusing on the central issue taken up by Cook and Wehmeier, the issue of what exactly Frege meant by the 'sharp boundaries' requirement on functions. While this issue is, as Cook and Wehmeier explain, to some extent tangential to the overall logicist project, having strictly speaking to do with the semantics of function-terms, nevertheless the issue is an important one, I think, for understanding Frege's general views of the nature of mathematics and of mathematical objects.

Frege says, in a large number of places, that functions must be defined over 'all arguments.' A sample of the relevant texts is as follows.

> [A] first-level function with one argument always has to be so constituted that an object results as its value, whatever object is taken as its argument...[1]

> A definition of a concept (a possible predicate) must be complete. It has to determine unambiguously for every object whether it falls under the concept or not (whether the predicate can be applied to it truly). Thus, there must be no object for which, after the definition, it remains doubtful whether it falls under the concept...[2]

He also says some things that are arguably incompatible with what he's said in these passages. For example, in "Function and Concept," he notes that

> So long as the only objects dealt with in arithmetic are the integers, the letters *a* and *b* in '*a* + *b*' indicate only integers; the plus-sign need be defined only between integers.[3]

Frege also notes in his *Grundgesetze* discussion of criteria for acceptable definitions that while it is impermissible to first define a function-term over merely part of one's domain, and later go on to use the thus partially-defined term to extend the definition to the rest of that domain (the practice of 'piecemeal definition,' as he calls it), it is nevertheless unproblematic to define a function-sign just over a specific domain of objects that does not include all of the mathematical objects there are. One must simply make sure, in Frege's view, to confine the use of that sign to the theory of that domain over which it is defined. See for example *Grundgesetze* II section 61:

> It is, moreover, very easy to avoid multiple explanations of the same sign. Instead of first explaining it for a restricted domain and then using it to explain itself for a wider domain, that is, instead of employing the same sign twice over, one need only choose different signs and to confine the reference of the first to the restricted domain once and for all, so that the first definition is also complete and draws sharp boundaries.[4]

The question, then, is what to make of these apparently-conflicting pronouncements of Frege's. The view for which I've argued briefly in the book, and at more length in my [2012][5], is that when Frege claims that functions must be defined over 'every argument,' he has in mind what would have seemed entirely commonplace to his contemporary audience, the idea that a given mathematical theory is typically a theory of a particular domain, e.g. of the reals, or of the integers, or of geometric constructions in Euclidean space, etc.

The talk of 'every argument' is, in such a context, talk of every object, or function, under discussion in the theory. When Frege rails against his contemporaries for violating his requirement, the practice he objects to is in each case a practice in which function-signs are defined over only a portion of the mathematical domain in question.

Frege's reasons for the requirement of total definition are straightforward, and have entirely to do with his overriding concern with rigor. As against the practice of piecemeal definition, Frege notes that the practice leaves open the possibilities (i) of failing to account for some cases, thereby giving rise to well-formed sentences with no truth-value; and (ii) of contradicting, via later stipulations, what one has laid down with earlier. The difficulty noted in (i) is the central concern throughout Frege's discussions of definition and of the semantics of formal languages. In order to ensure rigor, as Frege sees it, it is essential that we never have well-formed sentences that lack a truth-value, since such gaps will interfere with the cardinal requirement on inference-rules that they be truth-preserving. This requirement on sentences carries with it the general requirement that no well-formed term lack a reference, and hence that every function referred to by a function-term be defined over every object, function (of appropriate level), or n-tuple thereof to which it is possible to refer in the theory.

The requirement that every well-formed piece of language have a reference, which I've elsewhere called the requirement of 'linguistic completeness,' does not imply the strong requirement sometimes attributed to Frege, the requirement that every function be defined over every object, function, or n-tuple whatsoever. It does not, most importantly, imply that those functions referred to in a formal language must be defined over nonmathematical objects, or even over mathematical objects not treated by the theory. In what are arguably the most easily-misleading passages dealing with the requirement of linguistic completeness, Frege would seem to contravene this claim, apparently holding the bizarre view that mathematical functions must be defined over such objects as the sun and the moon. The strangeness of this requirement is seen not just in the fact that no other mathematicians' function-definitions satisfy it, but that even Frege's own definitions fail to do so. It is important to notice here, though, that Frege's point is the conditional one: if our language contains a sign for a given celestial body, then we are required to lay down what value each of our first-level functions delivers for this object as argument. The requirement is once again that of linguistic completeness, not of definition over objects alien to the theory in question.

The requirement on function-signs comes out clearly in Frege's construction of his own system in *Grundgesetze*, and in his proof that his own function-signs meet it. The requirement is, as Frege puts it, that each function-term has reference. And the relevant sufficient condition for having reference, as applied to a first-level function of one argument, is that each result of completing that function-term by a referring object-name itself have reference.[6] And now one might ask: which referring object-names does Frege mean? Does he mean all such names, including e.g. names of celestial bodies, French architectural marvels and as-yet undiscovered mathematical objects? Or does he mean the names in the language of *Grundgesetze*? The answer here is unequivocal: Frege demonstrates that each first-level one-place function-term of *Grundgesetze* refers by demonstrating that the result of its completion by a referring object-name of *Grundgesetze* itself refers. (Similarly for function-names of higher type.) There is no requirement that the functions referred to be defined over outlying objects; and if there were such a requirement, Frege's functions would very plainly fail it. Similar considerations arise with respect to the definitions of the function-signs themselves, on which Frege's proof of their referentiality turns. The semantics of Frege's value-range operator is partly governed by the requirement of its consistency with Basic Law V, which determines the truth-condition of each identity sentence of the form

(1) $\varepsilon'\Phi(\varepsilon) = \alpha'\Psi(\alpha)$

namely that it have the same truth-value as does

(2) $\forall x(\Phi(x) = \Psi(x))$

In *Grundgesetze* I section 10, Frege notes that this requirement does not determine truth-conditions for those sentences of the form

(3) $a = \varepsilon'\Phi(\varepsilon)$

or

(4) $\varepsilon'\Phi(\varepsilon) = a$

where the singular term a is not of the form $\alpha'\Psi(\alpha)$. That is to say, without some further stipulation, function-terms of the form "$\varepsilon'\Phi(\varepsilon) = ...$" and "$... = \varepsilon'\Phi(\varepsilon)$" would fail the requirement of linguistic completeness. Frege's way of augmenting Basic Law V so as to meet this requirement is what's of most interest to us: he famously meets the requirement by providing stipulations that fix truth-conditions for those instances of (3) and (4) in which the term a is a name of a truth-value. That's it. No other terms are considered, and the functions in question are defined over no other objects. Frege's functions, in short, are not defined over any objects other than those that can be referred to in the quite restricted language of *Grundgesetze*. As Frege says, if one were to use the framework of *Grundgesetze*'s language for another purpose, one that involved reference to additional objects, one would simply make stipulations, at the outset of one's specification of the semantics of the language, sufficient to cover these cases as well.

My view, in short, is that Frege's view of mathematical languages involves these theses: (a) that each such language is interpreted via a specific collections of thoughts, which collection of thoughts is often the theory of a given mathematical domain; (b) that rigor requires that each well-formed function-term and object-term of such a language have a determinate reference; and hence (c) that each function referred to by a function-term of such a language must be defined over every object (or function of appropriate level) in the domain of the theory. The view that mathematical functions must be defined over objects not treated of in the theory in question, so that e.g. the function referred to by 'square-root of' in a language dealing with real numbers must be defined over the Eiffel Tower, is a thesis that, in my view, Frege never held.

2.2 Reply to Cook

Roy Cook raises the following very interesting issue. If, as I claim, Frege understands formal languages in such a way that the first-level quantifiers of two languages might have different domains, then Frege is, as Cook puts it, "saddle[d] ... with an unattractive (for him) sort of relativism." The "relativism" arises with respect to those terms whose reference depends on the range of the quantifiers. For example, as Cook points out, the ordered pair $< a, b >$, where a and b are objects, is, according to *Grundgesetze*, the value-range of that concept under which fall all and only the value-ranges of those two-place functions that map arguments a and b, in that order, to the True. If the second-level quantifiers in L1 and L2 have different ranges (as will generally be the case if their first-level quantifiers have different ranges), then the collection of functions figuring in L1's version of that ordered pair will be different from the collection of functions figuring in L2's version, so that the ordered pair serving as, say, $< T, F >$ in L1 is not the same object as is that serving as $< T, F >$ in L2. And perhaps worse, as Cook puts it, "there is nothing special about this example other than its sim--plicity on the varying domains approach the identity of the cardinal numbers (i.e. which 'collections' of value-ranges of functions are identified with 1, 2, ...) will also vary from domain to domain." And finally, via the same line of reasoning, we find that no 'logical object' (assuming these to be, as in *Grundgesetze*, value-ranges) can appear in the domain of two languages L1 and L2 if the domains of L1 and L2 differ in any way whatsoever. In addition to the general strangeness of this domain-relativity is the arguably unpalatable fact that from this position it follows that there is no universal domain, a seriously un-Fregean sounding conclusion.

In reply, let me begin with a certain amount of agreement: any position from which it is impossible or even difficult to understand universal first-level quantification (or universal *n*-level quantification for any *n*) is indeed one that we should not attribute to Frege. Because this thesis follows from the claim that all cross-theory identity-statements involving mathematical objects are false, this latter thesis too is one that does not fit well with Frege's general mathematical or semantic views, and ought not to be attributed to him.

As to disagreement: The first thing to note is that there are no cross-theory identity sentences of the kind Cook has in mind. No such sentence is well-formed in any Fregean language.

Nevertheless, one might think, we, the speakers of something other than (say) the *Grundgesetze* language, ought to be able to raise, in our language, the identity-questions Cook discusses, and ought to be able to go through just the kind of reasoning Cook outlines for us, to the effect that the relevant sentences (even if not Frege's) are in fact false. We ought to be able to say, for example, "The singular term *t* of L1 refers to the same object as does the singular term *t* of L2," and what we say here should, one might think, be true or false.

The last point just suggested, i.e. that sentences of the kind in question have a truth-value, is just the point that I believe to be unfounded on Frege's view. To make the point clearly, let's consider a real example, the example of how we might compare Frege's 1884 treatment of cardinal numbers with his 1893 treatment. The number 2, for example, in *Grundlagen*, is taken to be the extension of a particular second-level concept, one that we'll call "F2". (This is the concept under which fall all and only those first-level concepts of the appropriate cardinality.) In *Grundgesetze*, by contrast, 2 is taken to be the value-range of a first-level function, specifically that function that delivers the value True for the value-ranges of those functions that fall under F2. Call this first-level function "F2*". Ignoring subtle differences over the evolution of Frege's semantic and technical views in the intervening years, let's consider a formalized version of the *Grundlagen* view, on which 2 is the value-range of F2, and ask the following question: Is *Grundlagen*'s number 2 the same object as *Grundgesetze*'s 2? As we'll abbreviate it, our question is as to the truth of this sentence:

(5) $2_{Gl} = 2_{Gg}$

When dis-abbreviated, this formula becomes essentially:

(5*) The value-range of F2 = the value-range of F2*

In accordance with Frege's principles, the truth-value of (5*) should be the truth-value of the statement to the effect that F2 and F2* return the same values for the same arguments. But here we hit a roadblock: F2 and F2* do not take the same arguments, since F2 is second-level and F2* first. In keeping with Frege's principles and in line with his own practice, we could, if confronted with a language in which (5*) appears, simply make a stipulation about the truth-values of such value-range identity statements. But barring such a stipulation, and provided that our context is one in which non-equivalent such stipulations are consistent with the rest of the semantic rules (as would be the case in the kind of addition envisioned here), there is no fact of the matter about the truth-value of (5*), and hence no fact of the matter about the truth-value of (5), which is to say that there is no fact of the matter about whether *Grundgesetze*'s numerals co-refer with *Grundlagen*'s.

This is not an isolated point about numerical terms. When it comes to value-range terminology generally, the truth-conditions of identity-statements of the form 'the value-range of Φ = the value-range of Ψ' are determined by the intended role of the term-forming operator 'the value-range of ...,' which is to say that they are determined by the requirement of conformity to Basic Law V, just in those cases in which Φ and Ψ are functions of the same level and arity, and are defined over the same range of arguments. Other identity statements involving value-range terms require specific additional rules if they are to be provided with truth-conditions or truth-values. And when the identity-question at issue is not one that has been settled by such rules, as will typically be the case when the question is not expressed by a well-formed sentence of a clearly-regimented language, as in the kinds of examples discussed by Cook, the question has no answer.

The idea that there is no fact of the matter about some cross-theory identity claims sits uncomfortably with a robustly platonic view of the nature of the objects involved in those claims. But it is the natural position to take, as I see it, if one shares Frege's view of the nature of discourse about those objects, and specifically about such objects as value-ranges and numbers. On that view, value-ranges are those objects identity-claims between which are equivalent to universally-quantified generalizations about associated functions; cardinal numbers are those objects identity-claims between which are equivalent to statements of equi-cardinality between concepts, and so on. Given this understanding, determinate reference to a number or to a value-range does not require determinacy of truth-value for other, unrelated identity-sentences; such a requirement, when present, is pressed upon the theorist only by demands of rigor, and covers only the well-formed sentences of the language in question. That typical languages involving number-terms or value-range terms will leave most such identity-questions undecided is exactly what we should expect, given a Fregean view of reference to such objects. Finally: the existence of languages whose quantifiers range over absolutely everything is not in any straightforward way ruled out by the Fregean picture sketched here.

2.3 Reply to Wehmeier

Kai Wehmeier raises another interesting and important collection of issues surrounding the topic of Frege's requirement of totality for functions. Wehmeier's worries about my reading of Frege on this point concern not so much the ontological implications, but rather the issue of how best to interpret some critical passages in *Grundgesetze*. Most important here is *Grundgesetze* I,10, in which the following things happen:

(i) Frege notes that while Basic Law V determines that (1) and (2) above are coreferential, "we have ... by no means yet completely fixed the reference of a name such as '$\varepsilon'\Phi(\varepsilon)$'."

(ii) Frege notes that the only kinds of Grundgesetze sentences whose truth-conditions haven't been determined by Basic Law V are those identity-sentences of the form "$\varepsilon'\Phi(\varepsilon)$ = ..." or "... = $\varepsilon'\Phi(\varepsilon)$", where the ellipsis is filled in by a sentence, i.e. by a name of a truth-value. (This is because all of the other function-signs in the language are definable in terms of identity, value-range abstraction and the logical resources of the theory, and the only singular terms in the language, other than those of the form "$\varepsilon'\Phi(\varepsilon)$," are names of truth-values.)

(iii) Frege proves that there's more than one way in which one might remove, or "resolve" the indeterminacy, consistent with the semantic rules already in place. (iv) Frege chooses one such way of resolving the indeterminacy: he stipulates that sentences of the form "$\varepsilon'\Phi(\varepsilon) = ...$" and "$... = \varepsilon'\Phi(\varepsilon)$" shall refer to the True if either (a) Φ is a concept under which only the True falls and the ellipsis is filled in by a term that refers to the True, or (b) Φ is a concept under which only the False falls and the ellipsis is filled in by a term that refers to the False; such sentences shall refer to the False otherwise.

The question, for us, is that of what Frege is doing in step (iv). When he claims to be "resolv[ing]" an "indeterminacy," I take it that he means that, prior to the stipulation in question, there is no fact of the matter about whether e.g."$\varepsilon'(\varepsilon = (\forall x)x = x)$" co-refers with"$(\forall x)x = x$". That's what he means when he says, in our (i) above, that, prior to the stipulation to be given in (iv), we have "by no means completely fixed the reference of" value-range terms. We haven't completely fixed their reference because we haven't said enough to do what's required for reference-fixing in Frege's official story (see *Grundgesetze* I, 29): we haven't said enough to provide truth-conditions for each well-formed sentence in which the term appears. The point of the step (iv) stipulation is to supplement the incomplete semantic information provided by Basic Law V, and make it the case that each well-formed *Grundgesetze* sentence has a truth-value.

Wehmeier's view, on the contrary, is that the indeterminacy left by Basic Law V is an epistemic one: it's not that the sentences in question lack truth-value; it's rather that we just don't know what the truth-value is. On this view, the introduction of the value-range term-forming operator, together with Basic Law V, settles determinate reference on value-range terms, and hence settles truth-values on all identity-sentences involving those terms; it's just that we, in our limited state of knowledge, don't know what these truth-values (or even truth-conditions) are.

The argument Wehmeier offers in favor of the epistemic reading of the indeterminacy turns on Frege's permutation argument at Grundgesetze I, 10; our (iii) above. The permutation argument is this: Suppose X is a 1-1 function on objects. Then for arbitrary objects α and β, $X(\alpha) = X(\beta)$ if and only if $\alpha = \beta$. From this it follows that the identification of the truth-value of "$\varepsilon'\Phi(\varepsilon) = \alpha'\Psi(\alpha)$" with that of "$(\forall x)(\Phi(x) = \Psi(x)$", as given by Basic Law V, is compatible with two distinct ways of assigning reference to terms of the form "$\varepsilon'\Phi(\varepsilon)$", given only the supposition that there is such a non-trivial function X. Wehmeier's view is that this argument presupposes that, prior to the stipulation of our (iv), the value-range terms have reference, in a way that fixes the truth-values of all identity-sentences in which they appear. But notice that no such comprehensive presupposition is relevant: the argument presupposes simply that the truth-value of "$\varepsilon'\Phi(\varepsilon) = \alpha'\Psi(\alpha)$" is the same as that of "$(\forall x)(\Phi(x) = \Psi(x)$", as given by Basic Law V; no presumption is made about the truth-values of other sentences. Indeed, as I see it, the point of the argument is precisely to establish that there are additional identity-statements whose truth-values have not yet been fixed.

The stipulation itself, our (iv) above, is difficult to square with Wehmeier's reading of I, 10. On that reading, the truth-values of all identity-sentences involving value-range terms are fixed prior to the Section-10 stipulation, with the result that the stipulation made there is either redundant (since it's already the case that the term "$\varepsilon'(\varepsilon = (\forall x)x = x)$" co-refers with all true sentences), or is in conflict with the semantics of the language (since it's already the case that such terms do not co-refer with all true sentences). Neither of these options makes sense of Frege's claim to be eliminating, via stipulation, an indeterminacy left prior to I, 10.

A second important passage to which Wehmeier draws attention is *Grundgesetze* I, 34, at which Frege defines the application function sufficiently broadly to cover arguments that are not value-ranges, and to the immediately-succeeding sections 35 and 36, in which Frege discusses the possibility of arguments that are not value-ranges. In each of these cases, as Wehmeier points out, Frege's discussion of how to treat arguments that aren't value-ranges makes it clear that he envisions the use of the sentence-forms in question in theories whose domain includes objects other than value-ranges. The right thing to say about these passages, as I understand it, is that Frege is here setting up the formal apparatus that he's about to wield in a context that happens to include only value-ranges (as we can tell from his proof of referentiality in I, 30-31). But he clearly takes it that the framework established in the early sections of *Grundgesetze* is to be broadly applicable in the formalization of other, broader theories, as he suggests at the end of I, 10, when noting that the introduction to a language of this kind of new objects will bring with it the requirement of new stipulations so as to maintain linguistic completeness.

Further passages examined by Wehmeier include the p. 18 footnote. Here the matter is straightforward: Frege is not rejecting in this note the possibility that all terms refer to value-ranges, but instead the possibility that all terms refer to singleton value-ranges, so that each object is identical with the value-range of a concept under which it alone falls. As Frege points out, this idea is in conflict with the existence of concepts under which more than one thing falls.

Regarding full interpretation: Frege's view of mathematical work is that it involves the use of fully-interpreted languages, languages each of whose well-formed sentences expresses a single determinate thought. One might take it, and Wehmeier does take it, that this view is in conflict with the idea that Frege tolerates "variable domains." Here, I take it that there is simply a terminological unclarity. Frege does not tolerate variable domains in the modern sense in which a given language (or theory) is susceptible to reinterpretation. The sense in which he (on my view) tolerates the standard mathematical practice of investigating theories of the reals, theories of the integers, and so on is just that he holds that different mathematical languages, and hence different theories, can be about different domains. While this does mean that a given typographical complex of symbols can in one theory have a different sense and reference than it has in another, this is no difficulty for Frege, but simply an instance of distinct but typographically similar languages. Theories, for Frege, are collections of thoughts, each of which has a single, determinate domain.

Wehmeier raises the interesting point that, if two theories have different mathematical domains, then their Basic Law V sentences will involve quantification over different collections of functions, with the result that each theory will have a different version of Basic Law V. As to the status of cross-theory identity statements, see my reply to Cook, above, for the view that things are not quite as simple as they might appear. Nevertheless, it is true that when a theory has a non-universal domain, its version of Basic Law V will be a restriction of the fully-general V to the domain in question. This is, I take it, exactly as it should be. Wehmeier raises the worrying possibility that this state of affairs undermines the claim of Basic Law V to be a law of logic. For if there is a domain without the required collection of value-ranges, then V will be false when affirmed of this domain. And, one might think, a law of logic is not the kind of thing that can be "false of" any domain. But here, the apparent difficulty is not a difficulty for Frege. Frege does not subscribe to the modern view of the laws of logic as "true in" every domain: his laws of logic quite explicitly contain existential commitments falsified by, e.g., any finite domain, any domain lacking value-ranges, and so on. Basic Law V is a self-evident truth, in Frege's view, and a law of logic; this means that if one's theory is to satisfy the laws of logic, then its domain must contain a value-range for every function, and the value-ranges must obey Law V.

2.4 Reply to Rossberg

Marcus Rossberg draws attention to a central issue in Frege's logicist project, the issue of how best to understand the justification for Basic Law V. Frege characterizes the inference from the right-hand side to the left-hand side of V as a "conversion" that is legitimated immediately by a "basic logical law."[7] We now know, sadly, that he was mistaken here: the ontological demands of the identity-statement on the left are steeper than are those of the universal generalization on the right, and the term-forming operator appearing on the left (unlike anything on the right) provides the means for forming names to which no object can correspond. This makes especially vivid the question of why Frege took the law to be a fundamental law of logic, and of how he understood the connection between the two sides. The issue here concerns not just the law itself, but each "conversion" justified by it: when we infer an identity-statement regarding value-ranges from a universally-quantified statement, or vice-versa, as we must do regularly in Frege's development of arithmetic, how should we understand the connection between premise and conclusion?

One possibility is to understand Frege as holding that the two sides of Basic Law V express the same sense: to affirm the identity of the value-ranges of a pair of functions is just to affirm, in different terminology, exactly what's affirmed when we say that those functions take the same value for each argument. If this is how Frege understands it, then Basic Law V is of course a law of logic, since each instance of it is a biconditional linking a given sense to itself. Frege himself seems to claim as much in one (but only one) passage, the infamous remark in "Function and Concept" (p. 10, quoted by Rossberg above).

As Rossberg observes, however, this way of understanding Frege's view of Basic Law V does not fit well with the understanding of sense that Frege expresses in most of his post-1891 work, including *Grundgesetze*. The picture one cannot fail to take from most of Frege's remarks about sense is that the informativeness of an identity-statement (or a biconditional), which is to say its non-vacuousness, turns on its not expressing the same sense on each side, but on linking terms or statements with different senses. Because instances of Basic Law V are clearly (and were clearly understood by Frege to be) non-vacuous, there are strong reasons to take Frege's considered opinion to be one on which the two sides of V express, despite the stray remark in 1891, distinct senses. As Rossberg observes, a further reason in favor of this reading is that, despite a large number of straightforward opportunities to do so, Frege never again claims that the two sides of V express the same sense, and even seems carefully to avoid doing so.

My own view is that Frege does not have a clear position on the issue, largely because he does not have clear criteria of sense-identity. He does lay down some clear and important parameters, for example that if two pieces of language have the same sense, then they have the same reference, and that if two pieces of language express the same sense, then their substitution one for another preserves logical equivalence. He also relies on the considerably less-clear connection noted above between informativeness and distinction of sense. But this leaves a lot of gray area, an area into which Law V falls quite squarely. Reading between the lines, one gets a sense of Frege's being pulled in two directions on this issue: he is convinced that, in some sense, a statement of value-range identity is merely a restatement of the corresponding universal generalization, which provides reason to treat Law V as justified by the sense-identity of its two sides; on the other hand, he cannot fail to recognize that the move from generalization to value-range identity is non-trivial, providing reason to think of the sides as expressing distinct (though logically equivalent) senses.

In the end, Frege's silence on this issue is, I think, justified by the fact that the important question for him is that of whether or not Law V is clearly a law of logic, independently of the answer to our sense-identity question. And here he presumed that his audience would simply take Law V as they would take e.g. Law I, namely as quite clearly a law of logic, needing no further justification. If one can make sense of imagining the logically impossible, one can imagine that, had the law not led to inconsistency, Frege would on this point have been right.

Let me close by once again expressing my gratitude to Roy Cook, Kai Wehmeier, and Marcus Rossberg for having raised what I think are an extremely interesting collection of issues in the interpretation of Frege's work, and for having afforded me the opportunity to say a few words about them. Thanks also to Richard Zach for organizing both the APA event and this presentation of its content.

Patricia Blanchette
Dept. of Philosophy
University of Notre Dame
blanchette.1@nd.edu

Notes

[1] *Grundgesetze* II sec 63

[2] *Grundgesetze* II sec 56

[3] *FC* p. 19 (*CP* 148, KS 135)

[4] See also *Grundgesetze* II section 58

[5] "Frege on Shared Belief and Total Functions," *Journal of Philosophy* CIX 1/2 (Jan/Feb 2012) 9-39

[6] *Grundgesetze* I section 29

[7] *Grundgesetze* II sec 147