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Critical Remarks on *Frege's Conception of Logic* by Patricia Blanchette

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1. Introduction

Patricia Blanchette's study *Frege's Conception of Logic* is a major event in Frege research. With painstaking care, rigor, and exegetical sympathy, Blanchette investigates how Frege approached his logical enterprise, how his views differ from ours, and to what extent tenets of his that have not survived into the contemporary mainstream have something to offer us.

That this would be a splendid book is of course no surprise to those who have followed Blanchette's work on Frege over the years; her seminal article on the Frege-Hilbert controversy (Blanchette 1996) alone (a topic taken up and enlarged upon in chapter 5 of the work under discussion) have made her stand out as an expert on Frege's take on logic. Another remarkable feature of the book is Blanchette's informed and even-handed discussion, in chapter 7, of Frege's engagement with metatheory. I am inclined to think, tentatively at least, that she has gotten this exactly right. In any case, there is no doubt that Frege's Conception will become an instant classic and a must-read for anyone hoping to enter the field of Frege studies.

There is, however, one issue that Blanchette, in my view, gets flat-out wrong, to wit, Frege's attitude toward the requirement of sharp boundaries for concepts or, more generally, that of the totality of functions. Chapter 3 of *Frege's Conception* is devoted to this topic. In keeping with my task as a critic, I will focus in what follows on what I take to be Blanchette's fundamental misinterpretation of Frege's text in this chapter. Needless to say, this should in no way detract from my admiration for and appreciation of the work as a whole.

2. The Issue

In numerous places, some of which Blanchette cites, Frege insists that functions must be defined for all arguments, and concomitantly, that it must be determined for any object whatsoever—Julius Caesar, the Moon, the Earth's axis, England, the number One, what have you—whether it falls under a given (first-level) concept. Blanchette asks us not to take Frege by his word. He doesn't mean what he says, she claims, and indeed, even with respect to the logical system of his own *Grundgesetze*, Frege does not require functions to be total: According to Blanchette, "the functions referred to in *Grundgesetze* are very clearly not total" (p. 58). Moreover, the quantifiers of *Grundgesetze* do not range over all objects whatsoever; rather, they range only over value-ranges and truth-values (p. 74).

On the face of it, this is an extraordinary exegetical claim, since Frege not only never says that the *Grundgesetze* functions are partial (or even "very partial", as Blanchette puts it on p. 60), but rather, as noted, frequently and emphatically asserts just the opposite. Blanchette seems to feel the tension in the reading she offers, noting that Frege makes some (from her perspective) "puzzling pronouncements" (p. 65), accusing him of an "incautious way in which he sometimes puts the requirement of sharp boundaries" (p. 68), finding that "Frege is simply not very clear" (p. 72) and attributing to him a "notable lack of comprehensive precision" (p. 74).

Of course Frege's writing isn't *always* as lucid as one might wish, but it typically is. Thus having to posit major confusion on

Frege's part in order to make a particular exegetical line palatable should always be a red flag for an interpreter. Yet Blanchette contends, with respect to the conventional view that Frege requires all first-level functions to be defined for all arguments whatsoever, that "the texts just don't bear this out" (p. 68) and points to "Frege's practice" (p. 62) which, according to her reading, shows that functions need only be defined for arguments nameable in the language of *Grundgesetze*, that is, for value-ranges and truth-values.

Given the extraordinary nature of her contention, one is somewhat taken aback to discover that Blanchette's only evidence for her claim about *Grundgesetze* appears to be a remark in §10 of volume I. I'll discuss this passage in a moment, but I first want to point out how implausible the proposed reading is in the face of Frege's practice in other parts of *Grundgesetze*.

Consider, to begin with, Frege's lengthy footnote 1 on p. 18 of *Grundgesetze* I, toward the end of §10. Its point is to examine the prospects of a proposal to "generalise our stipulation so that every object is conceived as a value-range." The stipulation Frege refers to is the identification, justified in §10, of the two truth-values with particular value-ranges, a topic to which we'll return. What's important about this footnote is just that, according to Blanchette's reading, such a proposal would have been entirely pointless: If the only objects in the domain of the theory at hand are value-ranges and truth-values, as Blanchette makes out, and the truth-values have already been identified as certain value-ranges, then there is neither need nor scope for an additional proposal to identify every object with a value-range, since every object in the domain already is a value-range.

There are other places where Frege's practice contradicts Blanchette's reading. In §34 of *Grundgesetze* I, he introduces the application function $\xi \cap \zeta$ with the words: "The object $\Phi(\Delta)$ appears as the value of the function $\xi \cap \zeta$ – with two arguments, Δ as the ξ-argument, and $\dot{\varepsilon}\Phi(\varepsilon)$ as the ζ-argument." He immediately adds: "However, $\xi \cap \zeta$ must be defined for all possible objects as arguments." This addition would be entirely unnecessary if the domain of the theory of *Grundgesetze* consisted only of value-ranges (which, after the stipulation of §10, subsume the truth-values), since all possible arguments would then already have been considered. That this is not just an oversight on Frege's part becomes clear when he reiterates the point after presenting his definition A of $\xi \cap \zeta$, explicitly noting that

> two cases have to be distinguished in order for the value of the function $\xi \cap \zeta$ to be determined. When the ζ-argument is a value-range, then the value of the function $\xi \cap \zeta$ is the value of that function whose value-range is the ζ-argument for the ξ-argument as argument. If on the other hand the ζ-argument is not a value–range, then the value of the function $\xi \cap \zeta$ is $\dot{\varepsilon}(\mathrm{\top}\, \varepsilon = \varepsilon)$ for every ξ-argument.

According to Blanchette's reading, the case where the ζ-argument is not a value-range simply cannot occur, and it would be mysterious why Frege feels compelled to mention it.

There are more places like this, e.g. §35 of *Grundgesetze* I (p. 54), where Frege notes that if "an object is taken as argument of the function $2 \cap \xi$ that is not a value-range, then we have no corresponding argument for the second-level function $\varphi(2)$ and so the mutual representability of functions of the first and second level lapses." Similarly in the final paragraph of §36 (p. 55): "One can still ask what $\Gamma \cap (\Delta \cap \Theta)$ is when Θ is not a double value-range but merely a simple value-range or not a value-range at all." Clearly it is a live option for Frege that some objects within the purview of his theory fail to be value-ranges.

Apart from Frege's practice in *Grundgesetze*, there are other reasons issuing from Frege's theoretical commitments that militate against Blanchette's reading, according to which the range of Fregean quantifiers can vary with the theory in which they are being applied. One such commitment that Blanchette herself again and again, rightly, emphasizes is that, for Frege, all begriffsschrift sentences are fully interpreted. This just isn't the case on Blanchette's reading, since, for her, when we consider Theorem 32

$$\begin{array}{l} \vdash\!\!\top\!\!\!\top\; \not{\wp} u = \not{\wp} v \\ \quad\;\, \llcorner\; u \frown (v \frown \rangle q) \\ \quad \llcorner\; v \frown (u \frown \rangle \not{\mathfrak{I}} q) \end{array}$$

in the context of *Grundgesetze*, the thought it expresses entails nothing about the knives and forks on my dinner table, whereas if we consider it as part of a joint theory of arithmetic and medium-sized empirical objects, it entails the thought that the number of forks is the same as the number of knives. We thus have here a single begriffsschrift sentence that expresses different thoughts depending on the theoretical context within which it is being used— a decidedly un-Fregean notion.

It may be worth noting here that, under Blanchette's interpretation, not even all value-ranges can be in the domain over which *Grundgesetze*'s first-level quantifiers range. Consider, for instance, the value range $\dot{\varepsilon}$ (ε = Caesar) of the concept ξ = Caesar. It cannot be in the domain, for otherwise; the result \ $\dot{\varepsilon}$ (ε = Caesar), i.e. Caesar, of applying Frege's description function $\backslash\xi$ to it would also be in the domain, contradicting the assumption that the domain consists only of value-ranges and truth-values. Thus presumably only "pure" value-ranges, whose members, members of members, etc., are all value-ranges, can be admitted.

Now Frege holds that Basic Law V is a law of logic. But which Basic Law V? The version in which the first-level quantifier ranges over all objects whatsoever, the version in which it ranges only over the objects in the scope of the theory of *Grundgesetze*, whatever that is, the version in which it ranges over the objects in the scope of a mathematical theory of celestial mechanics? All of them? It seems that, if Frege really allows the quantificational parts of his begriffsschrift sentences to be reinterpretable according to theoretical context, he quite obviously cannot just assume without argument that Basic Law V is a logical law no matter what the domain of first-order quantification.

3. Blanchette's Textual Evidence

Blanchette tells us that Frege, in §10, "discusses the fact that course-of-values names have not yet been assigned determinate reference via the stipulations given to this point" (p. 58). Her uncharacteristically vague formulation makes one wonder: What does it mean that these names have not been assigned *determinate* reference? Do they have *in*determinate rather than determinate reference? If so, what is it to have indeterminate reference? Frege's own words (quoted by Blanchette) are:

> By presenting the combination of signs '$\dot{\varepsilon}\Phi(\varepsilon) = \dot{\alpha}\Psi(\alpha)$'as coreferential with '$\smile\!\!\!\!\!\!^{\mathfrak{a}}\;\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', we have admittedly by no means yet completely fixed the reference of a name such as '$\dot{\varepsilon}\Phi(\varepsilon)$'. (GGI, p. 16)

Note that Frege doesn't say anything about no reference, or only "indeterminate reference", having been assigned to value-range terms. In fact, he speaks quite comfortably about *the* reference of a value-range term, which, given his sensitivity to the problem of empty singular terms, we may well the take as an indication that these terms, for him, very much do have a reference. Blanchette, however, apparently reads Frege's formulation that "we have (...) by no means yet completely fixed the reference of a name such as '$\grave{\varepsilon}\Phi(\varepsilon)$'" as meaning that this term has *no* reference. This is simply not warranted by the text, as becomes even clearer in the sentences immediately following the above quotation:

> We have only a way always to recognise a value-range as the same if it is designated by a name such as '$\grave{\varepsilon}\Phi(\varepsilon)$', whereby it is already recognisable as a value-range. However, we cannot decide yet whether an object that is not given to us as a value-range is a value-range or which function it may belong to (. . .). (GGI, p. 16)

What's at issue is thus not that value-range terms haven't been assigned a reference, or only an indeterminate one; rather, the issue is that the validity of Basic Law V, that is, the co-reference of the begriffsschrift expressions '$\grave{\varepsilon}\Phi(\varepsilon) = \grave{\alpha}\Psi(\alpha)$' and '$\smile^{\mathfrak{a}}\, \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', underdetermines which particular injection from first-level functions to objects the value-range function is. That is why we cannot *decide* whether an arbitrary object not presented to us by means of a value-range term is a value-range or not; the reason is not that value-range terms somehow lack "determinate" reference. The (first version of the) permutation argument, which immediately follows but is not quoted by Blanchette, makes this quite clear:[1]

> If we assume that $X(\xi)$ is a function that never receives the same value for different arguments, then exactly the same criterion for recognition holds for the objects whose names have the form '$X(\grave{\varepsilon}\Phi(\varepsilon))$' as for the objects whose signs have the form '$\grave{\varepsilon}\Phi(\varepsilon)$'. For then '$X(\grave{\varepsilon}\Phi(\varepsilon)) = X(\grave{\alpha}\Psi(\alpha))$' too is co-referential with '$\smile^{\mathfrak{a}}\, \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$'. From this it follows that by equating the reference of '$\grave{\varepsilon}\Phi(\varepsilon) = \grave{\alpha}\Psi(\alpha)$' with that of '$\smile^{\mathfrak{a}}\, \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', the reference of a name such as '$\grave{\varepsilon}\Phi(\varepsilon)$' is by no means completely determined (...)

In other words, the logical source of knowledge tells us, or so Frege believes, that the value-range function is an injection from first-level functions into objects (that's the content of Basic Law V), but it doesn't tell us anything more about this injection. In particular, from this piece of logical knowledge it is underdetermined whether some arbitrarily given object falls within the range of the value-range function. That quite clearly does not entail that the value-range function is not total on the space of all unary first-level functions, only that we limited beings do not know, of each object *a*, whether it falls into the function's range and if so, for which first-level function as argument *a* is the value of the value-range function.

Indeed, it is hard to see how the permutation argument itself could even make sense in the absence of references for value-range terms. For if '$\grave{\varepsilon}\Phi(\varepsilon)$' lacks reference, surely '$X(\grave{\varepsilon}\Phi(\varepsilon))$' does, too, and Frege is no more entitled to speak of *the objects whose names have the form* '$X(\grave{\varepsilon}\Phi(\varepsilon))$' than he is of *the objects whose signs have the form* '$\grave{\varepsilon}\Phi(\varepsilon)$'; yet that is precisely what he does.

Blanchette is impressed by the fact that Frege, in §10, "does not bring it about that the concept-phrase '$\underline{\quad\quad} = \grave{\varepsilon}(\varepsilon = \varepsilon)$' determinately holds or fails to hold of each object," and that he rather "brings it about that every way of completing this concept-phrase with a singular term of the language (. . .) has a determinate truth-value" (p. 59). But on the straightforward reading of §10, there is no pressure on Frege in the first place to ensure that '$\underline{\quad\quad} = \grave{\varepsilon}(\varepsilon = \varepsilon)$' "determinately holds or fails to hold of each object", for it already does.

A final point: Blanchette makes much of a passage in *Funktion und Begriff* (Frege 1891, 19/32) which she reads as endorsing the claim that, as long as only numbers are under discussion, the addition function need only be defined for numbers. According to her, Frege in this passage claims that "an arithmetic that deals just with the integers can quite happily incorporate an addition sign defined over just the integers" (2012, 68), and "as long as we're talking just about e.g. the integers, it's entirely coherent to define the addition sign just over the integers" (2012, 74). But it is not at all clear that the passage can bear this interpretive weight. The one sentence Blanchette actually quotes from the paragraph in question reads as follows:

> So long as the only objects dealt with in arithmetic are the integers, the letters *a* and *b* in "$a + b$" indicate only integers; the plus-sign need be defined only between integers.

But given what follows, it seems quite likely that what Frege is doing here is not *endorsing* the position that a partial definition of addition is fine, but rather laying out the naïve view he is about to reject. For the text continues (all emphases by the author):

> Every extension of the field to which the objects indicated by *a* and *b* belong obliges us to give a new definition of the plus-sign. *It seems to be demanded by scientific rigour that we should have provisos against an expression's possibly coming to have no reference*; we must see to it that we never perform calculations with empty signs in the belief that we are dealing with objects. People have in the past carried out invalid procedures with divergent infinite series. It is thus necessary to lay down rules from which it follows, e.g., what
>
> '$\odot$+1'
>
> stands for, if '$\odot$' is to stand for the Sun. What rules we lay down is a matter of comparative indifference; but *it is essential that we should do so—that 'a + b' should always have a reference*, whatever signs for definite objects may be inserted in place of '*a*' and '*b*.'

There is at least a suggestion here that restricting the possible values of '*a*' and '*b*' to the integers, and accordingly defining addition only for numbers, runs counter to the demands of scientific rigor; and in the end, Frege seems to say explicitly that it would be a mistake to rely on such a restriction.

In any case, it appears that Frege could not have endorsed the view Blanchette imputes to him on the basis of the passage in *Funktion und Begriff*, since it runs counter to basic principles of his logical theory. For, once it is acknowledged that there are numbers, Frege must acknowledge that there are numerical concepts, and hence that there are value-ranges of numerical concepts. But these value-ranges are objects, just like the numbers, and hence possible arguments to the addition function. So the problem of having to contend with empty terms—in this case, terms for sums of value-ranges that are not themselves numbers—arises even on the assumption that we're trying to restrict the domain to the integers.[2]

4. But does it matter?

Blanchette worries about the totality requirement arise mainly from her desire to obviate Joan Weiner's interpretation of Frege, according to which there is, strictly speaking, no conceptual analysis of informal mathematical notions involved, because informal mathematical discourse, ostensibly violating the totality requirement, fails to express thoughts. She wants to establish, that is, that Frege certainly does conceive of informal mathematical talk as dealing in thoughts. Her chapter 3 is devoted to the argument that failure of totality cannot possibly condemn mathematical discourse to the realm of non-thought-expressing practice because that would entail that the begriffsschrift theorems of *Grundgesetze* themselves would fail to express thoughts, given the (purported) failure of totality in *Grundgesetze*.

As I have argued, this may not have been the most felicitous strategic move on Blanchette's part, as it appears that Frege did maintain the totality requirement in *Grundgesetze*. But how worried does she really need to be in the face of Weiner's challenge? Not very, I should think.

This is not the place to discuss alternative replies at any length, but let us briefly consider a very simple example mentioned by Blanchette, the phrase "the eldest child of". Since there is no function that maps each person to their eldest child, so that the totality requirement is violated, one might think that no sentence containing the phrase can express a thought (p. 57). But there are several ways in which one can avoid that conclusion.

One would be to maintain that the thought expressed by a sentence of the form "$\mathcal{C}$(the eldest child of X)" is that expressed by "for every total function f with the property that, whenever x is a person who has an eldest child, y, f maps x to y, it is the case that $\mathcal{C}(f(X))$".

Another possibility is to claim that different speakers may associate different total functions with the phrase "the eldest child of", just as they often associate different senses with a proper name like "Aristotle", but that the thoughts associated with typical sentences containing the phrase are sufficiently similar for Frege's purposes.

Either way would seem to accord with Frege's pronouncement that "[w]hat rules we lay down [for the don't care cases] is a matter of comparative indifference" (Frege 1891, 20–21/33). In any case, I don't think anything of consequence in Blanchette's book ultimately depends on denying that Frege maintained the totality requirement in *Grundgesetze*, so my criticism here in no way undermines her larger interpretive project.[3]

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Notes

[1] For an extended discussion of the arguments in §10, see Wehmeier and Schroeder-Heister 2005.

[2] Note, incidentally, that Frege could not have assumed that the numbers themselves might simply serve as the value-ranges of arithmetical concepts, for he was well aware, by the time of *Grundgesetze* at least, of Cantor's Theorem. See (Wehmeier 2004, 250).

[3] Thanks to Patricia Blanchette, Roy Cook, Marcus Rossberg, and Matthias Schirn for discussions regarding the material presented here, and to Richard Zach for organizing the APA book symposium on Frege's Conception of Logic, and for inviting me to participate.

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