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Blanchette on Frege on Analysis and Content
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1. The Analysis of 'Cardinal Number'

With *Frege's Conception of Logic*, Patricia Blanchette presents a detailed and comprehensive analysis of Gottlob Frege's notions of logic, logical consequence, and conceptual analysis, and the role these play in his logicist project. Blanchette's investigation is of fundamental importance for Frege scholarship. It will prove to be consequential not only exegetically, but also in providing contemporary (not just Fregean) philosophy with a better understanding of distinctive contributions Frege made that have unjustly been forgotten but provide enticing alternatives to current conceptions of logic and analysis.

There is little to complain about in Blanchette's fabulous book. The friendly suggestion I want to make pertains to her account of Frege's notion of analysis in his mature logicist project, and the relation of this notion of analysis to her explanation of Frege's conception of logical consequence. Blanchette investigates the question what exactly Frege's logicist reduction of arithmetic amounts to. Crucial here is the "recarving of content" that is said to be achieved by Hume's Principle. Frege's claim in *Grundlagen* is that identity statements for cardinal numbers of the form:

(N) $NxFx = NxGx$

have the same content as equinumerosity claims: that F is equinumerous with G (i.e., that there exists a bijection between the concepts):

(EQ) $F \approx G$

As Frege puts it: "We carve up the content in a way different from the original way, and this yields us a new concept."[1] Equivalence statements of cardinal numbers thus receive their content and truth conditions.

The *Grundlagen* analysis employs extensions. The cardinal number of the concept F is defined as the extension of the second-level concept *equinumerous with F*, $ext(\Phi \approx F)$, thus the target analysis, to be demonstrated to follow logically from (EQ) is

(N_{Gl}) $ext(\Phi \approx F) = ext(\Psi \approx G)$

Blanchette makes the case that the logicist reduction proceeds by stepwise analysis. (N) is analyzed as (EQ). By purely logical inter-derivability of (N_{Gl}) and (EQ), these too have the same content. Therefore, by transitivity, the sameness of content of (N) and (N_{Gl}) is established.

The mutual derivability (N_{Gl}) and (EQ) is demonstrated using a principle that allows us to transform propositions of the form

(UG) $\forall x(Fx \equiv Gx)$

into propositions of the form

(EXT) $ext(F) = ext(G)$

and *vice versa*. In *Grundlagen*, Frege does not explicitly state this principle while clearly relying on it and its obviousness. By the time of *Grundgesetze*, Frege has this transformation formulated in full generality. Instead of extensions, he employs value-ranges. All functions, not just concepts, are now equipped with these corresponding objects. Frege gives the following explanation (Frege 2013, vol. I, §3, p. 7):

I use the words

> "the function $\Phi(\xi)$ has the same *value-range* as the function $\Psi(\xi)$"

always as co-referential with the words

> "the functions $\Phi(\xi)$ and $\Psi(\xi)$ always have the same value for the same argument."

Frege continues using the term "extension" for value-ranges of concepts, since for all intents and purposes this is what they are. This "transformation of the generality of an equality into a value-range equality", as Frege calls it (Frege, 2013, vol. I, §9, p. 14), was dubbed the "Initial Stipulation" by Richard Heck (2011). It gives rise to the ill-fated Basic Law V in the *Grundgesetze* system of concept-script:

$$(\text{V}) \quad \vdash \left(\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha) \right) = \overset{\mathfrak{a}}{\backsim} \left(f(\mathfrak{a}) = g(\mathfrak{a}) \right)$$

Note that '$\dot{\varepsilon}$' (and likewise, '$\dot{\alpha}$') denote the second-level function that takes first-level functions to their value-range, so that $\dot{\varepsilon}f(\varepsilon)$ is the value-range of function *f*; that '$\overset{\mathfrak{a}}{\backsim}$' is Frege's universal quantifier, and '$\mathfrak{a}$' the variable it binds; and that Frege uses '$=$' for both identity and biconditional. We will get back to the latter.

The analysans of (N) in *Grundgesetze* thus becomes

$$(\text{N}_{\text{Gg}}) \; \dot{\varepsilon}\big(\exists H(\varepsilon = \dot{\alpha}H(\alpha) \wedge H \approx F) \big) = \dot{\varepsilon}\big(\exists H(\varepsilon = \dot{\alpha}H(\alpha) \wedge H \approx G) \big)$$

that is, the cardinal number of *F* is now analyzed as the extension of the *first-level* concept under which fall all and only those value-ranges of functions that are equinumerous with *F*. Again, (N) has the same content as (N$_{\text{Gg}}$) by the inter-derivability of the latter and (EQ), this time explicitly mediated by Basic Law V.

Blanchette argues forcefully and in detail that Frege's project is best understood in this way. The reduction of ordinary mathemat-

ics to logic is carried out by the analysis of (N) as (EQ), which in turn is logically equivalent to (N$_{\text{Gg}}$). Blanchette's central claim is that logical consequence is preserved over Frege's analysis. Accordingly, Frege not only shows that the laws of arithmetic can be derived from pure logic for his cardinal numbers, but also that the same holds true for the cardinal numbers of our ordinary lives, those that mathematicians have proved theorems about all along and continue to do so.

I will not take issue with Blanchette's characterization of the reduction involved in Frege's project. I find it plausible that Frege must have had something very much like this in mind, albeit probably not in as much beautiful detail as Blanchette provides. What I disagree with, however, is the exact role Blanchette assigns to Basic Law V in her reconstruction of Frege's project.

2. Sense and Reference and Basic Law V

Grundgesetze contains a second consequential change to Frege's project. Between *Grundlagen* and *Grundgesetze*, Frege drew the distinction between sense and reference. As he puts it in the Foreword to *Grundgesetze*:

> [C]ontent I called judgeable content. This now splits for me into what I call thought and what I call truth-value. This is a consequence of the distinction between the sense and the reference of a sign. In this instance, the thought is the sense of a proposition and the truth-value is its reference. (Frege, 2013, vol. I, p. x)

So what has become of the sameness of content after content has thus fallen apart into sense and reference? Is sense preserved between (N), (EQ), and (N$_{\text{Gg}}$), or is it merely reference? As explained above, Blanchette suggests an intriguing alternative: logical entailment is all that matters.

Let us thus consider what Blanchette presents as Frege's conception of logical consequence. It is neither proof- nor model-theoretic, but operates on *thoughts*, i.e., on the senses of (interpreted) sentences. If logical consequence thus operates on sense, what of Basic Law V, which plays the all important role in the logicist reduction? If by the Initial Stipulation we may infer an identity of value-ranges from a generality of an equivalence and vice versa, then the thoughts expressed by these logically entail each other. This may suggest that Basic Law V states that its right- and left-hand side have the same sense, or express the same thought.[2]

Indeed, Blanchette seems sympathetic to this view. She argues that Frege held on to what is now usually called the "multiple decomposition" of content after the introduction of the sense–reference distinction: "After the advent of the mature semantic theory, the entities multiply decomposable are thoughts" (Blanchette, 2012, p. 41). If this is so, Frege might have taken the left-hand side of Basic Law V, decomposed to involve value-ranges, to express the same thought as the right-hand side, decomposed not to involve value-ranges. The trouble is, nowhere in *Grundgesetze* does Frege say that the two sides of Basic Law V have the same sense or express the same thought. Our questions thus are: To what extent are thoughts multiply decomposable in Frege's view? Does Frege take Basic Law V to express the sameness of sense for its two sides? If the answer to the latter is negative, does this pose a problem for Blanchette's reconstruction of Frege's logicist project?

Blanchette provides three examples in which Frege, after the introduction of the sense–reference distinction, commits to multiple decomposition. Her second example (p. 41), which we will consider first, can be found in the short 1906 piece from Frege's *Nachlass*, "A Brief Survey of my Logical Doctrines" ("*Kurze Übersicht meiner Logischen Lehren*", 1906, p. 218). Frege writes that

one and the same thought can be split up in different ways and so can be seen as put together out of parts in different ways.

The example Frege cites here is however rather harmless. He considers splitting up the proposition '1^2 is greater than 2', and accordingly the thought expressed by it, in two different ways:

(1) ξ is greater than 2

(2) ξ^2 is greater than 2

This is obviously much simpler than the recarving or multiple decomposition suggested for Basic Law V. Also Blanchette's first example seems less radical. This example originates from "On Concept and Object" ("*Ueber Begriff und Gegenstand*"). Blanchette quotes Frege (1892b, p. 200):

It is thus not impossible that one way of analyzing a thought should make it appear as a singular judgment; another, as a particular judgement; and a third, as a universal judgement.

But the operative word here is "appear" ["*erscheint*"]. The sentence immediately preceding the one quote above is:

Language has means of presenting now one, now another, part of the thought as as the subject; one of the most familiar is the distinction of active and passive forms.

Frege's examples are:

(3) There is at least one square-root of 4.

(4) The concept *square-root* of 4 is satisfied.

(5) The number 4 has the property that there is something of which it is the square.

Frege's point is this: "The thought itself does not yet determine what is to be regarded as the subject", where the notion of subject in question is that of the grammatical subject. Note that "language" in the latter quote above means natural language—for instance, English or German—not the representation of the thought in concept-script. (3)–(5) (arguably) are all of the logical form:

(6) $\exists x\; x^2 = 4$

albeit with different emphases.

These examples are much more closely related to those Frege discusses as early as *Begriffsschrift*, which Blanchette too describes as less radical than the recarvings encoded in Basic Law V or Hume's Principle (Blanchette, 2012, p. 41). The former example can be seen as neglecting some structure that would be available in the decomposition of the thought. In such a case, the decomposition is superimposed on the finest-grained structure ('ξ is greater than ζ' would be another option for the analysis). The latter examples regard features of natural language, like active and passive voice, grammatical subject, etc., which are not seen as logically relevant and thus will not survive the analysis in concept-script; think of the notorious 'Cato killed Cato', 'Cato was killed by Cato', 'Cato killed himself', etc. (Frege, 1879, §9).[3]

While we could easily reject these examples as tangential to the question of whether Basic Law V asserts the sameness of sense of its right and left hand sides, this will not do as a response to Blanchette's third example, the passage from *Function and Concept*, where Frege claims that

$x^2 - 4x = x(x-4)$

and

$$\dot{\varepsilon}\big(\varepsilon^2 - 4\varepsilon\big) = \dot{\alpha}\big(\alpha(\alpha - 4)\big)$$

express "the same sense, but in a different way" (Frege (1891), p. 10; compare Blanchette (2012), p. 42).

The 1891 lecture *Function and Concept* is the place where Frege first introduces his distinction between sense and reference. It could be pointed out that the relevant passage occurs before the part where Frege draws the distinction, but we should probably not reject taking "sense" seriously here merely on this ground.[4]

Had the phrase "the same sense, but in a different way" occurred in a Fregean writing before 1891, we would probably read it as a first, terminologically confused flickering of the sense–reference distinction: "the same sense" should really be "the same reference"; "a different way" should be "a different sense".[5] But Frege does say that the two propositions express the same sense, i.e., the same thought—a reference is not expressed. Nonetheless, it just does not fit Frege's attitude as we find it in *Grundgesetze* or any other writing after *Function and Concept*. The best, I think, we can say is: Frege should not have written this.

Frege is clear, in *Grundgesetze* and elsewhere, that for instance '2²' and '2 + 2' have the same reference but different senses (Frege, 2013, vol. I, p. IX). Indeed, Frege introduces the sense–reference distinction precisely because he is seeking to make sense of informative identity statements (Frege, 1892a). If ''2² = 2 + 2' is informative (and thus not a statement of sameness of sense, but merely of sameness of reference), surely, the sameness of reference of the two formulae displayed above must count as informative too, and thus neither they should have the same sense. The same goes for Basic Law V. Biconditionals in *Grundgesetze* are lumped together with identities, since the role of both is to express the sameness of reference. (In the case of propositions, this reference is the True or the False.) So, Basic Law V states that its two sides

have the same reference. But we should not conclude that the sense too is the same. Basic Law V is an informative identity statement if ever there was one.

Blanchette observes (p. 44) that Frege is coy about committing to Basic Law V's stating a sameness of sense. This may be understated. The impression one get is rather that Frege carefully avoids suggesting that the sense is the same. In *Grundgesetze*, Frege repeatedly states that the two sides of Basic Law V are co-referential, but never that they express the same sense. So too in the course of the infamous permutation argument in §10.[6] Frege starts the section by describing the problem he is about to tackle: "By presenting the combination of signs '$\dot\varepsilon\Phi(\varepsilon) = \dot\alpha\Psi(\alpha)$' as co-referential with '$\mathbin{-\!\!\!-}\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', we have admittedly by no means yet completely fixed the reference of a name such as '$\dot\varepsilon\Phi(\varepsilon)$'." Only a few lines down, regarding an assumed permutation X, he writes that "'$X(\dot\varepsilon\Phi(\varepsilon)) = X(\dot\alpha\Psi(\alpha))$' too is co-referential with '$\mathbin{-\!\!\!-}\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$'", adding in a footnote: "Thereby it is not said that the sense is the same." Some commentators have interpreted this footnote as evidence that Frege held that, in contrast, the sameness of sense does obtain for Basic Law V.[7] Perhaps it suggests that Frege would have liked the two sides of Basic Law V to have the same sense. But if he had thought that he could claim this sameness of sense, it would be hard to see what prevented him from stating this claim explicitly, either in the first sentence of the paragraph or in the footnote.

I must submit: if Frege was not sloppy or confused, then he changed his mind between 1891 and 1893.[8] This explanation is simpler than explaining why, in the ten years and probably more that Frege held on to value-ranges, he never again, despite ample opportunity, stated that an identity of value-ranges has the same *sense* as the corresponding generality of an equivalence. In 1891,

Frege had been tempted to think that the sameness of sense obtained. In 1893, he had realized that his theory of sense and reference does not yield this result.

3. Definition and Explanation

The principal way for Frege to preserve sense, in *Grundgesetze* and elsewhere, is by definition. But Basic Law V is of course not a definition, but a basic law of logic. Neither is the "Initial Stipulation" in §3 of the first volume of *Grundgesetze* a definition, as Frege emphasizes in volume II in his discussion of the §3 stipulation: "This conversion is not to be taken as a definition" (Frege, 2013, vol. II, §146). Also in general, Frege cautions his readers, "we should not regard the stipulations about the primitive signs in the first volume as definitions. Only what is logically composite can be defined; what is simple can only be pointed to." (Frege, 2013, vol. II, p. 148 fn. 1) So, the "stipulation" here is not a stipulative definition, and indeed no definition at all.

Austin's translation of *Grundlagen* (Frege, 1950, §64) calls the recarving of 'straight line *a* is parallel to straight line *b*' as 'the direction of *a* = the direction of *b*' a "definition". But the German word here is not "*Definition*"; it is "*Erklärung*": "explanation". In the passage regarding the equality of cardinal number in §62 of *Grundlagen*, the analogous translation of the cognate verb appears: Austin translates "*erklären*" as "define" instead of "explain". It is common and tempting to translate "*Erklärung*" in Frege in these kinds of context as "definition"; and there is indeed a use of "*Erklärung*" in mathematical literature to mean just this. But Frege keeps the notions carefully separate, although he does not lose a single word about the difference in *Grundgesetze*. He might have taken it to be too obvious to clarify.[9]

Explanation, for Frege, is a wider notion than definition. Defining is certainly an excellent way of explaining, given that a definition stipulates the sameness of sense for explanans and explanandum. But not all notions can be defined. For instance, the "*Erklärung*" of '$\underset{a}{\smile}$' in *Grundgesetze* (vol. I, §8, p. 12) surely *cannot* be a definition: '$\underset{a}{\smile}$' is primitive. Also in vol. I (§4, p.8), Frege writes: "The functions with two arguments, $\xi = \zeta$ and $\xi > \zeta$, always have a truth-value as value (at least if the signs '=' and '>' are explained in the appropriate way)." '=' is primitive and thus undefined; '>' would end up being defined. Frege subsumed both, the definition of '>' and the explanation of '=', under the term "explanation". Finally, Part 2 of vol. I of *Grundgesetze*, entitled "Definitions" (§26, p. 43), starts in its very first sentence with: "The signs explained so far will now be used to introduce new names." Indeed, as the title of the part correctly suggests, nothing has been defined in *Grundgesetze* before this, but only primitive signs have been *explained*.

Where Frege gives explanations, or writes about explanations of primitive signs, he frequently mentions reference, but never sense. In *Grundgesetze*, §18, Frege writes: "From the reference of the function-name $\backslash\xi$ (§11) $\vdash a = \backslash\grave{\varepsilon}(a = \varepsilon)$ follows", where §11 contains the explanation of the (primitive) concept-script substitute for the definite article, '$\backslash\xi$'. The explanation of '$\underset{a}{\smile}$' in §8, mentioned above, is all about reference, and so too is the explanation of 'value-range' and '$\grave{\varepsilon}\Phi(\varepsilon)$' in §§3 and 9. Perhaps most telling is a passage in §30 of the first volume of *Grundgesetze*, where Frege uses the phrase "explanation of reference".

In his "Comments on Sense and Reference", written between 1892 and 1895, Frege is clear that the focus of logic and mathematics is on reference: "the reference and not the sense of words [is] what is essential for logic [. . .] the laws of logic are first and fore-

most laws in the realm of reference and relate only indirectly to sense. [...] Reference thus proves to be what is essential for science." (Frege, 1892–1895, pp. 132–134), (Frege, 1979, pp. 122–123). Thus, what matters for logic and Frege's logicism is that the two sides of Basic Law V have the same reference. Since Basic Law V was to be a law of logic, the co-reference of its two sides is, of course, more than *mere* co-reference. It is sameness of reference as a matter of logic. But this is not to say, sameness of sense.

Blanchette's insightful account of Fregean analysis and its role in the logicist project should survive abandoning the thought that Basic Law V expresses sameness of sense. The alteration proposed here may indeed be seen to be in keeping with her suggestion regarding the central importance of logical consequence for Fregean analysis. Logical equivalence, and not sameness of sense, appears to take the role that the mysterious sameness of content played before the sense–reference distinction in almost all the important transitions, as befits a logicist project. The thought is that whatever the analysis of (N) as (EQ) guarantees, it is at least some strong, not merely contingent, co-reference for all instances, even if it falls short of sameness of sense. So, (N) and (N_{Gg}) too are guaranteed to have the same reference, by the logical equivalence of (EQ) and (N_{Gg}). If Blanchette's arguments that logical consequence is preserved over Fregean analysis can survive abandoning sense as a replacement for content in the analysis of (N) as (EQ)—an investigation as careful and detailed as Blanchette's would have to show—then (N) and (N_{Gg}) should still be logically equivalent, since (N)'s analysans (EQ) is logically equivalent to (N_{Gg}). Basic Law V sits in the middle of the derivation of (N_{Gg}) from (EQ), and of the latter from the former. The two sides of Basic Law V do not have the same sense; nor do they have to. Sameness of reference as a matter of logic suffices, and that is what Basic Law V would have delivered, had it not been for its inconsistency.[10]

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Notes

[1] Frege (1884)/(1950), p. 75. Ebert (2014) finds Austin's translation of Frege's word "*zerspalten*" as "carve up", and the subsequent use of "recarving" in the literature contentious. He suggests to follow Dummett (1991, p. 168) in using the literal translation "split up".

[2] That Frege held that abstractions principles like Hume's Principle or Basic Law V state the sameness of sense for the propositions on their two sides is argued, among others, by Beaney (2005), Burge (1990), Currie (1982), Milne (1989), and Simons (1992); see, e.g., (Dummett, 1991, ch. 14), Ebert (2014), or (Klement, 2002, ch. 3) for dissent.

[3] Also note the striking similarity of these examples with the different ways that Frege (1880–1881) lists of expressing '$2^4 = 16$' in natural language, using, for example, the phrases 'fourth root of 16' or 'logarithm of 16 to the base 2'. He continues: "We may now also regard the 16 in $x^4 = 16$ as replaceable in its turn, which we may represent, say, by $x^4 = y$. In this way we arrive at the concept of a relation, namely of the relation of a number to its 4th power. And so instead of putting a judgement together out of an individual as a subject and an already previously formed concept as predicate, we do the opposite and arrive at a concept by splitting up the content of possible judgement." (Frege, 1979, p. 17) Compare also Picardi (1993, p. 76) for her categorization of five different classes of pairs of propositions Frege seems to consider as equipollent at some point or another, and her discussion of which of these may qualify for sameness of sense.

[4] Klement (2002, p. 87) suggest that "sense" here is a mere slip, that Frege may not have had "fully mastered the distinction or the terminology" yet. Klement also (ibid., fn. 28) cites the fact that Frege's claim occurs before the introduction of the distinction (which appears a few paragraphs later) as possible evidence that he might not have used "sense" as a technical term in this passage. Simons (1992, p. 765) finds it "out of character for Frege to have made such a slip".

[5] Indeed, very similar phrases do occur before the distinction, for example in *Begriffsschrift*, §8: "the same content can be completely determined in different ways", or *Grundlagen*, §62: "we must reproduce the content of the proposition in a different way". (The proposition to which Frege refers is "the cardinal number which belongs to the concept *F* is the same as that which belongs to the concept *G*".) Whether "content" here should be understood as "sense", however, is our question.

[6] Frege (2013), vol. I, p. 16; see Heck (2012, ch. 4), and Wehmeier and Schroeder-Heister (2005) for discussion.

[7] See, e.g., Simons (1992, p. 764). Schirn (2006) and Ebert (2014), for instance, disagree.

[8] This is also suggest by Ebert (2014, §2.2, fn. 26).

[9] His discussion of the terms in his debate with Hilbert is a different matter—and complicated by the fact that Frege picks up on Hilbert's use of "*Erklärung*".

[10] I am indebted to Patricia Blanchette, Roy Cook, Philip Ebert, Walter Pedriali, Kai Wehmeier, and Richard Zach for helpful comments and discussions.

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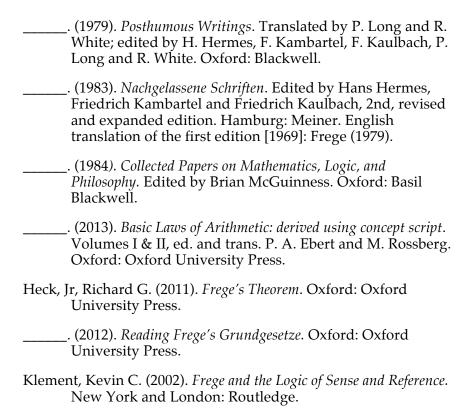
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