Journal for the History of Analytical Philosophy

Volume 2, Number 1

David Bostock, *Russell's Logical Atomism*. Cambridge: Oxford University Press, 2012. ISBN: 9780199651443

Reviewed by Gregory Landini

Review: *Russell's Logical Atomism* by David Bostock

Gregory Landini

Throughout the twentieth-century, Bertrand Russell influenced a wide community of intellectuals with his many philosophical positions and analyses. In 1911 he came to describe a new scientific method in philosophy which applies the new mathematics and the new logic to solve (dissolve) philosophical problems. He called it "Logical Atomism." Bostock offers an engaging book which consists of three main parts, each of which concerns various aspects of Russell's Logical Atomism from its early formation to the neutral monism of *The Analysis of Matter* (1927). Part I offers a critical introduction to Russell's logic and a detailed history of its development from *The Principles of Mathematics* (1903) through *Principia Mathematica* (1910-1913). Bostock realizes that this history is important for the subsequent parts of his book, and his expertise as a logician shows in his discussions. Part II concerns Russell's theory of knowledge and Part III offers an overview of Russell's metaphysics.

Bostock does not discuss Wittgenstein's logical atomism because he thinks it is "clearly rather different from Russell's" (p. viii) and warrants a separate account. This difference was once clear to many, but some recent interpretations place Wittgenstein as Russell's ally in a research program—a Logical Atomism understood as a non-empiricist analytic program according to which the only necessity is logical necessity. The last two decades have produced something of an upheaval of the orthodox interpretation that Russell's philosophy belongs in the empiricist tradition and that Russell embraced a ramified and type regimented ontology of entities ("propositional functions"). The upheaval was spurred by the appearance of Russell's voluminous work notes which are catalogued at the Russell Archives at McMaster University, Ontario, Canada. Bostock remarks in his Preface (p. *viii*) that all of Part I endured "extended criticism." Though an iconoclast, this may explain why Part I rejects the new perspectives on Russell's philosophy of logic. Fortunately, this does not impact the whole of Bostock's interesting book.

Bostock nicely sums up Russell's logicism as "… making two claims (*a*) that the concepts of mathematics can be analyzed in terms of the concepts of pure logic, and (*b*) that the *truths* of mathematics then turn out to be no more than truths of logic" (p.2). It is refreshing to see that Bostock does *not* take Russell's logicism to be committed to the existence of a *derivation* of all mathematical truths from a consistent, recursively axiomatizable and semantically complete system for logic. Russell, Frege, and many others at the time, thought that logical truths (and also mathematical truths) are consistently recursively axiomatizable. Gödel showed they are not. But Gödel's results are irrelevant to logicism. Unfortunately, it doesn't take long for the deductive thesis to creep back into the discussion and we find Bostock suggesting that the logicist claim is "… that all of mathematics could be reached from a starting point which comprised pure logic and nothing else. (p. 19).

Bostock nicely points out that the new mathematics used multi-placed relations to emulate the structural field properties of integers, rationals, reals, and complex numbers, had shown Russell the way to logicism (p. 17). Bostock observes that the notion of "logic" in Russell's logicism involves more than just modern "elementary logic" (or quantification theory). But Bostock would have us believe that Russell was never quite sure just in what way it needs to go beyond it (p. 15). Be this as it may, I feel sure that Bostock agrees that Russell knew precisely what was involved in the new conception of logic that makes it an *informative* science. He knew that it involves impredicative *comprehension*. Let me call this "^{cp}Logic" (for comprehension principle logic).

Bostock remarks that Frege embraced the comprehension of a simple hierarchy of *levels* of functions (where *concepts* are those functions that yield truth-values) and grounded it in his ontological distinction between *functions* (which are essentially unsaturated) and *objects* which are saturated (p. 13). Bostock doesn't discuss the central role Frege assigned to higher-level functions in his logic. Natural numbers are objects correlated with Frege's *second-level* numeric concepts. But Bostock well understands that the structure of the Fregean extensional hierarchy of *levels* of functions (given with Frege's structured variables x, fx, $M_x fx$, $\Sigma_f(M_x fx)$, etc.) parallels the structure of a Russellian impredicative simple *type*-theory of attributes ... in intension (p. 57). Indeed, he astutely points out that *Principia* construes arithmetic in terms of numerical quantifiers (p. 109). Moreover, Bostock is quite correct that Russell could not accept Frege's function/concept *versus* object distinction. Nonetheless, Bostock resists putting all this together and does not see the development of Russell's logic as an effort to *emulate* the structure (without the ontology) of an impredicative simple type-theory of attributes in intention. Russell's long struggle toward *Principia*—a struggle involving his ingenious substitutional theory of propositional structure— was his quest for a type-free recasting of logical first principles that *emulates* simple impredicative type theory.

Bostock gives in to the historians who think, dogmatically, that free variables were anathema to Russell's logic. The dogma dismisses the obvious *syntactic* fact (of which Bostock is fully aware p. 14) that *free* variables are essential to Russell's actual deductive systems. In the 1908 paper "Mathematical Logic as Based on the Theory of Types" (*ML*), Russell even cites Frege as first to point out the essential importance between free versus bound variables, writing that "… we have to pass from the apparent to the real variable, and then back again to the apparent variable. This process is required in all mathematical reasoning…" There is every reason to believe that Russell held that free variables are essential to deduction. (Russell eventually changed his mind, adding to the second-edition of *Prinicipia* a plan for deduction without free variables that anticipated Quine's system by some fifteen years.) Of course, Russell (and indeed everyone at the time) was incompetent at formal semantics—i.e., Tarski's semantics for free variables using denumerable sequences. But this has no bearing whatsoever on the syntactic matter of whether there are free variables in Russell's logic.

It is no less obvious that *schemata* appear in Russell's formal systems for logic. The dogmatic historians try to spin this, and Bostock defers his own good judgment to their talking points (p.14): "Russell did not think of his logical formulae as containing schematic letters, open to any number of different interpretations. On the contrary, his view is that in logic one makes genuine assertions, that no genuine assertions can contain what is merely a schematic letter, and consequently that these letters should always be understood as variables bound by the tacit occurrence of an initial universal quantifier." As Bostock well knows, in modern characterizations of the axioms of a formal system, we often use schematic letters for *wffs* that help us to gather together *wffs* of a syntactic form that are to be the axioms of the system. This is perfectly compatible with the view that in a ^{cp}logic, where predicate variables can be bound (such as standard second-order logic and simple type-theory of attributes), we make assertions such as $(\exists F)(x = x \equiv_x Fx)$ and $(\exists R)(x = y \equiv_{x,y} R(x, y))$ that are about something— namely, properties and relations. The most straightforward interpretation is that *Principia* uses the letters φ, ψ, χ, *f*, *g* schematically for *wffs*. Thus, for example, φ*x* is for a *wff* in which the individual variable "*x*" occurs free. In contrast, the letters φ!, ψ!, χ!, *f!*, *g!* of *Principia* are bindable object-language predicate variables of its formal language. Type indices are dropped under the convention of typical ambiguity, but if restored they are clearly just simple-type indices (See *Principia*, p. 165). Whitehead and Russell were among the very first to set out explicit *comprehension* axiom schemas for their formal system for logic. Consider this:

$$\text{*12.1} \quad (\exists f)(\varphi x \equiv_x f!x).$$

Here we see that φx is schematic for a *wff* of the object-language. For instance, in the proof of *14.171, Whitehead and Russell appeal to

$$(\exists f)(x = x \equiv_x f!x)$$

as an instance of *12.1. To be sure, schema *12.1 is not itself an assertion, but it gathers together for us many *wffs* of a form which are axioms for the system.

Indeed, *12 is *impredicative*. It allows instances of the schema φx that involve *wffs* without restrictions on the type of the predicate variables (bound or free) occurring in it. Indeed, the above instance is impredicative for $x = x$ is defined in *Principia* as

$$\text{*13.01} \quad x = y =df \quad (\psi)(\psi!x \,.\!\supset\!.\, \psi!y).$$

Poincaré railed against this sort of thing, portraying it as if it were a viciously circular "definition" of the attribute *f!* because the quantifier (ψ)(….) includes all attributes in its range—including *f!* In excluding it, Poincaré hoped to undermine Cantor's revolution in mathematics. Cantor's diagonal arguments, the foundation of his revolution, require such impredicative instances of comprehension. Frege's definition of the ancestral, which enables logic to prove mathematical induction, also essentially requires impredicative comprehension. It is often said that Russell agreed with Poincaré. In truth, Russell defended Cantor's work and lampooned Poincaré's VCP as self-undermining—a demand that we are to mention that certain things are not to be mentioned which is no way to avoid a painful topic. (See Russell's *ML*).

The formal system of the historical *Principia* is that of a simple impredicative type-theory of attributes in intension and its axioms of reducibility *12 are its comprehension principles. At times Bostock is inclined to agree, but he hastens to add that this must have been done for practical purposes only (p. 54). Church, who invented the modern system of *r*-types (ramified types) with its nonpredicative predicate variables and Reducibility axiom, at least warned readers that he has intentionally abandoned the historical *Principia.* Church ("A Comparison of Russell's Solution of the Semantical Antinomies with that of Tarski") offered a reconstruction, on the basis of his reading of *ML*, that seemed to him to have been what is "… clearly demanded by the background and purposes of Russell's logic". But *ML* was Russell's substitutional theory retrofitted with orders of propositions and it embraced a language of *r*-type regimented predicate variables merely as a notational convenience. At that time, Russell imagined that the substitutional theory would appear in an appendix to *Principia*. It never happened. Bostock knows that *Principia* abandoned the substitutional theory (p. 54) but he doesn't quite realize that this entails that he abandoned *ML* and with it the language of *r*-types.

Bostock emphasizes the centrality of Russell's doctrine that the variables of logic are unrestricted (p. 14). It is important, however, to understand that Russell's doctrine was not semantic. It was a syntactic regulative principle constraining the formulation of any formal calculus that claims to capture ᶜᵖlogic. It is a syntactic demand that a formal system for ᶜᵖlogic shall have only one kind of genuine variable—i.e., the *individual* (entity) variable. This is precisely what we find in Russell's substitutional theory. The curious thing about *Principia,* in contrast to Russell's formal logical systems that preceded it, is that it adopts a simple type regimented syntax of predicate variables. This shift requires an explanation, and to this day it animates debate. Church's theory of *r*-types soon became the canonical interpretation in spite of Russell's explicit statements to the contrary that, e.g., "Whitehead and I thought of a propositional function as an expression containing an undetermined variable and becoming an ordinary sentence as soon as a value is assigned to the variable" (See e.g., "On the Notion of Cause 1912, *Introduction to Mathematical Philosophy* 1919, *My Philosophical Development 1959*). Church maintains that Whitehead and Russell intended a realist (objectual) semantics for *Principia's* predicate variables. Bostock concurs with this (p. 72, 249). But as we see, there is ample evidence that they intended a substitutional ("nominalistic") semantics for the predicate variables of *Principia* which makes them "internally limited by their significance conditions" while the *individual* variables of lowest type are genuine (objectual). This explains why Whitehead and Russell offer a recursive definition of a hierarchy of senses of truth and falsehood. To facilitate the recursion, the definitions of the quantificational system of section *9 define subordinate quantifiers in terms of *wffs* in which all quantifiers are initially placed. For example, we find

$$(x)\varphi x \vee (\exists y){\sim}\psi y \;=\!df_{*9.07}\; (x)(\exists y)\,(\varphi x \vee {\sim}\psi y),$$

where x is not free in the wff ψy and y is not free in the *wff* φx. If the *wffs* φx and ψy are quantifier-free, then the recursion maintains that

$(x)(\exists y)\,(\varphi x \vee {\sim}\psi y)$ is true 1.2 *iff*

for all individuals x, $(\exists y)\,(\varphi x \vee {\sim}\psi y)$ is true 1.1.

Eventually, we reach

$\varphi x \vee {\sim}\psi y$ is true 1.0 *iff* either φx is true or $({\sim}\psi y)$ is true.

$({\sim}\psi y)$ is true *iff* ψy is not true.

For the base case of the recursion at atomic *wffs*, the multiple-relation theory of judgment is to be applied. *Principia's* "systematic ambiguity of truth and falsehood," admits of fine grained distinctions since the number of quantifiers matters as well as the simple type-indices on the sorts of variable they bind. *Principia* says that these fine grained features can be safely ignored in practice (*PM*, p. 162). Bostock acknowledges all this, but somehow imagines the existence of such fine grained distinctions to be evidence against our interpretation that the notion of *order* is philosophically grounded in *Principia's* hierarchy of senses of "truth" and "falsehood" (pp. 88, 218).

Bostock points out that Russell accepts Cantor's power-theorem and proves it in *Principia*. He is concerned, however. He writes that "… statements about the real numbers may always be paraphrased as statements about the propositional functions from which he [Russell] 'constructs' them. So this must presume that there are non-denumerably many propositional functions. But then propositional functions cannot be merely linguistic expressions" (p. 249). I fear that this confuses a substitutional semantics for the predicate variables of *Principia* with a realist (objectual) semantics for predicate variables whose domain consists of linguistic expressions. It should be noted that a consistent first order set theory that includes a proof of Cantor's power-class theorem has a *denumerable* model. And there are denumerable non-standard Henkin models even for higher-order systems.

A substitutional semantics is not a realist semantics whose domain consists of linguistic entities. Be this as it may, Bostock is justifiably concerned. The substitutional semantics Whitehead and Russell offer *fails* to validate all instances of *12.1. Ramsey convinced Russell of this, and likely it was in virtue of this that, by 1919, Russell came to admit, *12 is unacceptable. The substitutional semantics validates only predicative instances of *12. This led Church and his followers to "improve" *Principia*. Church adds comprehension axioms and a grammar for non-predicative predicate variables with order indices that can be above the order of the simple type symbol, and he distinguishes an explicit Reducibility axiom. Ramsey's "improvement," in stark contrast with Church, left *Principia* as it is and offered a new substitutional semantics for its predicate variables that allows infinitely long conjunctions and disjunctions— a semantics that he hoped would validate *12. Bostock accepts none of this and perpetuates the mistaken view that Ramsey advocated a simple type ontology of entities (p. 219).

Ramsey is characterized by Bostock as having distinguished semantic paradoxes from the logical paradoxes. Russell, in contrast, is characterized as holding that they all stem from a common source—a violation of Poincaré's VCP. In his efforts to illustrate the alleged commonality of the paradoxes, Bostock neglects many significant distinctions among the paradoxes (p. 75). To make Berry's paradox of "the least integer not nameable in fewer than nineteen syllables" akin to the indexical Liar ("*This* sentence is false"), he presents it as a paradox of "the least integer not named by *this* name." It doesn't work. No name contains an indexical and definite descriptions are not names for Russell. Russell's paradox is strangely portrayed as akin to the Grelling which involves the property *Heterological* (Het). An adjectival expression has the property Het *iff* it names an attribute that the adjective does not exemplify. For example, "red" is Het since it not red. To make it appear as if this situation were analogous to the Russell paradox of the set of all sets not members of themselves, Bostock characterizes Het as a property an adjective has *iff* it is not true of itself (p, 23, 75). The analogy limps. Set membership ∈ is a primitive relation in an ontology of sets, while "being true of" is certainly not primitive and must be defined with the help of semantic notions such as "naming" or "denoting." All this might be forgiven, but for the fact that the thesis of the commonality of the paradoxes is flatly rejected in Russell's writings on his substitutional theory. (See "On 'Insolubilia' and Their Solution by Symbolic Logic" (*InS*) and "On the Substitutional Theory of Classes and Relations" (*STCR*).) Indeed, we find Russell *dismissing* the paradoxes of "nameability" and "definability" (employed in the Berry, the König-Dixon and Richard paradoxes) as confused *viciously circular* notions. In dismissing such paradoxes as confusions, Russell explains why Richard's paradox does not jeopardize the well ordering theorem. In contrast, Russell did not dismiss the paradoxes of sets and attributes (such as Cantor's paradox of the greatest cardinal, the Burali-Forti, and Russell's paradoxes of classes and attributes) which require, as he put it, "some elaborate re-statement of logical principles."

The elaborate restatement was originally to be Russell's substitutional theory of propositional structure. Russell's early logical particle signs are quite different from what appears in *Principia*. Bostock seems uninterested in this fact (p. 203). But it is important. Russell held that the horseshoe sign "⊃" stands for a universal—the relation of 'implication.' In *Principia,* the horseshoe sign is the modern sign for "if … then" which is flanked by *wffs* to form a *wff.* During the era of Russellian propositions, the sign is flanked by terms to form a *wff*. To make this clear, let us use the sign Ɔ allowing α Ɔ β as a *wff* where α and β are any terms. Where A is any *wff* of the language, {A} is a term. Thus, Russell allows x Ɔ {y Ɔ x} or for convenience x . Ɔ. y Ɔ x. Then Russell has:

~α =*df* α Ɔ *f*
α ∨ β =*df* α Ɔ β . Ɔ. β
α • β =*df* ~ (α Ɔ ~β)
α ≡ β =*df* (α Ɔ β) • (β Ɔ α).

The difference is not insignificant. It is essential to the intelligibility of Russell's substitutional theory.

Bostock speaks of Russell's substitutional theory as a "no properties theory" (p. 48) and seems unaware that substitutional theory embraces the existence of type-free properties and relations. We have already noted that Russell embraces a relation of 'implication,' and in *InS* "~" stands for a property. The expression "$p/a;b!q$" says that the entity q is structurally exactly like p except for containing entity b wherever a is contained in p. This is a four-place relation of *substitution*. (Every lower-case letter of the English alphabet is an individual variable; there are no special "propositional variables.") The expression $p/a;b$ abbreviates the definite description $(\iota q)(p/a;b\,!\,q)$. The expression $s/t,\ w;p,\ a$ abbreviates the definite description $(\iota q)(s/t,\ w;\ p,\ a\,!\,q)$ where such multiple substitutions are defined in terms of a series of cleverly crafted single substitutions. A few examples show us how a simple impredicative type theory of attributes in intension is emulated. In place of $\varphi^{(o)}(x^o)$ the substitutional theory has

$$(\exists q)(p/a;x!r \equiv_r r = q \,.\bullet.\, q).$$

In place of $\sim\varphi^{(o)}(x^o)$ the substitutional theory has $\sim(p/a;x)$. Here are some examples emulating comprehension:

$$(\exists\varphi^{(o)})(\varphi^{(o)}(x^o, y^o) \equiv_{x^o, y^o} x^o = y^o),$$

$$(\exists\, p, a, b)\,(p/a, b;x, y \equiv_{x,y} \{x = y\}).$$

For the next simple type, consider the following

$$(\exists\psi^{((o))})(\psi^{((o))}(\varphi^{(o)}) \equiv_{\varphi^{(o)}} (x^o)\sim\varphi^{(o)}(x^o))$$

$$(\exists\, s, t, w)\,(s/t,w;p,a \equiv_{p,a} \{(x)(\sim(p/a;x)\}).$$

Note that it is not just the number of substitutions that emulates the type; it depends on the proposition in which those substitutions are made.

Russell's paradox, as Bostock notes (p. 22), was discovered when Russell investigated Cantor's power theorem as applied to the universal class V. Now every class is a member of V, but there is a function f (the identity function) from a sub-group of objects in V (namely, the class of all classes) onto the classes of those objects. In that case, the function f of Cantor's diagonal class is identity. Thus, the class C such that $(x)(x \in C \equiv (\exists y)(xfy \;\&\; x \notin y))$ is the Russell class. Bostock fails to note, however, that Cantor's diagonal class essentially requires the *impredicative* comprehension of a class (or of an attribute). In striking contrast, Russell's paradoxes of classes and attributes do *not* require impredicative comprehension. In any case, Russell was in earnest to emulate an impredicative simple type theory of attributes. This led him to his substitutional theory of propositional structure. And in stark opposition to Poincaré's diatribe against Cantor about viciously circular "definitions," Russell demanded that any solution must *preserve* Cantor's work.

Now in a letter to Couturat of 1904, Russell offers a general syntactic form for the generation of paradoxes of classes and attributes. He reveals that he knew that where f is a one-one function, we get a contradiction from each of the following:

$$(\exists w)(x \in w \equiv_x (\exists y)(x = fy \;.\&.\; x \notin y))$$

$$(\exists\varphi)(\varphi x \equiv_x (\exists\psi)(x = f\psi \;.\&.\; \sim\psi x))$$

For example, we get the Russell paradox of classes when $fz = z$, and we get a paradox if $fz = \iota z$. Bostock points out that Russell realized that this utterly destroys the naïve notion of a class (p. 23). Russell knew that such paradoxes are blocked by a simple-type theory of classes. But type-theories of *entities* were anathema to

Russell's metaphysics. The impredicative simple type theory of attributes (and thereby classes) must be emulated by a type-free theory of entities.

Russell arrived at the general form above by studying an interesting result concerning a pairing of his ontology of propositions with a theory of classes (or attributes). He first set it out in Appendix B of *Principles.* Bostock insightfully calls attention to this paradox (p. 23):

$$(\exists w)(x \in w \equiv_x (\exists y)(x = \{p \in y \supset_p p\} \bullet x \notin y)).$$

In 1904, Russell knew of the analog for attributes

$$(\exists \varphi)(\varphi x \equiv_x (\exists \psi)(x = \{\psi p \supset_p p\} \bullet {\sim}\psi x)).$$

These paradoxes, Russell observed, are *not* blocked by simple type-theory. Observe that the general form of the Appendix B paradoxes is the same as the above, namely the following (respectively):

$$(\exists w)(x \in w \equiv_x (\exists y)(x = fy \ .\&.\ x \notin y))$$

$$(\exists \varphi)(\varphi x \equiv_x (\exists \psi)(x = f\psi \ .\&.\ {\sim}\psi x)).$$

There is, however, an important difference. The Appendix B paradoxes are unique in that they arise because Russellian propositions assure the existence of a one-one function *f* that violates Cantor's power-theorem. Cantor's theorem establishes that there can be no function from any group of objects onto the attributes (or classes) of those objects. Given the strong identity conditions for propositions, there is a one-one function *f* such that $fy = \{p \in y \supset_p p\}$. Its inverse, therefore, is a function in violation of Cantor's theorem. Similarly, there is a one-one function *f* such that $f\psi = \{\psi p \supset_p p\}$. Russell concluded that he cannot embrace a theory of propositions together with a simple type theory of attributes (or classes). From 1905 through 1908, Russell chose propositions. His substitutional theory of propositional structure is wholly type free and endeavors to emulate a simple impredicative type theory of attributes in intension. (Through scope, Russell constructs extensional contexts from intensional contexts and emulates simple types of classes.)

Bostock observes (p. 53) that in its original form the substitutional theory was inconsistent. He correctly recounts that its inconsistency was due to a paradox (which I have called the "p_o/a_o paradox") akin to the paradox of Appendix B of *Principles*. But when it comes to the details, Bostock's formulation has problems. Let us begin with the correct formulation:

$$(x)(p_o/a_o;x \equiv \{(\exists p, a)(x = \{p/a;b!q\} \ .\bullet.\ {\sim}(p/a;x)\}).$$

By universal instantiation, we get:

$$p_o/a_o;\{p_o/a_o; b!q\} \equiv$$
$$\{(\exists p, a)(\{p_o/a_o;b!q\}=\{p/a;b!q\} \ .\bullet.\ {\sim}(p/a;\{p_o/a_o;b!q\})\}).$$

Now Russell's intensional logic of propositions assures the following strong identity conditions for propositions:

$$\{p_o/a;b!q\} = \{p/a;b!q\} \ .\supset.\ p_o = p \bullet a_o = a.$$

Hence we arrive at:

$$p_o/a_o;\{p_o/a_o;b!q\} \equiv {\sim}(p_o/a_o;\{p_o/a_o;b!q\}.$$

Note that this is akin to the Appendix B paradox because it involves a one-one function *f* such that $f(p, a) = \{p/a; b!q\}$ and this violates Cantor's power theorem. Its inverse is a function from objects of the form $\{p/a; b!q\}$ to pairs *p, a* which are used in the substitutional technique to emulate attributes in intension of objects. Unlike Cantor, however, Russell cannot deny the existence of such a function. It is assured by Russell's strong identity conditions for propositions.

Unfortunately, the essential role of the *strong* identity conditions for propositions is lost in Bostock's attempt (p.53) to formulate the paradox as follows:

$$(\forall x)(p_o/a_o ; x \leftrightarrow (\exists p, a)((x = p/a ; x) \;\&\; \sim(p/a;x))$$

For the sake of convenience, we might forgive Bostock's use of modern logical particles (double arrow, &), but Bostock's clause $x = p/a; x$ does *not* render a function that violates Cantor's theorem. There is no contradiction. We noted that "$p/a;b$" is used by Russell to abbreviate the definite description. Russell's intensional logic does *not* have the following as a theorem:

$$p/a;b = r/c;b \;\; .\supset .\; p = r \bullet a = c.$$

Suppose $a \neq b$. It follows that $\{a = a\} \neq \{b = a\}$. But

$$\{a = a\} / a; b = \{b = a\} / a; b \;.$$

That is, the proposition, namely, $\{b = b\}$, which is the result of substituting b for a in $\{a = a\}$ is identical with the proposition which is the result of substituting b for a in $\{b = a\}$.

Bostock recognizes the importance of the p_o/a_o paradox. This is laudable, but these details are important. The p_o/a_o paradox is produced by the same diagonal method that produces Cantor's power-theorem. Liar paradoxes are not diagonal paradoxes violating Cantor's results. The p_o/a_o paradox is not among the semantic paradoxes involving notions of 'defining,' or 'naming' that Russell dismissed. It is not a paradox involving *truth*, and it cannot arise in Russell's consistent (albeit self-referential) quantification theory of propositions. Russell accepts Cantor's work and the *impredicative* diagonal methods that generate it. But he must find a way to block Cantor's diagonal method from applying to propositions and generating the p_o/a_o paradox.

In *InS* Russell thought the solution was to modify the substitutional theory by abandoning general propositions. There remain general *wffs*, but now only a quantifier-free *wff*, A, can be nominalized to make a term {A} for a proposition. Russell came to realize, however, that this did not work. His new approach was *ML* which was finished by July of 1907. General propositions return, but propositions are split into orders. We can nominalize a formula A of the language of substitution to make a term $\{A\}_v$ in accordance with the following rule: If n is the highest order index on any variable occurring in A, then $v = n + 1$ if the variable is bound and $v = n$ if the variable is free. In *ML*, Russell doesn't say what orders are required in the *wff*

$$p_m / a_u \,;\, b_v ! q_n \,.$$

But after trying many experiments in a 1907 manuscript called "On Types," he decides that one must have $m = n$ and $u = v$. This is the substitutional theory retrofitted with orders of propositions. The addition of orders to the substitutional theory has the consequence that what is emulated is no longer a simple impredicative type theory of attributes. What is emulated is a ramified type theory of attributes. For example, we get the following theorem. We have pm

$$(\exists p_m, a_v)(x_v)(p_m / a_v ; x_v \equiv \{Ax_v\}_m),$$

Where p_m and a_v are not free in the *wff* A. To preserve Cantor's work in the context of orders of propositions, Russell added new axioms of *Reducibility* applied to propositions in extensional contexts. For example, Russell has:

$$(q_n, b_0)(\exists p_1, a_0)(x_0)(p_m / a_0 ; x_0 \equiv q_n / b_0 ; x_0).$$

According to *ML*, the new substitutional theory (with its orders of propositions) was to be in an appendix to *Principia.* The practical working language would employ the "technically convenient" notations of a *ramified*-type theory of attributes. Laudably, Bostock seems to sense that *Principia* is not the theory of *ML* (pp. 53, 205, 246).

The story of the origin and nature of *Principia* is complicated. No one can be blamed for an incomplete grasp of Russell's many twists and turns, and difficulties with any interpretation are sure to exist. In spite of the deleterious influence of some of the dogmatisms, Bostock has made a solid contribution to the history. His negative assessment of Russell's type theory, however, is unwarranted. Bostock holds that *Principia* resorts to a non-logical "infinity axiom" (*Infin ax*) for the theory of progressions (p. 99). But in fact there is no such axiom in *Principia*. Instead, *Infin ax* forms an antecedent clause of some theorems. Bostock sides with the traditional metaphysicians of mathematics (p.103), maintaining that this antecedent is untoward and must be removed. But *Principia* stands against the metaphysicians: the infinity of the natural numbers may (epistemically) be a logical truth, but it is *not* an arithmetic truth!

In spite of Bostock's concerns, I see no telling objection to simple type theory. The inconveniences it produces are easily overcome by Whitehead and Russell's technique of typical ambiguity. Bostock worries that the effect of the grammar of types is to hamstring both logic and mathematics" (p. 107). He worries, for example, that one cannot say that a relation *R* is transitive in every level [type]. But quite to the contrary, we can say that a homogeneous relation $R^{(t,t)}$ of any type *t* is transitive. The schematic type indices *(t, t)* does the trick; and similarly we can say that the non-homogeneous relations $R^{(t,(t))}$ and $R^{((t),t))}$ are transitive no matter the type *t*. Volume II of *Principia* is filled with all sorts of wonderful theorems concerning non-homogeneous cardinals of relative types. Whitehead and Russell's logicism makes no proscription that mathematicians work in a language adorned with type indices. *Principia* is itself completely free of any type indices! Whitehead and Russell do have a proscription, however. It is that mathematics is the study of all the relational structure kinds there are. And all relational structure *kinds*, whether a given structure is simple type regimented or not, can be represented in simple-type theory. Thus, even if there are metaphysical structures that violate simple types (e.g., the von Neumann progression Λ, {Λ}, {Λ, {Λ}}, …, the Zermelo progression Λ, {Λ}, {{Λ}}, … it is only their structure type, *progression*, that matters to mathematics and this can be studied in simple type theory.

Let us turn now to Part II and Part III of Bostock's engaging book. The striking theses of Part II are that (1) Russell anticipated Gettier's problem for the definition of *knowledge* as justified true belief (p. 137) and proposed as a solution that in "derivative knowledge" the propositions which form one's justification must themselves be *known intuitively*; (2) Russell anticipated Goodman's new riddle of induction (p. 141), and (3) that Russell abandoned neutral monism (p.175, 197). Bostock's discussion is insightful. It should be noted, however, that Gettier rejected the *Theaetetus* definition of *knowledge* as justified true belief, demanding a fourth condition. Russell's discussion of why true belief is not sufficient for knowledge in *The Problems of Philosophy,* and in *Enquiry into Meaning and Truth* (1940), and *Human Knowledge: Its Scope and Limits* (1948) suggests that he accepts the *Theaetetus* definition.

In *Our Knowledge of the External World as a Field for Scientific Method in Philosophy* (*KEW*), Russell endorses acquaintance with sense-data and *sensibilia* (unsensed sense-data) understood as transient *physical* particulars. This period forms the apex of Russell's commitment to the relation of *acquaintance*. Bostock draws attention to Russell's struggle in his abandoned book *Theory of Knowledge* (1913) to give an account of what are to be the objects of acquaintance when we have knowledge of logical notions. He concludes that "… it was not a good idea on Russell's part to suppose that all parts of any propositions that I understand must be things that I am acquainted with. This evidently does not apply to the logical notions involved, and I have argued that it does not apply to predicates either" (p. 133). In sympathy with Russell, we can at least understand the seriousness of the problem of our knowledge of logical notions.

In *KEW*, Russell was following Whitehead's suggestion that the physical laws of *matter* (i.e., continuants persisting through time) are, as Bostock puts it, to be constructed in terms of a "… temporally ordered and continuous series of classes of nearly simultaneous (actual and possible) experiences [i.e., sense-data and sensibilia] (p. 165). In *Problems*, Russell had not offered a construction, but instead imagined material continuants as inferred—a form of knowledge by description— as the best explanation of the data of sense. Bostock prefers this tack (p. 168), but he gives a nice discussion of Russell's construction of matter and the identity of a bit of matter "at a time" and "through time" in (Chapter 9). Central to Bostock's discussion is the ontological status of *sensibilia* in Russell's constructions. Bostock maintains (p. 156) that at each place unoccupied by a perceiver there actually are various physical particulars (e.g., light-waves), that will cause a sense-datum (appearance), but they are not "the same kind of things" and are not "in themselves like appearances." Bostock admits that Russell does not say this explicitly and does call both "appearances." In his eagerness in *KEW* to make the transient sense-datum play the role of something non-mental and yet subjective, Russell says that sense-data and sensibilia depend on the perceiver. The key issue, however, is whether in *KEW* a sense-datum's existence requires states of sense-organs, nerves, and the brain of a perceiver. Surprisingly, Bostock thinks that it does, in spite of its being an *object* of an independent *act* of sensing.

Bostock thinks that when Russell converted to neutral monism, the sense-datum is subsumed into the *act* of sensing (p. 137). This, however, is far from clear. It forces a Spinozistic one-to-one parallelism upon Russell's neutral monism that, in embracing *images*, Russell seems quite clearly to reject. In the *Analysis of Mind*, Russell lampooned behaviorist attempts to replace imagistic thinking with, e.g., micro-movements in the larynx. In Russell's view, images are transient physical particulars that occur *only* in series that constitute minds, while "sensations" can appear in both series that constitute minds and those that constitute matter. Bostock says that this "… departs from the official theory of neutral monism" (p. 175). But perhaps it is just a departure from Bostock's parallelist version of neutral monism.

Russell's neutral monism, it should be noted, was not very "neutral" since his view is that *physical* transient particulars (in some works infelicitously called "sensations") occur in both series that constitute minds and those that constitute matter. But Bostock has a different notion of neutrality in mind. He writes that "Russell's version of neutral monism was never properly 'neutral' or 'monistic'. Most of the ingredients from which minds are constructed do not also occur in matter, and conversely most of the ingredients from which matter is constructed do not occur in minds" (p. 190). In Russell's *Analysis of Mind*, *some* physical transient particulars are *images* and occur only in minds. I don't know why parallelism should be counted as the "proper" form of neutral monism.

The deeper question with Russell's neutral monism, it seems to me, is the question as to how a transient particular (whether an image or otherwise), or a series of them, takes on a *qualitative* nature merely by being a part of a series that constitutes a mind. If it has such a qualitative nature intrinsically (which Russell explicitly denies in *Analysis of Mind*, p. 297), then it is different in kind from those transient particulars that do not have such a qualitative character and so there is, after all, no genuinely neutral transient particulars. This may be the sort of concern Bostock means to raise in his discussion of Russell's view that images occur only in minds. Perhaps images have an intrinsically qualitative character. In *Outline of Philosophy* (1925) Broad pushed Russell, raising what amounts to Jackson's "Mary argument." Russell wonders whether qualitative character is an emergent property of those series that constitute minds. Unfortunately, the philosophical culture of the time was to be entirely dismissive of Cartesian notions such as *knowledge* (as opposed to a capacity for reacting appropriately to stimuli), *life*, *subjectivity* and what we nowadays call "*qualia*".

Bostock points out (p.174) that Russell's early rejection of neutral monism had been based on the problem of giving an account of the facts of *presentation* (the inverse of the relation of *acquaintance*) —that a mind can be presented with *this*. Indexicals ["emphatic particulars"] such as *I*, *this*, *here*, *now*, had pushed Russell away from neutral monism in the 1913 *Theory of Knowledge* (from which his paper "On the nature of Acquaintance was excerpted), though he entertained the thesis that only the indexical "*this*" need be primitive. For example, "I" is defined as "the subject of this experience." Russell was concerned that there must be a 'self' [momentary subject] if there is *presentation* (rendered as the inverse relation involved in me being *acquainted* with *this*). Oddly, Bostock thinks that Russell just abandoned the concern. But in *Analysis of Mind* (p. 179*f*) Russell did try to address the indexical "*this*" and *presentation*, and he explicitly derides what it regards as confused notions of, *consciousness*, *introspective* knowing and even Brentano's principle of *intentionality.*

Ultimately, Bostock concludes that there was a "demise" of neutral monism (i.e., that Russell came unconsciously to abandon it). His argument is that in the *Analysis of Matter*, a bit of physical matter is constructed as a *cause* of appearances and not as a series of appearances (p.195). This is an interesting concern worth further research, but it is far from clear that Bostock has made a telling case in its favor.

Part III is significant because it is here (pp. 220ff) that Bostock allows himself to speculate on what may have been in Russell's mind when he was developing his multiple-relation theory of judgment in the aborted *Theory of Knowledge* book. Wittgenstein's criticisms of the theory are addressed here as well. The innovations are new and worth further study. In his chapter on Russellian facts, Bostock goes on to offer a discussion of Russell's view in his lectures on logical atomism--- views which seem to be a liberal departure from his conception of facts as truth-makers in *Principia*. It is here that Russell entertained, in addition to the abstract fully general logical facts of *Theory of Knowledge*, general (but not fully general) facts, and the *negative facts* which "nearly produced a riot." Bostock briefly touches on Russell's conception of the logically perfect language and questions of extensionality.

Bostock also returns to Russell's theory of definite descriptions and offers the surprising thesis that Russell "… could have defined 'E!*a*,' meaning that *a* exists, as short for '$(\exists x)(x = a)$' in either case, i.e., when '*a*' is a simple name or a definite description" (p. 264). This proposal undermines the fact that no definition in *Principia* applies to definite description expressions since the latter are not terms of the formal language. Definitions involving individual variables, such as *13.01 cannot be applied to expressions involving definite descriptions such as "$x = \iota y \psi y$". One must first eliminate the description. I see no reason to think that "$(\exists x)(x = \iota y \psi y)$" says that "$\iota y \psi y$ exists". Quite clearly, it says

$$(\exists x)((\exists y)(\psi u \equiv_u u = y \;.\&.\; x = y))$$

i.e., there is something that is identical to something that is uniquely ψ. In *Principia*, ontological commitment derives from the use of variables. There is no *wff* in *Principia*, the predication of which generates an ontological commitment.

Bostock's book is essential reading whether one agrees or disagrees with details of his account. It is natural to focus on important points of disagreement in a review, but there is much to praise about the scope and depth of the book. It is refreshing to find that Bostock does not construe Russell's logical atomism as a form of British empiricism that makes acquaintance with atoms (construed as mind-dependent sense-data) at its foundation. Russell's theory of acquaintance afforded him a theory that explains our synthetic *a priori* knowledge of logic and mathematics. Bostock does not put it quite this strongly. But he strongly praises Russell's program of using logic in analysis, and thinks he was wholly successful in influencing philosophers to adopt his new methods (p. 281). Bostock does not go so far as to explicitly endorse Russell's remarkable thesis that *logic is the essence of philosophy* (p. 279). This thesis is definitive of Russell's logical atomism as a research program whose directive is to reveal, through an analysis that embraces the new sciences of mathematics (logic) and physics, that the only necessity is logical necessity. It is this, and not empiricism, that is the import of Russell's "supreme maxim of scientific philosophizing" which supplants the method hypothesis of (dubitable) entities with the method of construction. Perhaps, Bostock would agree.

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