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## Frege's Curiously Two-Dimensional Concept-Script

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In this paper I argue that the two-dimensional character of Frege's Begriffsschrift plays an epistemological role in his argument for the analyticity of arithmetic. First, I motivate the claim that its two-dimensional character needs a historical explanation. Then, to set the stage, I discuss Frege's notion of a Begriffsschrift and Kant's epistemology of mathematics as synthetic *a priori* and partly grounded in intuition, canvassing Frege's sharp disagreement on these points. Finally, I argue that the two-dimensional character of Frege's notations play the epistemological role of facilitating our grasp of logical truths (foundational and derived) independently of intuition. The rest of this paper critically evaluates Frege's view and discusses Macbeth's (2005) account.

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# Frege's Curiously Two-Dimensional Concept-Script

Landon D. C. Elkind

*What may I regard as the Result of my Work? It is almost all tied up with the concept-script.* (Frege 1906b [1979], 184)

## 1. Frege's Curiously Two-Dimensional Concept-Script

Frege makes hay over his Begriffsschrift ("concept-script").[1] His notations are distinctive partly because they are two-dimensional: practically all of Frege's contemporaries and intellectual descendants preferred one-dimensional notations. Despite their limited uptake, Frege's two-dimensional Begriffsschriften are vital to his self-image as a logician: he points in part to his Begriffsschrift in distinguishing himself from others like Peano and Boole. Kluge (1980, 140) makes no overstatement in claiming, "If Frege's philosophical endeavour can be said to have a conceptual focus, it is his notion of a Begriffsschrift."

Why did Frege use two-dimensional notations rather than one-dimensional ones? They were not easier to typeset, as Frege implicitly acknowledges in a jab at Peano:

> In the Peano conceptual notation the presentation of formulas upon a single line has apparently been accomplished in principle. To me, this is a gratuitous renunciation of one of the main advantages of the written over the spoken. After all, the convenience of the typesetter is certainly not the *summum bonum*. (Frege 1897b [1984], 236)

Frege is the last person who should be so unappreciative of typesetters. As Bynum (1972, 34) notes, Frege had trouble finding any publisher for *Grundgesetze* given the reception of *Begriffsschrift*. Finally, Hermann Pohle courageously assumed the financial risk for Volume I. Frege says in the Foreword to *Grundgesetze*'s Volume I that Volume II's publication "will depend on the reception of this first volume." (Frege 1893 [2013], V) Sales were poor despite Frege's plea, so Frege had to pay for the publication of Volume II. Again, Frege concedes that Peano showed one-dimensional scripts can work "in principle." So, as Macbeth (2005, 1) asks, why did Frege still prove in, and even pay to publish in, his two-dimensional Begriffsschrift? As Dunning (2018, 4) says, "Why was he so committed to writing this particular way?"

Here I argue that two-dimensional character in particular of Frege's Begriffsschrift has a substantial philosophical purpose. The two-dimensional character of Frege's Begriffsschrift serves as an anti-Kantian weapon to show that arithmetic is epistemologically independent of intuition and so is, as Frege would say, "analytic." Note that Frege's use of the word "analytic" differs significantly from ours, so we will need to dust off the archaism to grasp his point. Note also that I will be using "arithmetic" in Frege's broad sense to include analysis.

In particular, the role of two-dimensionality in Frege's Begriffsschrift is connected with Frege's account of judgment as recognition of a thought's truth once that thought has been grasped, as will be discussed below. The two-dimensional layout facilitates our recognition of logical truths by separating, in a given judgment, the logical relationships between contents and the contents

[1] Below I italicize "Begriffsschrift" only when it stands for the title of his 1879 work, *Begriffsschrift*, and leave it unmodified when it stands for Frege's concept-script. In quoted material I leave the original style unchanged. As Barnes (2002, 65n 1) and Schlimm (2018, 53–54) note, Frege sometimes uses "Begriffsschrift" as a common noun, for a concept-script in general, and other times as a proper noun for his 1879 work. Here I use "Begriffsschrift" only for Frege's concept-script. Since two-dimensionality is a common feature of the Begriffsschrift in *Begriffsschrift* and that in *Grundgesetze*, I will use the singular "Begriffsschrift" without investigating the differences between the Begriffsschriften in 1879 and in 1893–1903.

themselves. On the left occurs a judgment-stroke, quantifiers with the variables they bind, negation-strokes as short vertical strokes, and conditional-strokes. On the right, arranged vertically one below another, occur the contents. The content-stroke of course occurs on the left and right, since the turnstile affixes to it and and it prefixes the logical contents aligned vertically on the right. Humans being physiologically as they are, Frege believes this layout facilitates our successively grasping thoughts independently of sensible intuition, and they do this better than do one-dimensional notations.

The two-dimensionality of Frege's Begriffsschrift also helps show arithmetic judgments are epistemologically independent of sensible intuition. Now human beings practically need notations because, among other things, our memory is limited, as Frege explicitly recognizes. Notations are furthermore sensible, whether visually or audibly, and must be given how humans are physiologically. Yet notations are theoretically dispensable for, say, an omniscient being, or so Frege says. Frege is not troubled by our physiological need for notations. But defenders of Kant might object that arithmetic judgments still depend epistemologically on sensible intuition, whether through counting fingers or numbering scope markers. As I argue here, Frege thought that the two-dimensionality of his Begriffsschrift advanced his anti-Kantian view of arithmetic as wholly independent epistemologically of sensible intuition, and did so better than did its one-dimensional rivals, even despite our physiological need for notations.

## 2. Frege's View of what a Begriffsschrift is

In this section, I discuss Frege's notion of a Begriffsschrift. This has been somewhat unhelpfully polluted by his frequent comparisons of his Begriffsschrift with Leibniz's notion of a universal characteristic language that would also serve as a logical calculus.[2] It will help us distinguish the two to reconsider Leibniz's project and Frege's view of what a Begriffsschrift is for.

### 2.1. Frege's local Begriffsschrift, not Leibniz's global one

Briefly put, Leibniz's dream language was global: his idea of a *characteristica universalis* was, as Antognazza (2009, 92) puts it, "the universal formal language designed to eliminate the ambiguity and fluctuation of natural language, reducing it to an 'arithmetical calculation' and thereby allowing the peaceful resolution of all manner of controversies." This bold project is epistemologically global in scope in that it aims to cover all (scientific and non-scientific) knowledge. As Mugnai (2018, 177–78) notes, Leibniz (1666 [2020], 2) wanted a complete classification of all terms into a list of logically simple, indefinable terms, which themselves could be explained but not defined. Leibniz intended to design a language consisting of a finite list of simple terms such that any complex term could be analyzed into a combination of simple ones, thereby creating an alphabet of all human knowledge. The list of simple terms would then be used to define all other terms by resolving them into the simple terms that are their constituents. Leibniz admittedly wavered on various key questions, such as whether the list of simple terms is finite or infinite, and whether humans could ever identify genuinely simple terms (Mugnai 2018, 178). But from his 1666 *Dissertario de arte combinatoria* onward, Leibniz returned to his grand project of developing *ars characteristica*, "the art of forming and arranging characters so that they agree with thoughts." (Mugnai 2018, 178) This characteristic art of decomposing concepts into their simple constituents (analysis) and then composing them again from simple ones (synthesis) was only one part of his even grander project of developing a *characteristica universalis*, a universal lan-

[2]Danielle Macbeth argues, in (2005, 97) and more recently in (2018, 1419), that Frege's Begriffsschrift is a Leibnizian universal language. As we will see, Frege's language is not global in scope like Leibniz's dream language was.

guage with *characters* (literally, "marks" or "molds") for every simple concept, an alphabet of everything necessary to express all thoughts. (Mugnai 2018, 178)

By 1678, shortly after his 1666 *Dissertario*, Leibniz knew that his global project required a prior study of all human knowledge, whether *a priori* or *a posteriori*. Leibniz therefore hoped to create an encyclopedia containing all previously discovered scientific and non-scientific knowledge (Pelletier 2018, 162–63). This grew into the remarkable ambition of developing a *scientia generalis* ("general science"), which would include knowledge of scientific principles and their applications, plus an alphabet of simple concepts and ways of combining them to form others, and which would be an instrument of understanding old scientific principles afresh and of discovering new ones (Pelletier 2018, 168). Scholarly debate continues over the exact relationships of these various notions—*ars characteristica*, *characteristica universalis*, and *scientia generalis*—but it is clear that Leibniz's project encompassed all knowledge and depended upon developing an alphabet of simple (to us, if not unqualifiedly simple) concepts and ways to combine these into complex ones, plus methods for reasoning about (simple and complex) concepts and (old and new) scientific principles. Leibniz's dream notations were thus to be used globally, that is, in every branch of human knowledge.

Unlike Leibniz's dream language, Frege's Begriffsschriften are local: Frege says they are designed only for those branches of science wherein valid proofs are important, including logic, arithmetic, analysis, geometry, and (potentially) theoretical physics and philosophy. (Frege 1879 [1997], 50–51) Frege has no intention of creating an alphabet for human thought, nor of witnessing his Begriffsschrift applied in disciplines wherein validity of proofs are not vital. Frege in fact criticizes Leibniz's aspiration for a global symbolism as overambitious:

> Leibniz too recognized—perhaps overestimated—the advantage of an appropriate symbolism. His conception of a *calculus philosophicus* or *ratiocinator*, was too grandiose for the attempt to realize it to go further than the preliminaries. (Frege 1879 [1997], 50)

In particular, Frege never insists on creating an encyclopedia of all human knowledge. (Kluge 1980, 147) Frege admittedly, following Trendelenburg, hints hopefully that a localized, piecemeal approach can achieve Leibniz's dream language. (Sluga 1980, 52) But Frege only claims that his Begriffsschrift fulfills Leibniz's dream in logic specifically, and that his confidence in the possibility of extending his Begriffsschrift only covers those disciplines wherein validity of proofs is vital. (Frege 1879 [1997], 50)[3] Hence Frege remarks that his Begriffsschrift is devised for "particular scientific purposes" (Frege 1879 [1997], 49) and not all of them, much less for non-scientific ones, and colorfully illustrates the point with his microscope analogy.[4]

This is important because it refines our inquiry. We asked why Frege insisted on writing two-dimensionally. Now we can see that the answer must be that Frege thought two-dimensional notations did better than one-dimensional ones with respect to proof specifically.

If this interpretation of Frege's view of what a Begriffsschrift is is right, then it departs significantly from the well-known view that Frege's view of what a Begriffsschrift is involves its being a universal language. This view arises from van Heijenoort's influential gloss on Frege's Leibnizian locution of "*calculus ratiocinator*" and "*lingua characterica*," so I next discuss why van Heijenoort's gloss is to be rejected in favor of the reading espoused here.

[3] Frege's understanding of Leibniz ("Frege's Leibniz") is mediated by Trendelenburg's "On Leibniz's Project of a Universal Characteristics," from which Frege lifted the name "Begriffsschrift" for his conceptual notations. (Sluga 1980, 49) So Frege should be read cautiously as criticizing the version of Leibniz found in Trendelenburg's works. Indeed, Frege's criticism of Leibniz in the preface to Begriffsschrift cites Trendelenburg's essay, and Trendelenburg similarly argued that Leibniz's project was too wide in scope. (Sluga 1980, 51)

[4] Frege says that the eye, like an ordinary language, is "inadequate" for scientific purposes, whereas a microscope, like a Begriffsschrift, is "perfectly suited for just such purposes, but precisely because of this is useless for all others." (Frege 1879 [1997], 49–50)

## 2.2. Against van Heijenoort's reading

Since van Heijenoort's 1967 "Logic as Calculus and Logic as Language," historians of logic have commonly situated some pioneering logicians in "universalist" or "algebraic" traditions. A *universalist* logician takes as their motif that logic is a "*lingua characteristica*," such that it is a domain-general, interpretation-independent, universal language; importantly, logic so-conceived is allegedly incompatible with metalogical inquiry and techniques like model-theoretic semantics for interpreting the logical language.[5] The idea here is that logic is given by completely generalized axiom schemata whose instances are all true, and perhaps by some elucidations that suffice for understanding the intended interpretation of one's logical language.[6] Once understood and accepted, this system becomes the new framework in which all scientific inquiry is conducted. One of van Heijenoort's key takeaways is that, because logic is universal, it is misguided to investigate its metalogical features like completeness and consistency or its various interpretations over different domains. On this view, any such investigation of logic undermines the universality that logic must have.

In contrast, an *algebraic* logician takes as their motif that logic is a "*calculus ratiocinator*," such that it is a domain-specific, interpretation-dependent, particular language; importantly, metalogic is compatible with logic so-conceived, and it is permissible to use modern semantic techniques of interpreting a language over different domains to establish metalogical facts.[7] This is because logic on this conception is a tool for reasoning that can be deployed for different ends. So there is no universal, purpose-independent meaning to the logical languages used: the symbols even in one single language can be reinterpreted. This opens the way to using metalogical interpretations over models to investigate properties of logical systems. In summary, a universalist logician sees logic as what one thinks in, as the medium of thinking; an algebraist sees logics as a tool for excavating thoughts that are independent of any given logic.

As van Heijenoort (1967, 326–27) notes, a consequence of the universalist view as he describes it is that "nothing can be, or has to be, said outside the system" and that, as a logical language, the Begriffsschrift on van Heijenoort's reading "supplants the natural language." If van Heijenoort's account is right, then viewing Frege's Begriffsschrift as designed for scientific disciplines wherein valid proof is vital and not as a Leibnizian universal language supplanting natural language or extending to all human knowledge, as I argued for doing above, is sorely mistaken. So I next argue that van Heijenoort's gloss is wrong.

This is not to say that van Heijenoort's proposal is not useful. In fact, besides being a normative tool for evaluating views of logic, the universalist-algebraic narrative has been quite fruitful for the history of modern logic.[8] Hintikka (1988, 1) called it "[t]he most important background factor in the development of twentieth-century logic." van Heijenoort's hypotheses seemed to explain much, such as why Frege, Russell, and Whitehead did not systematically investigate the semantic completeness of their logics: a universalist commitment would account for that.[9] It also would explain the Hilbert-Frege controversy over the need for consistency proofs.[10] It further would explain why Frege held that comprehending his logical system relied on the prior under-

[5]See van Heijenoort (1967, 325–26), Hintikka (1988, 1–2), and Kusch (1989, 2–3).

[6]Frege's theses in *Grundgesetze* are arguably not axiom schemata, but axioms with rules for uniform substitution. The situation in Begriffsschrift is less clear. For a discussion see Landini (2012, 19–21).

[7]See van Heijenoort (1967, 327–28), Hintikka (1988, 3–4), and Kusch (1989, 4–5).

[8]As noted in Kusch (1989, 4) and Floyd (2009, 179–80).

[9]They did show that their axiom schemata sufficed for demonstrating much of mathematics, which some have called "experimental" completeness proof. (van Heijenoort 1967, 427) See also Goldfarb (2001, 30–31).

[10]See Hintikka (1988, 8–9) and Ricketts (1997, 169–70). For background, see Blanchette (2018, §2–§3).

standing of his reader.[11] It also would explain Frege's insistence that logical laws have content and are not merely formal. (Goldfarb 2001, 28–29)

Still, despite its explanatory power and simplicity, the universalist-algebraic narrative is a coarse cut. One reason for this is that actual logicians do not neatly fall into either univeralist or algebraic camps. For example, Peirce, who is typically classified an algebraic logician, in fact developed many allegedly univeralist techniques, including a propositional logic in papers from 1879–1880 and 1884–1885, a higher-order quantificational logic in a project begun in 1867 and finalized in 1883–1885, and an axiomatic characterization of number theory in 1881, all seemingly independently.[12] Granted, it is not known to what extent Peirce's writings were influenced by Frege's work because Peirce's knowledge of Frege's writings remains under investigation, but questions about priority and independence are not the issue here: the point is that Frege's "universalist" techniques are developed at length in Peirce's "algebraic" writings.[13] For that matter, Peirce and Schröder developed Boole's "algebraic" logic in tandem with their "universalist" elements, so that even near descendants of the "algebraic tradition" headed by Boole included characteristically "universalist" features. (Peckhaus 2004, 6–7) All this historical data muddies van Heijenoort's seemingly clear distinction.[14]

The universalist-algebraic narrative also does not explain what it was introduced to explain. van Heijenoort (1967, 324; see also 325) offered the universalist-algebraic narrative to account for Frege's distinction between *calculus ratiocinator* and *lingua characteristica*. But the universalist-algebraic distinction cannot correspond to Frege's terminology. For if van Heijenoort is right, then universalists and algebraists are in some sense contraries: one logician cannot be both. We saw that this distinction cuts too coarsely when applied to some pioneering logicians like Peirce, who do not clearly fall into just the universalist or just the algebraic tradition. Worse still for van Heijenoort's narrative, Frege's Leibnizian terminology is such that a *calculus ratiocinator* and a *lingua characteristica* are not opposed at all. Frege firmly identifies his Begriffsschrift as being both:

> In Leibnizian terminology we can say: Boole's logic is a *calculus ratiocinator* but not a *lingua characterica*; Peano's mathematical logic is in the main a *lingua characterica*, and at the same time also a *calculus ratiocinator*; whereas my conceptual notation is both, with equal emphasis. (Frege 1897b [1984], 242)

Frege's use of "*calculus ratocinator*" and "*lingua characterica*" does not imply opposition, whereas "universalist" and "algebraic" logicians are in some sense opposed. Two non-contraries should not be interpreted in a manner that makes them into contraries.[15] All of this weighs against van Heijenoort's admittedly fruitful gloss on Frege's uses of Leibniz's famous phrases.

### 2.3. Thinking with a two-dimensional Begriffsschrift

In the last subsection of Section 2 I discuss the nature of a Begriffsschrift in Frege's view. For Frege's Begriffsschrift is a language, one designed for thinking with and seeing, and designed so as to not be for speaking in. According to Frege, a Begriffsschrift is, in a phrase, characteristically two-dimensional, whereas a one-dimensional logical notation is a Begriffsschrift in name only.

[11] See van Heijenoort (1967, 326), Hintikka (1988, 1–2), and Goldfarb (2001, 27).

[12] See Anellis (2012, 246–47, 255–56, 259–60).

[13] See Anellis (2012, 265–67). See also Zeman (1986, 1) and Houser (1987, 425, 436).

[14] Quine walked back his claim that Frege alone developed the new quantification theory. (Anellis 2012, 254–55) Hintikka had to argue at some length that Peirce, despite developing quantification theory, was "a major representative of. . . the tradition of 'logic as calculus'." (Hintikka 1988, 28–29)

[15] Bynum (1972, 15) also incorrectly says that Frege's Begriffsschrift approaches "a universal language, rather than. . . an abstract logic or *calculus ratiocinator*." This suggests an incompatibility that is not there.

This somewhat unusual view of what a Begriffsschrift must be like stems from Frege's well-known distaste for using ordinary language in logic. Frege had much to say about the logically polluted character of natural language. In *Begriffsschrift*'s Preface, Frege comments:

> So that nothing intuitive could intrude here unnoticed, everything had to depend on the chain of inference being free of gaps. In striving to fulfill this requirement in the strictest way, I found an obstacle in the inadequacy of language: however cumbersome the expressions that arose, the more complicated the relations became, the less the precision was attained that my purpose demanded. Out of this need came the idea of the present *Begriffsschrift*. (Frege 1879 [1997], 48)

Frege thought that ordinary language was so logically polluted that he deliberately picked out unusual symbols in Hermann Pohle's shop for typesetting his *Grundgesetze*. Here is how Green, Rossberg, and Ebert put it:

> However, Frege chose the mathematically unfamiliar symbols on principled grounds. As he explains in vol. II [§58] of *Grundgesetze*, new signs ought to be chosen for the newly defined terms to ensure that the reader (and author!) does not rely on extraneous, previously associated content. For Frege, definitions have to be complete and must fully explain the newly introduced signs without recourse to informal notions, to intuition, or to any other source. . . (Green, Rossberg and Ebert 2015, 19)

Frege went to admirably great lengths to "break the power of words over the human mind" and "free thought from the taint of ordinary linguistic means of expression." (Frege 1879 [1997], 50–51) But as Barnes (2002, 65) notes, a reader might well ask whether the cure is worse than the disease. Why can we not just ask the reader to forbear from all psychological associations with the logical symbols and continue using ordinary ones?

The answer is that Frege wanted his Begriffsschrift to be a "formula language of pure thought," that is, he wanted us to think in a Begriffsschrift's symbolism when following his proofs. Frege holds that logical and mathematical formulas express thoughts and that symbols are not their subject-matter. (1893 [2013], §32; 1903 [2013], §58, n. 1) Nonetheless, he wants symbols to express thoughts with logical perspicuity because symbolism is crucial to logical clarity. Indeed, he holds that given how human beings are physiologically, we practically must think in symbols:

> Symbols have the same importance for thought that discovering how to use the wind to sail against the wind had for navigation. Thus, let no one despise symbols! A great deal depends upon choosing them properly...for we think in words nevertheless, and if not in words, then in mathematical or other symbols. Also, without symbols we would scarcely lift ourselves to conceptual thinking. (Frege 1882b [1972], 84)

As Frege sees it, a Begriffsschrift proper is symbolic language in which we are to think. A Begriffsschrift is not a logical encoding of natural language judgments.[16] Rather, Frege views a Begriffsschrift as a language we think in, a medium for thinking, not a tool that we apply to our thoughts. This follows for Frege from the intimate tie between thinking and symbolizing.

Thus, a Begriffsschrift is not for clarifying natural language words picking out concepts. (Barnes 2002, 73) And it is not a symbolization of thoughts as expressed in any other natural language, but an entirely new language for writing concepts or ideas. (Barnes 2002, 75) Hence, as Barnes (2002, 76) notes, an apt translation of "Begriffsschrift" is "ideography" as Jourdain (1989, 241) has it in his 1912 essay on Frege. A symbolism is literally an *ideography* if it symbolizes ideas themselves, that is, symbolizes ideas directly, rather than symbolizing speech, that is, symbolizing ideas mediately. (Barnes 2002, 78)

[16] As Barnes (2002, 66) rightly puts it, "A Fregean formula is not like that: in order to understand a Fregean formula you do not have to find a corresponding English formula. . . "

It may strike us as bizarre that speech would be essentially different from (Begriffsschrift) writing for Frege. But there is little doubt this is his view:

> Speech often only indicates by inessential marks or imagery what a concept-script should spell out in full. At a more external level, the latter is distinguished from verbal language in being laid out for the eye rather than for the ear. Verbal script is of course also laid out for the eye, but since it simply reproduces verbal speech, it scarcely comes closer to a concept-script than speech: in fact it is at an even greater remove from it, since it consists in signs for signs, not of signs for the things themselves. A *lingua characterica* ought, as Leibniz says, *peindre non pas les paroles, mais les pensées* [paint not the words, but the thoughts]. (Frege 1882c [1979], 13)

For Frege, then, a Begriffsschrift is precisely an ideography.[17] Thus, it cannot be spoken, for it must be for expressing thoughts themselves and not for expressing thoughts indirectly by expressing instead words for thoughts. A Begriffsschrift is a silent language of unmediated thought.[18]

## 3. Frege's Begriffsschrift separates Form and Content

We saw in Section 2 that Frege wants a Begriffsschrift for the proof-centered sciences, and for expressing thoughts directly. As such, it cannot be spoken and must be written. That is, Frege thinks a Begriffsschrift proper must be two-dimensional. In Section 3 I explain this by arguing that Frege holds a Begriffsschrift must spatially distinguish logical form and logical content.

[17] Barnes (2002, 79) holds that Frege's conception of a Begriffsschrift differs between 1879 and his later works, such as two essays on Boole in 1880 and 1882. That issue is set aside here; see footnote 1.

[18] Hence the difficulty of teaching it discussed in (Dunning 2018, 22–31).

### 3.1. Arithmetic-language as Frege's model for two-dimensionality

In this subsection I argue that, in Frege's view, arithmetic-language implicitly distinguishes form and content. Numerous times, Frege defends his Begriffsschrift with its two-dimensional arrangement by claiming it agrees with the usual two-dimensional arrangement in arithmetic:

> In fact I am in complete accord with the usual practice; for in arithmetical derivation too we put the individual equations in succession one beneath the other. But every equation is a content of possible judgment, or a judgment, as is every inequality, congruence, etc. Now what I set beneath one another are also contents of possible judgment, or judgments. . . We thus make use of the advantage that a formal language, laid out in two dimensions on the written page, has over spoken language, which unfolds in the one dimension of time. . . it would be extremely difficult to grasp what was going on, if one wished subsequently to introduce whole formulae in place of those single letters. (Frege 1882c [1979], 46)

> The arithmetic language of formulas is a conceptual notation since it expresses directly the facts without the intervention of speech. As such, it attains a brevity which allows it to accommodate the content of a simple judgment in one line. Such contents—here equations or inequalities—as they follow from one another are written under one another. If a third follows from two others, we separate the third from the first two with a horizontal stroke, which can be read "therefore." In this way, the two-dimensionality of the writing surface is utilized for the sake of perspicuity. (Frege 1882b [1972], 88)

Indeed, Frege's Begriffsschrift is subtitled "a formula language, modeled upon that of arithmetic, for pure thought." What does Frege mean that it is modeled on arithmetic?

Here, as elsewhere, it pays to consider Frege's background, namely, the mathematical intellectual environment in which he was educated and writing: the philosophical and mathematical contexts are mutually influencing. (Tappenden 2013, §9.6) The two-dimensional character of Frege's Begriffsschrift are inspired by his broader mathematical context, as his pre-Begriffsschrift writings show. Consider one example (there are many)[19] of the two-dimensional layout of arithmetical formulas in his 1873 doctoral dissertation:

> We assume that the coordinate system in the plane $E$ with the axes $w||\zeta, z||n$ is such that
>
> $$w = \frac{\zeta - \eta'}{2}, \qquad z = \frac{\eta + \zeta'}{2}. \tag{10}$$
>
> ... The axes $u||\zeta$, $v||\eta$ of the coordinate system in the plane $U'$ may be assumed to be such that
>
> $$u = \frac{\zeta + \eta'}{2}, \qquad v = \frac{\eta - \zeta'}{2}. \tag{11}$$
>
> From equations (10) and (11) it follows that
>
> $$\begin{cases} \zeta = u + w, & \eta = z + w \\ \zeta' = z - v, & \eta' = u - w. \end{cases} \tag{12}$$
>
> (Frege 1873 [1984], 27)

By 1879 Frege realizes that we could cut the prose—and should for logical perspicuity—to get a sequence of three equations such that the third is implied by the first and second:

$$w = \frac{\xi - \eta'}{2}, \qquad z = \frac{\eta + \xi'}{2}. \tag{1}$$

$$u = \frac{\xi + \eta'}{2}, \qquad v = \frac{\eta - \xi'}{2}. \tag{2}$$

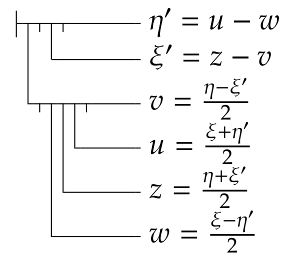
$$\begin{cases} \xi = u + w, & \eta = z + w \\ \xi' = z - v, & \eta' = u - w. \end{cases} \tag{3}$$

[19]One could similarly consider derivations among equations in, say, Frege (1874 [1984], 70–71).

The only things missing from the prose-free layout above are the indications of logical relationships between these arithmetical contents. Frege explicitly says that this is how he was inspired to the two-dimensional arrangement in *Begriffsschrift*:

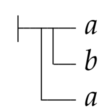
> Now I have attempted [here Frege cites *Begriffsschrift* in a footnote] to supplement the formula language of arithmetic with symbols for the logical relations in order to produce—at first just for arithmetic—a conceptual notation of the kind I have presented as desirable. (Frege 1882b [1972], 89)

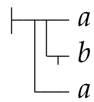
Frege took the arithmetic style of vertically arranging contents and introduced to the side symbols indicating logical relationships between them, that is, signs for logical form. Thus:

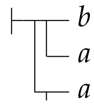



The above is missing some important qualifying conditions (and a proof). Still, one can see that Frege's point is that he is overlaying the existing arithmetic-language, which approximates a Begriffsschrift, with symbols for logical relations between contents, that is, with symbols for logical form (in Frege's sense of relations between contents). Thus, attending to Frege's mathematical context makes clear that Frege means to separate logical contents along one dimension and logical form along another.

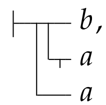
It is clear that this successive grasping of thoughts is lifted from arithmetic proofs without the prose. Compare the above two-dimensional formula with this proof (Frege 1893 [2013], 65):







The first formula is Basic Law I. The second formula is derived by substituting $\top b$ for $b$ and fusing horizontals. The final formula results from contraposing $\top b$ and — $a$, giving us



and then permuting the subcomponents — $a$ and $\top a$. This simple proof illustrates that Frege is copying the vertical arrangement of contents, often equations, that was and still is common in mathematics. Frege uses his two-dimensional arrangement to horizontally-indicate logical relations between, as was the mathematical style at the time, vertically-arranged contents.

### 3.2. Frege's Begriffsschriften have content

In this subsection I defend the claim that Frege's Begriffsschrift formulas have content. As we saw in Section 2, it was important to Frege that his Begriffsschrift approximate the Leibnizian dream of a *lingua characteristica* locally, that is, for the proof-centered sciences. Frege used Leibnizian locution to situate this local project in the storied tradition of providing a language for thought. Thus a kernel of truth in van Heijenoort's attribution to Frege of a "universalist" philosophy of logic is that Frege did indeed want his Begriffsschrift to be a language for thought (in proof-oriented sciences) and not just a formal language. Its formulas express thoughts with content. Frege makes this clear in his discussion of Peano's notations:

> I shall now inquire more closely into the essential nature of the Peano conceptual notation. It is presented as a descendant of Boole's logical calculus but, it may be said, as one different from the others. . . the fundamental idea has been altered entirely. Boole's logic is logic and nothing more. It deals solely with logical form, and not at all with the injecting of a content into this form—while this is exactly the intention of Mr. Peano. In this regard his enterprise more closely resembles my conceptual notation than it does Boole's logic. From another point of view, however, we can recognise a closer affinity between Boolean logic and my conceptual notation, in as much as the main emphasis is on inference, which is not stressed so much in the Peano logical calculus. (Frege 1897b [1984], 242)

This feature of the Begriffsschrift, that they concern content and form, helps explain why Frege sparred with Schröder over whose logic realized the Leibnizian ideal of *lingua characteristica*. (Peckhaus 2004, 9–10) As Frege saw it, the "domain-calculus" of Schröder was no logic, but a barren notation because it was pure formalism. (Frege 1895 [1984], 228) As such, he rather insisted that his Begriffsschrift, and not Schröder's "domain-calculus," was truer to Leibnizian ideals inasmuch as his Begriffsschrift, like Peano's notations, have content. Indeed, in nearly every place where Frege uses Leibniz's famous phrases, he is concerned to distinguish his Begriffsschrift from rivals' notations or to stress the novelty of his Begriffsschrift as having content. So while rejecting van Heijenoort's reading of Frege's Leibnizian locution, we build on his insight that Frege's Begriffsschrift formulas have content and form.

Against this view, Mezzadri (2019, 182) claims that (early) Frege denied that logic has content and affirmed that logic is formal. Mezzadri (2019, 188) argues that Frege means only that his Begriffsschrift is combinable with contentful signs but that its logical signs "do *not* on their own express" contents. A key passage occurs in Frege's *Begriffsschrift* §1:

> The symbols used in the general theory of magnitude fall into two kinds. The first consists of the letters, each of which repre-

> sents either a number left undetermined or a function left undetermined... The other kind consists of symbols such as $+$, $-$, $\sqrt{}$, 0, 1, 2, each of which has its own particular meaning. (Frege 1879 [1997], 52)

Frege clearly means to distinguish variables from constant signs. Mezzadri (2019, 189–90) takes this to mean that there is a formal, logical part of the Begriffsschrift and a contentful, non-logical (instead of a contentful and logical) part of it.[20] Mezzadri (2019, 194) sees this as weighing against the universalist view ascribed to Frege and influentially expounded in van Heijenoort's 1967 paper.

Like Mezzadri, I disagree with van Heijenoort's reading. But few would deny that (early) Frege, at least through the writing of *Grundgesetze*, held that arithmetic was part of logic, and yet that its theses are contentful. Misgivings about the universalist conception applying to Frege aside, it seems difficult to apply Mezzadri's claim consistent with Frege's Logicism. For instance, Frege (1882b [1972], 88) criticizes Boole, Grassman, Jevons, and Schröder, for developing a symbolism without content: it is thus ill-suited, Frege argues, to formalize proofs involving "analytic equations"—which, to a Logicist about arithmetic, would presumably be logical, contentful claims involving meaningful signs like those Frege mentions in Begriffsschrift §1.

However, Mezzadri's insight directs our attention to the form/content distinction in Frege's Begriffsschrift. In the last subsection of Section 3, building on Mezzadri's insight, I argue that Frege's Begriffsschrift separates form (to the left, aligned horizontally) and content (to the right, aligned vertically). They are thus separated along two different dimensions.

[20] See also Frege (1882c [1979], 13), Frege (1882b [1972], 89), and Frege (1882a [1972], 93).

### 3.3. A dimension for form, another for content

In this subsection I argue that Frege places form along one spatial dimension and content along another. This makes good sense of why he insists that a Begriffsschrift proper is two-dimensional.[21] It also accounts for why it is incapable of being spoken: speech is one-dimensional in a way that writing is not.[22] As our above discussion of *Begriffsschrift*'s subtitle "modeled upon that of arithmetic" suggested, Frege insists on two-dimensionality because he wants to separate content and form along different spatial dimensions:

> The "conceptual notation" makes the most of two-dimensionality of the writing surface by allowing the assertible contents to follow one below the other while each of these extends [separately] from left to right. Thus, the separate contents are clearly separated from each other, and yet their logical relations are easily visible at a glance. (Frege 1882a [1972], 97)

Two-dimensionality allows one to separate logical relationships between contents, or what we might call "form," and the contents themselves. Frege compares the logical contents to word stems and the relationships between contents to prefixes and suffices, and to "formwords {*Formwörter*} that logically interrelate the contents embedded in the stems." (Frege 1882a [1972], 93)

[21] "I would demand the following from a true conceptual notation: It must have simple modes of expression for the logical relations which, limited to the necessary, can be easily and surely mastered. These forms must be suitable for combining most intimately with a content. Also, such brevity must be sought that the two-dimensionality of the writing surface can be exploited for the sake of perspicuity." (Frege 1882b [1972], 88)

[22] "We may now ask which is preferable, audible symbols or visual ones... The spatial relationship of written symbols on a two-dimensional writing surface can be employed in far more diverse ways to express inner relationships {*innere Beziehungen*} than the mere following and preceding in one-dimensional time [characteristic of speech]... In fact, simple sequential ordering in no way corresponds to the diversity of logical relations through which thoughts are interconnected." (Frege 1882a [1972], 87)

Thus, in Frege's Begriffsschrift, contents occur in vertical arrangement on the right side of a judgment. The logical relations among contents, like those expressed by the letter-filled concavity for quantification, the short vertical negation-stroke, the judgment-stroke, the horizontal-stroke, and the conditional-stroke (despite splitting vertically), always occur on the left, and are read horizontally. So in short, my claim is that the two-dimensionality of Frege's Begriffsschrift spatially distinguishes logical content and logical form.

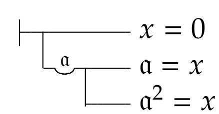
There is strong textual evidence for this reading. First, it explains well some of Frege's criticisms of his rivals' one-dimensional notations. Despite admitting that one-dimensional notations could work, Frege urges that Peano's one-dimensional notation sacrifices the chief advantage of the written over the spoken, namely, the two-dimensional layout:

> Because of the two-dimensional expanse of the writing surface, a multitude of dispositions of the written signs with respect to each other is possible, and this can be exploited for the purpose of expressing thoughts. In an ordinary written or printed text it is of course quite incidental which written sign happens to appear underneath another; in tabular lists, on the other hand, the two-dimensional expanse is utilised to achieve perspicuity. In much the same way I am trying to do this in my conceptual notation. I attain a clear articulation of the sentence by writing the individual clauses—e.g. consequent and antecedents—one beneath the other, and, to the left of these, by means of a combination of strokes, I exhibit the logical relation which binds the whole together. I mention this because efforts are now being made to squeeze each formula on to one line. In the Peano conceptual notation the presentation of formulas upon a single line has apparently been accomplished in principle. To me this seems a gratuitous renunciation of one of the main advantages of the written over the spoken. After all, the convenience of the typesetter is certainly not the *summum bonum*. For physiological reasons it is more difficult with a long line to take it in at a glance and apprehend its articulation, than it is with shorter lines (disposed one beneath the other) obtained by fragmenting the longer one—provided that this partition corresponds to the articulation of the sense. (Frege 1897b [1984], 236)

As Macbeth (2005) and Schlimm (2018, 76) have noted, Frege says here that his two-dimensional layout separates contents (subclauses of a sentence) vertically and the logical relationship judged to hold between them (using strokes) horizontally. The result of separating logical form and logical content along different spatial dimensions is a perspicuous articulation of the judgment.

That advantage matters because Frege's purpose is with valid proof: hence Frege (1879 [1997], 51) restricts himself in *Begriffsschrift* to a single mode of (primitive) inference. Frege (1897b [1984], 238) suggests that rivals like Boole and Peano are less concerned with inference, and *ipso facto* less concerned with valid proof. Two-dimensionality helps exhibit separately logical relationships between contents from contents themselves. If a logician has other aims, then two-dimensionality is perhaps irrelevant, as Frege (1897b [1984], 234) admits.

That Frege's two-dimensional notation separates logical form and logical content is further evidenced by his discussion of Boole. In his "The Aim of 'Conceptual Notation'," Frege illustrates the need to sometimes confine generality to a part of the judgment by discussing the following judgment as an instance of this need (1882a [1972], 99):




Frege then uses this judgment to exhibit the advantages of his Begriffsschrift:

> The "conceptual notation" makes the most of the two-dimensionality of the writing surface by allowing the assertible contents to follow one below the other while each of these extends [separately] from left to right. Thus, the separate contents are clearly separated from each other, and yet their logical relations are easily visible at a glance. . . I consider this mode of notation one of the most important components of my "conceptual notation,"

> through which it also has, as a mere presentation of logical forms, a considerable advantage over Boole's mode of notation. (Frege 1882a [1972], 97, 99)

Further, the symbolism of Frege's Begriffsschrift bears out my reading. Frege's separation of contents and logical relationships between contents is achieved using the *Formwörter*

$$\sqsubset\ ,\ \smile\ ,\ \text{—}\ ,\ \vdash\ ,\ \text{and}\ \top\ ,$$

along the horizontal dimension, and contentful signs along the vertical one, as in the above example. In contrast, one-dimensional notations or speech would give us

$$(\forall\mathfrak{a})(\mathfrak{a}^2 = x \rightarrow \mathfrak{a} = x) \rightarrow x = 0.$$

Here signs indicating logical form and logical content both occur along one dimension. Even in Polish notation, say, one has signs for form and content occurring along one dimension:

$$C\Pi\mathfrak{a}C(\mathfrak{a}^2 = x)(\mathfrak{a} = x)(x = 0).$$

Thus, as Frege rightly says, "simple sequential ordering in no way corresponds to the diversity of logical relations through which thoughts are interconnected." (1882b [1972], 87) One-dimensional speech or script mixes symbols for form with symbols for content, contrary to what Frege wants from a Begriffsschrift proper and achieved in his own Begriffsschrift.

In short, Frege insists that any "true conceptual notation" must, among other things, exploit "the two-dimensionality of the writing surface. . . for the sake of perspicuity." (1882b [1972], 88) Unlike its one-dimensional rivals, Frege's Begriffsschrift does just that by distinguishing logical form and logical content along different spatial dimensions. This explains Frege's insistence on the typesetters inconvenience, namely, on his two-dimensional Begriffsschrift.

## 4. Frege's Begriffsschrift in Anti-Kantian Context

In Section 2 I argued that Frege's Begriffsschrift is designed for those scientific disciplines wherein valid proof is vital. In Section 3 I argued that Frege's Begriffsschrift are two-dimensional to facilitate perspicuous representation (through spatial separation) of logical form and logical content.

Why, though, did Frege care about valid proof, or about logical perspicuity? Why insist on spatially separating content and form so sharply? There are many reasons: one is perspicuity for its own sake; another is demonstrating that Logicism with respect to arithmetic is true. However, in Section 4 I argue that a key and sufficient reason for this sharp spatial separation was to support Frege's anti-Kantian view of arithmetic as analytic. Contrapositively, Frege's anti-Kantian epistemology of arithmetic necessitated a sharp spatial separation of content and form (so as to show that arithmetic is independent of spatial intuition).

Just as the broader mathematical context is helpful in understanding Frege, so too is the broader Kantian (and neo-Kantian) context. Having valid proofs perspicuously represented in a way that is clearly independent epistemologically of sensible intuition underpins Frege's argument against the Kantian position. As we will see, the Kantian position was that arithmetic depended epistemologically on sensible intuition. Justifying arithmetic judgments depended epistemologically on the sensible intuitions occurring when one counts (and on abstractions that extend these intuitions to numbers we cannot count).

Frege rejected Kant's view, but how did he argue against it? Frege supported his claim that sensible intuitions are dispensable by exhibiting arithmetic in the Begriffsschrift, so that one can see, partly thanks to its two-dimensionality, that arithmetic is analytic, and so epistemologically independent of sensible intuition. To set the stage for making this connection between the two-dimensional character of Frege's Begriffsschrift and the Kantian context, I first consider Kant's notion of "analytic" followed by Frege's account. Then I take up Frege's account of judgment. Finally, this controversy over the correct epistemological account for arithmetic judgments is tied to Frege's two-dimensional Begriffsschrift.

### 4.1. Kant's notion of "analytic"

Neo-Fregeans are quite familiar with the debate over whether Hume's Principle is analytic.[23] In that debate Wright wrote, "But that was exactly the classical account of analyticity: the analytical truths were to be those which follow from logic and definitions." (1999, 8) Wright is right—if "the classical account of analyticity" means that of Frege. In *Grundlagen* Frege writes:

> If, in carrying our this process, we come only on general logical laws and on definitions, then the truth is an analytic one, bearing in mind that we must take account also of all propositions upon which the admissibility of any of the definitions depends. (Frege 1884 [1953], §3)

This, however, is not "the" classical account, that is, the view of Frege's predecessors. Frege's conception of analyticity is a significant departure from that of Kant. (Coffa 1991, 1–2) Behind that disagreement lurks another over their conceptions of logic: Kant held that logic was purely formal whereas Frege vehemently denied this. (MacFarlane 2002, 60)

In this section, I will discuss Frege's conceptual notations and his conception of analyticity in their Kantian context, discussing Kant's view of analytic judgments and epistemology of mathematics and Frege's response to them.

Kant's notion of analyticity as presented in the *Critique* and *Prolegomena* is as follows: an *analytic* affirmative categorical judgment such as "*A* is *B*," "all *A*s are *B*s," or "some *A*s are *B*s," or a negative one like "*A* is not *B*," "no *A*s are *B*s," or "some *A*s are *B*s," is such that the concept *B* is "contained" or "thought already" in the concept *A*.[24] Kant wrote:

> In all [categorical] judgments. . . either the predicate *B* belongs to the subject *A* as something that is overtly contained in this concept *A*; or *B* lies entirely outside the concept *A*, though to be sure it stands in connection with it. In the first case I call the judgment **analytic**, in the second **synthetic**. (Kant 1781-87 [1998], A6/B10)

> Analytic judgments say nothing in the predicate except what was actually thought already in the concept of the subject, though not so clearly or with the same consciousness. . . By contrast, the proposition: Some bodies are heavy, contains something in the predicate that is not actually thought in the general concept of body; it therefore augments my cognition, since it adds something to my concept, and must therefore be called a synthetic judgment. (Kant 1783 [2004], §2(a))

Kant's preferred example of analyticity, used in both texts, is the judgment "all bodies are extended," whereas a synthetic one is "some bodies are heavy." Kant claims extension is thought already when one makes a judgment about spatial bodies, even if this connection is unclear to the judger. Heaviness is not thought already in making a judgment about bodies.

So on Kant's account, analytic judgments merely explicate the subject concept by analyzing out a component concept into the predicate, whereas synthetic judgments amplify the subject concept. (1781-87 [1998], A7/B11) A corollary of Kant's take on the analytic-synthetic distinction is that all analytic judgments are *a priori* even when the concepts being judged are empirical like "gold is a yellow metal." (1783 [2004], §2(b))

Kant also claims that all analytic judgments depend on the law of non-contradiction, whereas synthetic judgments necessarily involve some additional principle. (1783 [2004], §2(c)) Kant then claims that judgments of experience are synthetic *a posteriori* while judgments of mathematics, of natural laws in science, and

[23] The best entry into that debate is still Wright (1999), reprinted in Cook (2007), which also contains other nice essays on the topic like Boolos (1997).

[24] See also (Coffa 1991, 13–14).

of metaphysics are all synthetic *a priori*. (1781-87 [1998], B14–B18) Because of the uninformative nature of analytic judgments on Kant's account, it is hard to see how he could have held otherwise; even Frege agreed that neither logic nor mathematics are analytic in Kant's sense. The point of deepest disagreement is Kant's claim that mathematical judgments are synthetic and as such epistemologically depend upon intuition. It is worth quoting Kant's full description of how pure mathematical judgments are and must be epistemologically justified using intuition:

> To be sure, one might initially think that the proposition "7+5 = 12" is a merely analytic proposition that follows from the concept of a sum of seven and five in accordance with the principle of contradiction. Yet. . . [o]ne must go beyond these concepts, seeking assistance in the intuition that corresponds to one of the two, one's five fingers, say, or (as in Segner's arithmetic) five points, and one after another add the units of the five given in the intuition to the concept of seven. For I take first the number 7, and, as I take the fingers of my hand as an intuition for assistance with the concept of 5, to that image of mind I now add the units that I have previously taken together in order to constitute the number 5 one after another to the number 7, and thus see the number 12 arise. . . The arithmetical proposition is therefore always synthetic; one becomes all the more distinctly aware of that if one takes somewhat larger numbers, for it is then clear that, twist and turn our concepts as we will, without getting help from intuition we could never find the sum by means of the mere analysis of our concepts. . . Help must here be gotten from intuition, by means of which alone the synthesis is possible. (Kant 1781-87 [1998], B15–B16)

As Kant says in the *Prolegomena*, "Therefore it is only by means of the form of sensory intuition that we can intuit things *a priori*." (1783 [2004], §10) This is not to say that Kant believes mathematics is grounded in actually intuiting things such that justifying $n + m = k$, say, requires perceiving $k$-many things for each $k$. Kant's use of "intuition" is broader and includes the necessary forms of sensibility to which all actual intuitions must conform. Thus although we in fact and must rely on our actual intuitions of things to justify mathematical judgments like "$5 + 7 = 12$" and "a straight line between two points is the shortest line between them," such *a priori* intuitions are "inseparably bound with the concept *before all experience* or individual perception." (1783 [2004], §7) Thus we can and must reflect on our actual intuitions, including perceptual and imagined ones, to give concrete meaning and applicability to mathematical concepts: such reflection on the forms of our intuitions can justify synthetic *a priori* judgments about the pure forms of sensibility. (1783 [2004], §8–§9)

Kant's account thus has mathematical judgments epistemologically depend on intuition. The role of symbolism in mathematics is to offer a helpful model for intuition. As Coffa said:

> Thus, in Kant's youthful opinion, the symbolism of mathematics was what he might have called an *Anschauungsschrift* ["intuition notation"], a symbolic system designed to display in sensible intuition a reliable model of the domain of mathematical discourse. (Coffa 1991, 64)

Where does this leave logic? Ignoring transcendental logic here, Kant says that *pure general logic* concerns "nothing but the mere form of thinking" in abstraction from all content. (1781-87 [1998], A54/B78) Pure general logic deals with the ways of combining concepts in judgments and inferential patterns that are possible just in virtue of forms of judgment. (Longuenesse 2006, 137) On Kant's account, pure general logic is entirely formal. So it does not and cannot ground pure mathematical judgments because these have content.

### 4.2. Frege's notion of "analytic"

As is well-known, Frege found much fault with Kant's view. But the similarities are also striking. Frege agrees with Kant that geometric judgments are synthetic *a priori* and epistemologically depend on intuitions in Kant's (narrower) sense. (Dummett 1982, 234)[25] Reading Frege's 1884 *Grundlagen* makes it easy to forget his points of agreement with philosophers like Kant: the text is a broadside against almost everyone—Mill, Hobbes, Boole, Aristotle, and Kant. Yet Frege respected Kant's achievements, calling him "a genius to whom we must all look up with grateful awe." (Frege 1884 [1953], §89) He accepted a view of geometry that built upon Kant's own, even where it departed from Kant's views.

Now the suggestion that Frege's epistemology agreed with Kant's views about geometry but not about arithmetic, without context, is liable to seriously mislead. (Bar-Elli 2014, 1) Kant's conception of logic as purely formal was a rejection of the neo-Leibnizian orthodoxy of his day; similarly, Frege's expansion of logic's scope and of analyticity was a rejection of Kant's epistemology. (MacFarlane 2002, §3–§4) All the same, Frege's philosophy of logic and arithmetic developed not just against this Kantian background, but upon it, and Frege's two-dimensional Begriffsschrift are a response to Kant's epistemology.

Frege develops a different take on the analytic-synthetic distinction itself. The upshot is that analyticity is expanded to include truths of logic, arithmetic, and analysis. As we saw above, Frege means by an *analytic* truth one provable step-by-step from logical truths, which are completely general, and definitions. In contrast, a *synthetic* truth is one not justifiable in this way:

> If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some special science, then the proposition is a synthetic one. (Frege 1884 [1953], §3)

Further, an *a posteriori* truth is one whose proof relies on non-provable, non-general truths about particular objects known by sensible intuition, that is, on "an appeal to facts," whereas an *a priori* truth is one provable from general truths that "neither need nor admit of proof." (Frege 1884 [1953], §3)

Frege thus divorces analytic truths and intuition while wedding synthetic truths and intuition. Further, all analytic truths are *a priori*, though Frege rejects the converse, holding that geometric truths are synthetic *a priori*. (Frege 1884 [1953], §13)

So Frege wants all logical truths, and also truths of arithmetic and analysis, to be epistemologically independent of intuition. As MacFarlane (2002, 38–40) argues, part of Frege's rationale for this is that logic and arithmetic must be fully general, whereas intuition cannot justify a fully general truth. As he says in the letter to Marty of 29 August 1882:

> I regard it as one of Kant's great merits to have recognized the propositions of geometry as synthetic judgments, but I cannot allow him the same in the case of arithmetic. The two cases are anyway quite different. The field of geometry is the field of possible spatial intuition; arithmetic recognizes no such limitation. Everything is enumerable... Thus the area of the enumerable is as wide as that of conceptual thought, and a source of knowledge more restricted in scope, like spatial intuition or perception, would not suffice to guarantee the general validity of arithmetical propositions. (Frege 1980, 100)

So truths of logic, arithmetic, and analysis are completely general, and so must be intuition-independent. Synthetic truths, whether *a priori* or *a posteriori*, are intuition-dependent. But Frege denies that arithmetic truths epistemologically rely on sensible intuition. Consequently, Frege holds that arithmetic truths are analytic:

> If, again, we compare the various kinds of truth in respect of the domains that they govern, the comparison tells once more against the supposed empirical and synthetic character of arithmetical laws. Empirical propositions hold good of what is physically or psychologically actual, the truths of geometry govern all that is spatially intuitable... The truths of arithmetic govern all that is numerable.

[25] See Frege (1884 [1953], §89). As discussed in Shipley (2015, §2), when Frege holds that geometry epistemologically depends on intuition, he uses "intuition" in the wider sense that contrasts with "concepts": here, intuitions are individual representations whereas concepts are general ones. (Frege 1884 [1953], §12) So for Frege geometric intuitions are "individuating," objective representations and not psychological, subjective ones. (Shipley 2015, 4)

> This is the widest domain of all; for it belongs not only to the actual, not only to the intuitable, but everything thinkable. (Frege 1884 [1953], §14)

How does Frege establish his view that arithmetic, like logic, is analytic, and so epistemologically independent of intuition? Frege admits that the entire argument of *Grundlagen* makes his view at most "probable" because, as Frege sees it, hardly any arithmetic truths are proven in what would be the correct manner, one free of intuition. Proofs in Frege's day generally proceeded "by jumps" even if the underlying inference is logically correct, which consequently makes it difficult to separate truths whose justification epistemologically depends on intuition from those depending on logic: "On these lines what is synthetic and based on intuition cannot be sharply separated from what is analytic." (Frege 1884 [1953], §90)

To eliminate intuition-reliant jumps in arithmetic proofs, thereby showing that arithmetical truths are analytic, Frege was led to develop Begriffsschrift. Thus Frege's notations are to be understood in their Kantian context and, in particular, as an anti-Kantian weapon designed specifically to eliminate doubts about the analyticity of arithmetic:

> From the preceding [§§1–108 of the *Grundlagen*] it thus emerged as a very probable conclusion that the truths of arithmetic are analytic and *a priori*; and we achieved an improvement on the view of Kant. We saw further [in §91] what is still needed to raise this probability to a certainty, and indicated the path which must lead to that goal. (Frege 1884 [1953], §109)

Academics love to tease us with their past and future work, and Frege was no exception. The tool "still needed" show the analyticity of arithmetic with certainty is his Begriffsschrift. Indeed, he points to Formula 133 in the *Begriffsschrift* as evidence that analytic truths can be genuinely informative. (Frege 1884 [1953], §91, note 2) So Frege explicitly identifies his Begriffsschrift as a tool in demonstrating his own view of arithmetic and resisting Kant's.

Thus, Frege insisted on two-dimensional notations, that is, on a Begriffsschrift proper, rather than using a logically viable one-dimensional script because Frege's wanted to demonstrate his anti-Kantian view of arithmetic. He wanted to use the Begriffsschrift to show sensible intuition was dispensable in arithmetic by making proofs in arithmetic logically perspicuous. This included, even practically demanded, spatially separating form and content, that is, it required a characteristically two-dimensional Begriffsschrift.

### 4.3. Two-dimensionality and grasping thoughts

Above I argued that Frege had an anti-Kantian purpose in writing in his characteristically two-dimensional Begriffsschrift. How, though, does spatially separating logical form and logical content facilitate a reader accepting Frege's anti-Kantian view of arithmetic? To answer this question it helps to recall Frege's account of judgment as recognition of a thought's truth. A two-dimensional script facilitates our direct, immediate grasping of thoughts, thereby facilitating our recognition of that thought's truth (if it is true). Let us see this in detail.[26]

Recall that Frege explains judgment as "the acknowledgment of the truth of a thought." (1893 [2013], §5)[27] Similarly, inference is made from a thought previously acknowledged as true to a thought newly acknowledged as true and justified by the thought previously acknowledged as true.[28] *Grundgesetze* represents this using "$\vdash$ ," which Frege calls the judgment-stroke. In

[26]For a very helpful treatment of Frege's account of judgment in its Kantian context, see Shieh (2019, §1.3). Here I set aside the issue of whether judgment is factive for Frege. See Shieh's (2019, §3.6) discussion.

[27]See also Frege (1879-91 [1979], 1–2, 7), Frege (1906a [1979], 185), and Frege (1980, 6 February 1917).

[28]"To make a judgment because we are cognisant of other truths as providing a justification for it is known as *inferring*." (Frege 1879-91 [1979], 3) As Pedriali (2017, 4) puts it, in "Fregean inference. . . we move from the acknowledgment of true thoughts to the acknowledgment of true thoughts." See Frege (1914 [1979], 204).

"Der Gedanke" Frege explains that a thought is "something for which the question of truth can arise at all. . . thoughts are senses of sentences." (1918 [1997], 328) For Frege, although agents perform the acts of thinking and judging, the thoughts grasped in thought or judgment are mind-independent: "A thought belongs neither to my inner world as an idea, nor yet to the external world, the world of things perceptible by the senses." (1918 [1997], 342)[29] Frege thus distinguishes (quoting Frege 1918 [1997], 329):

1. the grasp of a thought—thinking,
2. the acknowledgement of the truth of the thought—the act of judgment,
3. the manifestation of this judgment—assertion.

How do we come to grasp a mind-independent thought, whether true or false? For logic's subject-matter is "such as cannot be perceived by the senses." (Frege 1879-91 [1979], 3) Frege says:

> The thought, in itself imperceptible by the senses, gets clothed in the perceptual garb of a sentence, and thereby we are enabled to grasp it. We say a sentence *expresses* a thought. (Frege 1918 [1997], 328)

So it is through perception of sentences, whether auditory, visual, or tactile, that we grasp thoughts.[30] Having grasped them, we may then be in a position to consider their justification and to recognize them as true, or to not do so, accordingly.[31]

According to Frege (1879-91 [1979], 6), this explains why languages may have different grammars and why learning them can help us identify the logical kernel expressed in a sentence. A thought can be "clothed" in different ways linguistically. Distinct "dressings" of a thought facilitate our grasping it with varying degrees of success. This is why Frege can consistently claim both that sentences merely express thoughts and that mastering various languages, say, his Begriffsschrift, are still epistemologically useful to grasping of some logical thoughts:

> . . . when we see that the same thought can be worded in different ways, our mind separates off the husk from the kernel. . . This is how the differences between languages can facilitate our grasp of what is logical. . . For this reason, it is useful to be acquainted also with a means of expression of a quite different kind. . . [footnote:] In this connection mention might also be made of my concept-script. I would not be in a position to write this work on logic without benefit of my earlier endeavours to devise a concept-script. (Frege 1879-91 [1979], 6)

Natural language is a barrier to grasping thoughts because of its logical imperfections and psychological trappings. (Frege 1897a [1979], 149) So different languages and scripts as they differ in logical perfection facilitate our gasping the very same thoughts, and our subsequent recognition of their truth, to varying degrees.[32] This makes sense if we view judgment as depending on clearly grasping thoughts. One would of course want a simple, readily surveyed, unambiguous notation to facilitate directly grasping thoughts to the greatest possible extent.

[29] Frege says of such metaphorical language, "The metaphors that underlie the expressions we use when we speak of grasping a thought, of conceiving, laying hold of, seizing, understanding, of *capere*, *percipere*, *comprehendere*, *intelligere*, put the matter in essentially the right perspective." (1897a [1979], 137) Needless to say, for Frege the thought grasped is independent of our thinking activity. Here a helpful comparison can be made with the anti-Idealist act-object distinction deployed in Russell (1912, 65–67). Note that Russell, in contrast with Frege, thinks the logical form of judging differs from that of thinking, which is a species of awareness of an object or a complex. (Russell 1912, 69, 80, 211–12)

[30] "Since the sense itself cannot be perceived by the senses, we have perforce, in order to communicate, to avail ourselves of something that can be perceived. So the sentence and its sense, the perceptible and the imperceptible, belong together." (Frege 1899-1906 [1979], 167)

[31] "Whenever anyone recognizes something to be true, he makes a judgment. What he recognizes to be true is a thought. It is impossible to recognize a thought as true before it has been grasped. A true thought was true before it was grasped by anyone." (Frege 1915 [1979], 251)

[32] This can be seen from what Frege says about translation: "The sentence is of value to us because of the sense that we grasp in it, which is recognizably the same in translation, too." (Frege 1914 [1979], 206)

Frege is careful not to say that the usefulness of sentences in directing our thinking towards specific thoughts is a necessary condition for grasping thoughts at all. Frege acknowledges that it is logically possible that some beings could grasp thoughts without the need for a sensible (auditory, visual, or tactile) sentence; he goes on to say that beings like us do need a sensible sign to grasp thoughts, and that inasmuch as a language is shaped by "the logical disposition" in ourselves (and not, say, "the poetic disposition"), it will facilitate logical and mathematical thinking with greater success. (Frege 1924 [1979], 269)

This explains Frege's repeated insistence that his Begriffsschrift, in contrast with the notations of Boole, Peano, and Schröder are designed to express content.[33] On Frege's view they must be so-designed: Frege wants his Begriffsschrift to express thoughts "more exactly" than in other languages. This will, Frege thinks, enable us to more readily and easily grasp the thought express. His Begriffsschrift thus must express thoughts with content to fulfill their epistemological purpose, namely, to "paint not the words, but the thoughts."

How precisely do Frege's Begriffsschrift play this epistemological role? Frege tells us that they express thoughts in more perspicuously, by, say, using fewer logical forms:

> A few new signs suffice to present a wide variety of mathematical relations which it has hitherto only been possible to express in words. This of itself justifies their introduction, since the formulae are much briefer and more perspicuous than the equivalent definitions of the concepts in words. (Frege 1882c [1979], 27)

> And so I replace the logical forms which in prose proliferate indefinitely by a few. This seems to me essential if our trains of thought are to be relied on: for only what is finite and determinate can be taken in at once, and the fewer the number of primitive sentences, the more perfect a mastery can we have of them. (Frege 1882c [1979], 39)

This perspicuity in expressing contents assists Frege in, among other things, eliminating even the appearance of epistemologically relying on sensible intuition; this includes eliminating inferential gaps in proofs such as mathematicians by habit and custom relied on sensible intuition to close. (Frege 1882c [1979], 32) For similar reasons, prose itself is cut as much as possible.[34]

Most relevant to us is the role of two-dimensionality in achieving perspicuity. As we saw, Frege speaks of two-dimensionality as helping "achieve perspicuity" and "attain a clear articulation of the sentence," as "a mere presentation of logical forms," and as being "surveyed visually." The rich class of logical relations between contents is neatly displayed in Frege's two-dimensional Begriffsschrift. It is partly because of its characteristic two-dimensionality that Frege can say that, when asserting formulas in his Begriffsschrift, he is fulfilling the Leibnizian ambition of painting thoughts rather than words.

If Frege is right about his Begriffsschrift perspicuously depicting thoughts, and if his account of judgment is accepted, then it is hard to resist his anti-Kantian view of arithmetic. Pretending that we did not know Basic Law V is false, it is difficult to see where sensible intuition enters the Fregean picture: in that case, it is hard to deny that intuition is eliminated from his proofs except inasmuch as we rely on sensible notations to grasp thoughts.

Thus, when we consider Frege's account of judgment as acknowledging the truth of grasped thoughts, and we consider Frege's talk of a Begriffsschrift expressing thoughts directly

[33] "In contrast we may now set out the aim of my concept-script. Right from the start I had in mind the *expression of a content*. What I am striving for is a *lingua characterica* in the first instance for mathematics, not a *calculus* restricted to pure logic." (Frege 1882c [1979], 12)

[34] "I wanted to supplement the formula-language of mathematics with signs for logical relations so as to create a concept-script which would make it possible to dispense with words in the course of a proof, and thus ensure the highest degree of rigour whilst at the same time making the proofs as brief as possible." (Frege 1880-81 [1979], 47)

and perspicuously, it is hard not see the characteristically two-dimensional nature of Frege's Begriffsschrift in their Kantian—really, anti-Kantian—context.

## 5. Frege's Begriffsschrift Reconsidered

In this last section I critically evaluate Frege's view that two-dimensional notations are generally superior to one-dimensional ones. First, I discuss whether Frege's notations are unduly (or duly) difficult. Second, I discuss how sensible intuition plausibly enters into reading one-dimensional notations. Finally, I critically discuss Macbeth's account of Frege's notations.

### 5.1. The ready intelligibility of Frege's notations

It is important to the epistemological purpose of Frege's notations that they do not make our grasping thoughts or our acknowledging their truth epistemologically depend on intuition. Dunning and Toader are thus overstating the role of sensible intuition in their interpretations:

> For Frege, to write in his notation was to create visual evidence, airtight steps in an observable chain of reasoning that rebuilt mathematics drawing only on the laws of logic. (Dunning 2018, 5)

> . . . an advantage of Frege's diagram is that we can more efficiently visualize and grasp the logical structure of the inference. I claim that this is due to an appeal to our intuition. . . In Frege's case, I contend, we see (perceive) the diagram, and therefore are inclined to see further (intuit) through the diagram, into the objective domain of concepts (which is of course not to say that this is how we primarily get access to this domain). (Toader 2004, 24)

It is strictly incorrect to speak here of visual evidence or appeals to sensible intuition. Frege cannot allow sensible intuition to play an epistemologically role in our grasping thoughts: it would undermine his anti-Kantian thesis.[35] So Frege cannot be using his Begriffsschrift to spatially represent or map the structure of thoughts directly.[36] Different notations may play a causal role in directing our thinking, but we do not grasp thoughts through intuitions.

This demand of the anti-Kantian dialectic explains why, as Toader (2004, 22) says, "Frege offers some surprisingly psychological justifications to warrant the use of a two-dimensional symbolism." But Frege's ban on intuition in arithmetic requires that two-dimensional notations facilitate our successive grasping of thoughts, and subsequent recognizing of their truth, such that intuition plays no part in their epistemological function.[37] To succeed in doing that, Frege's notations must be readily intelligible. They must allow us to immediately grasp thoughts without epistemologically depending on sensible intuition. Frege's notations, however, were widely criticized as being a monstrous waste of space and hardly more intelligible than one-dimensional symbolism.[38] As Dunning (2018, 7) notes, this reaction was no doubt partly because they flout "human linguistic expectations."

Despite this, Frege's notations are, as Schlimm (2018, 54) says, "quite advantageous in terms of perspicuity and readability." Schlimm (2018, 61) draws a helpful comparison between Frege's Begriffsschrift and syntax trees in linguistics. Few would com-

[35] "So that nothing intuitive [*Anschauliches*] could intrude here unnoticed, everything had to depend on the chain of inference being free of gaps." (Frege 1879 [1997], 48)

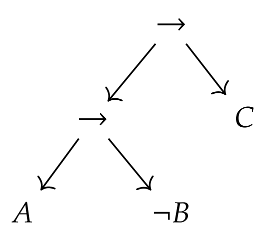
[36] Frege writes in the letter to Marty, "And to enable one to rely on intuition for support, it would not help at all to let something spatial represent something non-spatial in enumeration; for one would have to justify the admissibility of such a representation." (1980, 100)

[37] Frege wrote in a letter to Jourdain, "The need to exclude with certainty any tacit presuppositions in the foundations of mathematics led me to the conceptual notation of 1879." (1980, 73)

[38] See Vikko (1998, 415). Vikko (1998, 413, 420), however, persuasively argues that the reception of Frege's *Begriffsschrift* was less tragic than has been generally claimed, especially given that Frege was "a relatively unknown young mathematician."

plain that syntax trees are a befuddling way of presenting and even introducing logical form, or at least linguistic form. But the two-dimensionality of Frege's Begriffsschrift is similar in structure to syntactic trees, which are widely-agreed to be logically perspicuous, as their popularity in introductory logic texts (and use in disciplines like linguistics) shows. (Schlimm 2018, 62)

The analogy is imperfect because syntax trees do not separate logical relationships between contents and the contents themselves along different spatial dimensions. For example, Schlimm's example of $(A \rightarrow \neg B) \rightarrow C$ has the following tree:



Seeing where the negation-sign appears, this is not nearly the neat separation of logical form and logical content as Frege has in his Begriffsschrift. But Schlimm (2018, 61) rightly notes that no parentheses tracking is required in either case. This eliminates a defect of one-dimensional notations, one that potentially allows for sensible intuition to intrude into our analytic thoughts, as we will see below.

Thus, Frege's Begriffsschrift is, or at least plausibly approximates, a notation that can "exhibit" or "depict" logical relationships between thoughts. (Dunning 2018, 20) As Dunning (2018, 21) notes, two-dimensionality is absolutely crucial to Frege's claim here. This raises the question: how do one-dimensional notations fall short?

## 5.2. Kantians and one-dimensional notations

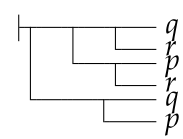
If I am right, the anti-Kantian ends of Frege's characteristically two-dimensional Begriffsschrift should be kept fully in mind. Banishing sensible intuition from arithmetic proofs is crucial to Frege's dispensability argument against the Kantian position that arithmetic is synthetic (in Kant's sense) rather than analytic (in Frege's sense).

How did one-dimensional notations fall short of banishing sensible intuition? One rationale would be that one-dimensional notations are harder, in Frege's view, to take in and survey. As we saw above, Frege clearly believed this. But we might be skeptical: one-dimensional notations, after all, are not insurmountable. They also conform to linguistic expectations. So why would one-dimensional notations be less suitable for Frege's anti-Kantian purpose? Frege is not perfectly explicit here. He offers some clues, though:

> For physiological reasons it is more difficult with a long line to take it in at a glance and apprehend its articulation, than it is with shorter lines (disposed one beneath the other) obtained by fragmenting the longer one—provided that this partition corresponds to the articulation of sense. (Frege 1897b [1984], 236)

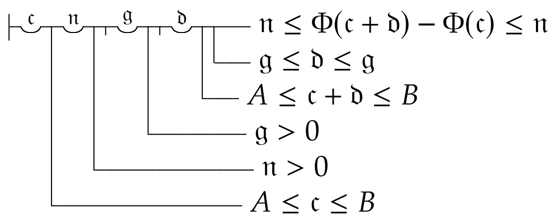
Frege seems to think that one-dimensional notations such as Peano uses fails to carve formulas at their senses. The result is that one may seem to rely on sensible intuition to support their parsing of a formula using, say, parentheses. (1897b [1984], 247)

A few examples will illustrate the manner in which sensible intuition might appear to reenter a proof. Consider this propositional logic formula in Begriffsschrift and in Peanese:

| *Grundgesetze* | Peanese/*Principia* |
| --- | --- |
|  | $p \supset q \mathbin{:.\supset:.} r \supset p \mathbin{:\supset:} r \supset q$ |

Even with conventions to reduce scope markers, one has to read ahead and back to identify the logical form of the formula. One has to count—using sensible intuition—the number of dots to identify the main connective. Frege's tabular, chunking presentation avoids this entirely: one reads from the turnstile down and around, wrapping around contents in sequential order. One grasps a Begriffsschrift formula and the thought it expresses merely by tracing around the outside and running counterclockwise. Peanese requires, or so it might seem, sensible intuition to parse, even where the formula is very simple.

Such examples might be multiplied indefinitely. Frege offers many excellent examples in an essay on Boole's notation and his Begriffsschrift. (1880-81 [1979], 21–27) Example (14) is the Begriffsschrift expression of "the real function $\Phi(x)$ of a real variable $x$ is continuous throughout the interval from $A$ to $B$:"




(Frege 1880-81 [1979], 24)

Notice again how we follow this formula in one direction, beginning with $\vdash$ and proceeding counterclockwise along the notation. We never retrace steps, no parsing of dots or parentheses, or of quantifier scopes, is required. In respect of logical perspicuity, the Begriffsschrift formula seems superior to its one-dimensional analogue (which has to be broken up to fit):

$$\forall\mathfrak{c}(A \le \mathfrak{c} \le B \,.\!\supset\!.\, \forall\mathfrak{n}(\mathfrak{n} > 0 \supset \neg\forall\mathfrak{g}(\mathfrak{g} > 0 \supset \neg\forall\mathfrak{d}(A \le \mathfrak{c}+\mathfrak{d} \le B :\!\supset\!: \mathfrak{g} \le \mathfrak{d} \le \mathfrak{g} \,.\!\supset\!.\, \mathfrak{n} \le \Phi(\mathfrak{c}+\mathfrak{d}) - \Phi(\mathfrak{c}) \le \mathfrak{n}))))$$

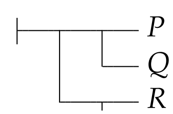
Even with conventions for reducing square dots and brackets, or for subscripting bound variables to material conditionals, it seems difficult to resist Frege's contention that the Begriffsschrift formula is easier to parse, and indeed needs no parsing. One needs only to grasp the thought it expresses by tracing it counterclockwise. Sensible intuition is not required to parse its syntax as it likely is for most reading its analogue in Peanese.

Thus, the two-dimensional character of Frege's Begriffsschrift perspicuously expresses thoughts by clearly exhibiting the contents and the logical relations between them. Crucially, they do this in a manner that is epistemologically independent of sensible intuition. Frege's Begriffsschrift thus serves as an anti-Kantian weapon: they serve a function in demonstrating the analyticity of arithmetic. In respect of their anti-Kantian purpose, the characteristically two-dimensional Begriffsschrift bests its one-dimensional rivals.

### 5.3. Against Macbeth's reading

In the previous section I described reading Frege's two-dimensional Begriffsschrift formulas in a single way. Further, I discussed Frege as separating spatially logical contents from the logical relations between those contents. This suggests that there is but one way to read Frege's two-dimensional notation, even if a given logical content is analyzable in various ways, that is, even if a given logical formula can be used to comprehend various functions.

Macbeth's excellent 2005 book *Frege's Logic* argues against this view. Macbeth argues that Frege's two-dimensional notation is, like a two-dimensional table, readable in multiple ways; otherwise, it would be no different from its one-dimensional rivals. (2005, 47) For instance, the Begriffsschrift formula



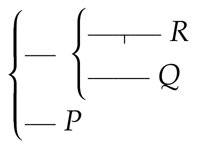

may equally be read as '$(\sim R \& Q) \supset P$,' which, according to Macbeth, justifies the two-dimensional character of Frege's Begriffsschrift. (2005, 2)

To put this another way, Frege in *Begriffsschrift* suggests that instead of taking the conditional-stroke as primitive, he might have adopted a conjunction-stroke,

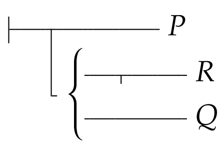
$$\begin{cases}\Gamma\\ \Delta\end{cases}$$

Frege (1879 [1997], 63)

In that case, on Macbeth's reading, the above formula might be read instead as

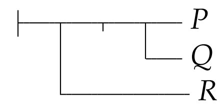



or, perhaps better, as




Macbeth claims that even with the conditional-stroke alone as primitive, a formula in Frege's Begriffsschrift might be read either using the conjunction-stroke or the conditional-stroke. Thus, "sentences in *Begriffsschrift* have a main connective only relative to an analysis." (2005, 50) Note that Macbeth (2005, 49) thinks that there is a difference here between the early and later Frege, but the real question before us is whether Macbeth is right about Frege's view of the Begriffsschrift notation at any stage. If so, then the reading advocated here of Frege's characteristically two-dimensional Begriffsschrift is mistaken.

There are, however, abundant reasons to separate the two-dimensional character of Frege's Begriffsschrift from his views about the multiply-analyzable character of thoughts. For one, as Schlimm (2018, 57) has noted, there are exceptions to the supposed alternative readings of a Begriffsschrift formula. For example, the Begriffsschrift formula




is not variously readable in the way Macbeth describes; one cannot export over a negation.

For another, as Angelelli (2007, 130) notes, Macbeth's thesis does not explain what it is partly introduced to explain: Macbeth's claim about the signs for logical forms does not in itself explain why Frege's Begriffsschrift is two-dimensional. For one can hold that the content $3 > 2$ is analyzable into (can comprehend) different functions by breaking it into saturated and unsaturated parts as 3 and $x > 2$ or as 2 and $3 > x$, or even as $>$ and $x > y$. (Frege 1891 [1984], 154) In the first two cases, we have comprehended two different functions that takes one argument; in the third case, we comprehended a function of two arguments. Further, even though the usual notation in modern logic has a fixed main connective in any well-formed formula, one could nonetheless hold that $\sim R(\supset Q \supset P)$ is identical in content with $(\sim R \& Q) \supset P$, so that the first formula can be read as one content or another. Being alternatively readable or multiply analyzable is not connected to two-dimensionality at all.

Further, Macbeth's proposed multiple readability of signs for logical relations runs against Frege's criticisms of Peano. Frege objects to Peano's twofold use of "$\varepsilon$" for subclass and membership, and of "$\supset$" for superclass and implication as here:

> Even if perhaps no mistake results from it, till the comprehensibility of the formulae suffers from it, when one always has first to call to mind how a sign is to be understood. It is especially disturbing when the same sign occurs more than once in the same formula with different uses. (Frege 1897b [1984], 242)

Macbeth's interpretation contradicts what Frege says here about the logical muddiness of Peano's symbols being readable in more than one way. Frege's objection applies quite generally to logically ambiguous symbols such as Macbeth argues Frege designed his own symbols to be. This weighs heavily against Macbeth's proposal.

Also, *pace* Macbeth, Frege's conditionals do seem to have a fixed logical structure. Even in two-dimensional notations, there is a logical difference between the two formulas below:


Yet Frege would be the first to say that the difference between these to formulas is that in one case the second conditional-stroke is a subcomponent, whereas in the second case the second conditional-stroke is a supercomponent. Examples like these suggest that Frege intends for his Begriffsschrift formulas to have fixed signs for logical relationships, and not to permit as Peano does variously reading the symbolism for different logical relations—even if no error resulted from doing so. If on the contrary Frege really did want us to variously read the conditional-stroke as, say, a conjunction-stroke, such that conversion was possible based on how we read the notation, then why does he not say so? In *Begriffsschrift* he in fact says that the conjunction-stroke might have been taken as primitive, but he did not do so: the fixed form of inference Frege wants to use is "expressed more simply" with the conditional-stroke. (1879 [1997], 63) If Macbeth is right, Frege's point here is practically moot: we can variously read the conditional-strokes as (partly, at least) conjunction-strokes. So why would it matter which is primitive? And why would Frege object when Peano follows suit?

Macbeth rightly draws scholarly attention to the two-dimensional character of Frege's Begriffsschrift and to Frege's insistence that this notation is significantly different from one-dimensional ones. If I am right, then Frege's interest in a characteristically two-dimensional Begriffsschrift is motivated by his desire for logical perspicuity. He wants to depict logical relationships between contents clearly. This makes good sense given the anti-Kantian context: the last thing he would want in a dialectic against Kantians about arithmetic is Peano-like logically ambiguity in his symbolism. Macbeth's reading, in contrast, undercuts the logical perspicuity that Frege insists is essential to a Begriffsschrift by making signs for logical relations logically multiple. It is not clear on Macbeth's reading how a characteristically two-dimensional Begriffsschrift relates at all to Frege's anti-Kantian purposes. Still less does it do so, on Macbeth's reading, in a way that favors Frege (or disfavors Peano).

## 6. Frege's Two-Dimensional Begriffsschrift Explained

Scholars have generally relaxed the presentation of Frege's theorems and proofs: they do not always insist on using Frege's two-dimensional Begriffsschrift.[39] Still, it seems to always be felt that something about Frege's view is distorted by this practice. (Ebert and Rossberg 2013, xxix–xxx) Cook for example merely asserts this without arguing for the point:

> any attempted translation from *Grundgesetze* to a contemporary formalism will, in the end, fail. Frege's system is not equivalent to any contemporary "living" formal system currently studied. . . (Cook 2013, A-1)

Cook's aversion to anachronism is laudable, but as we saw in Section 1, Frege concedes that Peano has shown one-dimensional

[39] This is a longstanding tradition. Couturat, for example, told Frege what surely must have pained him, ". . . I shall devote a separate article to your logico-mathematical theories, stripping them as much as possible of the symbolism which must have discouraged many of your readers, which in any case will certainly discourage mine, and which would even cause difficulties to the printer." (Couturat 1904, 14)

notations can present formulas in principle. Given what Frege says, it is unclear why Cook's claim should be accepted: why could we not devise a new one-dimensional script could not in principle be devised to translate Frege's notations?[40]

If the anti-Kantian context is connected to the characteristically two-dimensional Begriffsschrift, as I have argued it is, then this explains why Frege's notations are characteristically two-dimensional and why he insisted on writing in them despite the logical tenability of one-dimensional alternatives. A characteristically two-dimensional Begriffsschrift has a clear philosophical advantage in a dialectic with Kantians: by separating logical form and logical content they make thoughts perspicuous. Frege's Begriffsschrift thereby shows, he thinks, that arithmetic justification is epistemologically independent of sensible intuition: they avoid the need for counting scope markers, retracing formulas, and so on. One simply reads along one two-dimensional Begriffsschrift formula in one direction without ever needing to count. Frege's view is that a thought so-expressed in Begriffsschrift is thereby available our grasping it and judging it accordingly without epistemologically relying on sensible intuition.

It is no accident that Frege believed that the result of his work was "almost all tied up with the concept-script." If the result of his work was establishing the analyticity (in Frege's sense) of arithmetic by producing gap-free proofs between logically perspicuous formulas, then this result clearly was tied up with his characteristically two-dimensional Begriffsschrift: their two-dimensional character plays a substantial role in Frege's positive case for arithmetic being independent of sensible intuition. So scholars are right to hesitate before paraphrasing Frege's Begriffsschrift away, on pain of losing their Kantian—really, anti-Kantian—context.

[40] Landini (2012, 13) makes an interesting point that amalgamation of horizontals has no known valid analogue in one-dimensional notations. Attempting to design a one-dimensional notation that does preserve valid inferences made with amalgamation of horizontals would be a worthwhile endeavor. If one cannot codify amalgamation of horizontals in one-dimensional script, then Frege doubtless would have been pleasantly surprised to learn he was wrong about the in principle workability of one-dimensional notations.

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