

Journal for the History of Analytical Philosophy

Volume 9, Number 11

Editor in Chief

Audrey Yap, University of Victoria

Editorial Board

Annalisa Coliva, UC Irvine

Vera Flocke, Indiana University, Bloomington

Henry Jackman, York University

Kevin C. Klement, University of Massachusetts

Consuelo Preti, The College of New Jersey

Marcus Rossberg, University of Connecticut

Anthony Skelton, Western University

Mark Textor, King's College London

Editor for Special Issues

Frederique Janssen-Lauret, University of Manchester

Review Editors

Sean Morris, Metropolitan State University of Denver

Sanford Shieh, Wesleyan University

Design and Layout

Daniel Harris, Hunter College

Kevin C. Klement, University of Massachusetts

ISSN: 2159-0303

jhaponline.org

© 2021 Richard Lawrence

Frege, Hankel, and Formalism in the *Foundations*

Richard Lawrence

Frege says, at the end of a discussion of formalism in the *Foundations of Arithmetic*, that his own foundational program “could be called formal” but is “completely different” from the view he has just criticized. This essay examines Frege’s relationship to Hermann Hankel, his main formalist interlocutor in the *Foundations*, in order to make sense of these claims. The investigation reveals a surprising result: Frege’s foundational program actually has quite a lot in common with Hankel’s. This undercuts Frege’s claim that his own view is completely different from Hankel’s formalism, and motivates a closer examination of where the differences lie. On the interpretation offered here, Frege shares important parts of the formalist perspective, but differs in recognizing a kind of content for arithmetical terms which can only be made available via proof from prior postulates.

Special Issue: Frege in Philosophical-Historical Context

Edited by Sanford Shieh and Juliet Floyd

Frege, Hankel, and Formalism in the *Foundations*

Richard Lawrence

1. Frege and Formalism

We have come to think of Frege's program for the foundations of arithmetic as a kind of *logicism*, and to distinguish logicism from other competing foundational programs, such as formalism and intuitionism. But these boundaries have been drawn more sharply with hindsight. Frege never called his own view "logicism". He also developed his view in conversation with thinkers we now call "formalists", although "formalism" was neither their word nor Frege's. At the beginning of the *Foundations of Arithmetic*, for example, he refers to a "widely-held formal theory", which he reacts to at several points throughout the book (Frege 1884 [1953], X). And at the end of the book's most significant discussion of this theory, Frege says of his own view that "it too could be called formal", although he is quick to add that his view is "completely different" from the view he has just discussed (Frege 1884 [1953], §105 n 1).¹

This raises the question of what Frege's relationship was, at the time he wrote *Foundations*, to the view we now call formalism. Why does Frege think his own view could also be called "formal"? Is he just repurposing this label for the view we now call logicism, or is there a stronger connection between Frege's foundational program and the "formal theory" as he understood

¹Here and throughout, I have replaced "formalist" with "formal" in Austin's translations to more closely represent the original German. Frege refers to *eine verbreitete formale Theorie* and says of his own view that *man könnte sie auch formal nennen*. As I explain below, "formalism" is still an appropriate word for the view Frege is talking about here; but it is not Frege's.

it? Frege evidently saw this view as an alternative foundational approach, worthy of consideration and discussion but also distinct from his own. The question is, *how* distinct? What does Frege's own view have in common with it, and in what ways is it different?

There are a variety of reasons to look into Frege's engagement with formalism, both in the *Foundations* and in his later work. First and foremost, of course, it tells us something about how Frege saw his program for the foundations of mathematics. But it also has implications for an issue of more general interest, namely, how Frege understood his semantic categories. As we will see below, an important feature of the "formal theory" was that it identified numbers with signs, and denied that those signs had meaning by representing something else. (Thus the English word "formalist" is an appropriate label for this view, and I will continue to use it, although Frege did not use it himself.) Frege argued in different ways throughout his work that formalism does not offer an adequate account of the content of mathematical signs, and he took pains to distinguish his own view from the formalist one in this respect. Frege's discussions of formalism thus tell us quite a lot about how his understanding of content evolved over the course of his career.

This essay examines those broader issues through the lens of Frege's relationship to Hermann Hankel, his main formalist interlocutor in the *Foundations*. Frege's remarks about Hankel give the impression that he sees little merit in Hankel's view, and that the two authors have nothing in common. Indeed, as Tappenden (2019) shows in detail, Frege is extraordinarily uncharitable to Hankel, making no effort to represent Hankel's view accurately and presenting it so selectively that it seems incoherent. To counteract this impression, I will begin with an investigation of Hankel's view, which will reveal a surprising result: Frege actually has quite a lot in common with Hankel. Both are offering foundational programs with the goal of showing that arithmetic is analytic. The two authors share a common conception of what

it means to show this, and they pursue the same strategy for doing so, namely, showing that every arithmetic truth can be deductively derived from definitions of the concept of number, the arithmetic operations, and the individual numbers.

The similarities between the two views undercut Frege's claim that his own view is completely different from Hankel's, which raises the question of where exactly the differences lie. To answer that question, I will examine in detail Frege's argument against Hankel at the end of the *Foundations*, where Frege argues that formalism "fails to distinguish clearly between concepts and objects" (Frege 1884 [1953], §97). Frege's criticisms there show that he differs from Hankel in recognizing certain questions about the existence of meanings or contents for arithmetical terms. Although Hankel thinks that these terms have meanings, his philosophical framework prevents him from acknowledging these questions in the way that Frege understands them. While Frege allows that we can make *concepts* available through definitions like the ones Hankel gives, Frege thinks a question remains about whether there are any *objects* falling under them. He stresses that such questions arise in ordinary mathematics and need to be answered by mathematical proof. So Frege's argument against Hankel reveals that his conception of the contents of arithmetical terms—his conception of numbers as objects, as opposed to concepts—is closely connected with the demand to give such existence proofs in mathematics.

Hankel does not seem to have been given much attention in (English-language) Frege scholarship, with the exception of work by Jamie Tappenden, who describes Hankel as an important part of Frege's intellectual environment and often offers brief descriptions of his work (Tappenden 1995, 1997, 2005, 2008, 2019). I have yet to find a detailed philosophical exposition of Hankel's view in English, though.² Thus, I will begin with a fairly detailed

²Apart from Tappenden's work, there are details about Hankel's biography and mathematical contributions in Crowe (1972), Youschkevitch (1976), Detlefsen (2005), and Petsche (2009).

account of Hankel's formalism in Section 2. That will provide the background needed to explain, in Section 3, Hankel's argument that arithmetic is analytic, and what Frege's view has in common with Hankel's. It will also be crucial for my interpretation of what the differences between the two views are, which I will offer in Section 4.

2. Hankel's Formalism

Hankel lays out his formalism in an 1867 text called *Vorlesungen über die complexen Zahlen und ihre Functionen*. Frege frequently cites and quotes from this text in the *Foundations*; indeed, as Tappenden (2019) notes, Frege refers to Hankel more often than any other contemporary author. I will survey Hankel's view here and in the following section. We will see that Hankel's view anticipates Frege's in several important ways.

The most obvious way that Hankel anticipates Frege is that he too is offering a foundational program: he sees a need for an investigation and rigorous presentation of the basic concepts of arithmetic. This program is to proceed via an analysis of concepts, especially the concept of number.

Hankel is driven to this foundational investigation by a desire for a more rigorous understanding of *complex* numbers, which is his ultimate target in the *Vorlesungen*. In his introduction, he notes that historically, complex numbers were thought to be "paradoxical" or "impossible". They later became accepted; but that does not mean these worries were adequately addressed. Hankel stresses the need to address such worries by revisiting our explanation of the concept:

As the development of mathematical concepts and ideas generally goes historically through two opposed phases, so goes also that of the imaginary numbers. At first this concept appeared as a paradox, strictly speaking inadmissible, impossible; however, in the course of time, the essential services which it affords to science subdue all doubts of its legitimacy, and one is convinced in such decisiveness of

its inner truth and necessity, that the difficulties and contradictions which one noticed in it at the beginning are hardly felt. Today, the question of imaginary numbers is in this second stage;—however it needs no proof that the actual nature of concepts and ideas is only sufficiently clarified when one can distinguish what is necessary in them, and what is arbitrary, i.e., is put to a certain purpose in them (Hankel 1867, V–VI).³

Thus, Hankel sees a foundational investigation of the complex numbers as part of a general pattern in mathematics. Once a concept is better understood, we can give a better explanation of it and thereby clear up any initial difficulties that it presented.

Frege opens the *Foundations* with very similar words:

After deserting for a time the old Euclidean standards of rigour, mathematics is now returning to them. . . The concepts of function, of continuity, of limit, and of infinity have been shown to stand in need of sharper definition. Negative and irrational numbers, which had long since been admitted into science, have had to submit to a closer scrutiny of their credentials.

In all directions these same ideals can be seen at work—rigour of proof, precise delimitation of extent of validity, and as a means to this, sharp definition of concepts (Frege 1884 [1953], §1).

Frege, like Hankel, stresses the need to re-examine concepts in mathematics for the sake of greater rigor, even when they have

³Wie überhaupt die Entwicklung mathematischer Begriffe und Vorstellungen historisch zwei entgegengesetzte Phasen zu durchlaufen pflegt, so auch die des Imaginären. Zunächst erschien dieser Begriff als Paradox, streng genommen unzulässig, unmöglich; indess schlugen die wesentlichen Dienste, welche er der Wissenschaft leistete, im Laufe der Zeit alle Zweifel an seiner Legitimität nieder und es bildete sich die Ueberzeugung seiner inneren Wahrheit und Nothwendigkeit in solcher Entschiedenheit aus, dass die Schwierigkeiten und Widersprüche, welche man anfangs in ihm bemerkte, kaum noch gefühlt wurden. In diesem zweiten Stadium befindet sich die Frage des Imaginären heut zu Tage;—indessen bedarf es keines Beweises, dass die eigentliche Natur von Begriffen und Vorstellungen erst dann hinreichend aufgeklärt ist, wenn man unterscheiden kann, was an ihnen nothwendig ist, und was arbiträr, d.h. zu einem gewissen Zwecke in sie hineingelegt ist.

long since been accepted as useful. Like Hankel, he stresses that this is a general historical pattern in mathematics and part of its scientific process. And like Hankel, he sees “sharp definition” of those concepts as the means to this goal of greater rigor.

The emphasis that both authors place on analysis of concepts in foundational investigations also reflects a more telling way in which Hankel anticipates Frege: his foundational program is a response to Kant’s view of arithmetic. Hankel, like Frege, wants his program to show that arithmetic is *analytic*, rather than synthetic. It is supposed to yield an arithmetic which is based purely on concepts and logical deduction, and makes no essential use of Kant’s notion of pure intuition. I will examine Hankel’s argument against Kant, and the way Frege takes this argument up in his own foundational program, in Section 3. To see how this argument works, though, we first need to understand some of the details of Hankel’s formalism on its own terms. The rest of this section gives those details.

2.1. Formal vs. presented numbers

At the center of Hankel’s foundational program is a distinction between “formal” and “presented” (*actuelle*) numbers.⁴ Presented numbers are given to us in intuition, and “find their rep-

⁴Hankel’s word *actuell* is a bit difficult to translate. Its use seems to have been limited to philosophically-oriented texts in the nineteenth century; it appears neither in current German dictionaries nor in the Deutsches Textarchiv reference corpus going back to 1473. The word seems to be an alternate spelling of the German *aktuell* (which can mean “present” or “at hand”) and related to the French *actuel* (which can mean “present” and also “actual”, like German *wirklich*). When he introduces his distinction between *actuelle* and formal numbers, Hankel laments in a footnote that the most appropriate terms for them would be “real” and “ideal”, but those terms already have a defined and narrower meaning in mathematics. My translation of *actuell* as “presented” is intended to mark the idea that *actuelle* numbers, in contrast to formal numbers, are given in intuition. “Actual” would be a workable alternative, but it does not bring out the connection to intuition as clearly, and collides with the normal translation for *wirklich*.

resentation in the theory of actual (*wirklichen*) magnitudes and their combination" (1867, 7). Formal numbers (which Hankel also calls "transcendent", "purely mental", or "purely intellectual") are by contrast "not capable of any construction in intuition" (1867, 7). For Hankel, formal numbers are conceptual and independent of any intuitive representation. He sees the conceptual as defined by means of general laws or rules, and he thinks the only principle governing such rules is that they be consistent, i.e., not self-contradictory.

The distinction between presented and formal numbers thus runs parallel to Kant's distinction between intuitions and concepts. Hankel's goal is to build a foundation for arithmetic on formal numbers, relegating presented numbers to a secondary status. Since this is a move away from the intuitive toward the conceptual, it is also a move toward an arithmetic which is based purely on concepts and therefore analytic rather than synthetic.

Hankel does allow that formal numbers may have presented numbers *corresponding* to them, or that we can sometimes "attach" presented numbers as intuitive interpretations to our signs for formal numbers. When we work with a geometric interpretation of the complex numbers, for example, we are using intuitively-presented numbers which correspond to the formal complex numbers; but these presented numbers should be distinguished from the formal complex numbers themselves, whose properties are determined by a purely conceptual definition. Hankel thinks that while such a correspondence might be helpful, it is by no means necessary for working with formal numbers in mathematics. The formal numbers are prior to, and independent of, any presented numbers that we might put them in correspondence with.

Indeed, while intuitive representations can give us an initial grasp on a new type of number, Hankel holds that they are ultimately a barrier to mathematical understanding:

The condition for the establishment of a general arithmetic is therefore a purely intellectual mathematics detached from all intuition, a

pure theory of form, in which quanta or their images, the numbers, are not combined, but rather intellectual objects, thought-things, to which presented objects or relations of such objects can, but need not, correspond (Hankel 1867, 10).⁵

When Hankel later gives his general definition of formal number, he reiterates this point:

A different definition of the concept of the formal numbers cannot be given; every other definition must rely on ideas from intuition or experience, which stand in only an accidental relation to the concept, and the limitations of which place insurmountable obstacles in the way of a general investigation of the arithmetic operations (Hankel 1867, 36).⁶

Thus, Hankel expressly rejects a role for intuition in defining the formal numbers. He is working in a Kantian framework with Kantian terminology, but offering an anti-Kantian program: an arithmetic based on concepts, in which purely conceptual definitions suffice to ground arithmetical truths.

2.2. Defining the formal numbers

How does this program proceed? Hankel begins his book by describing a kind of genetic unfolding of arithmetic, in which we proceed from the natural numbers to wider systems of numbers in a series of stages. At each stage, we start with a domain of presented numbers, i.e., an intuitive grasp of those numbers and some operations defined on them. We then take the general

⁵Die Bedingung zur Aufstellung einer allgemeinen Arithmetik ist daher eine von aller Anschauung losgelöste, rein intellectuelle Mathematik, eine reine Formenlehre, in welcher nicht Quanta oder ihre Bilder, die Zahlen verknüpft werden, sondern intellectuelle Objecte, Gedankendinge, denen actuelle Objecte oder Relationen solcher entsprechen können, aber nicht müssen.

⁶Eine andere Definition des Begriffes der formalen Zahlen kann nicht gegeben werden; jede andere muss aus der Anschauung oder Erfahrung Vorstellungen zu Hilfe nehmen, welche zu dem Begriffe in einer nur zufälligen Beziehung stehen, und deren Beschränktheit einer allgemeinen Untersuchung der Rechnungsoperationen unübersteigliche Hindernisse in den Weg legt.

arithmetical laws which define the operations on those numbers as a conceptual definition, and work with that definition on its own terms. This allows us to recognize new formal numbers that are not part of our previous intuitive representation—in particular, numbers which provide *inverses* for the defined operations—which in turn forces us to abstract from that representation.⁷ When we find a new intuitive representation for the wider domain of numbers, this process can begin again.

Let's see how this works for the case of the negative numbers. Hankel imagines starting from an understanding of the natural numbers as measuring positionings or "puttings" of other, non-numerical objects. The number 3, for example, would be something like "putting an object thrice"; it corresponds to an intuitive presentation of objects in three different spatial locations.⁸ Addition of numbers then corresponds to putting distinct objects into the same presentation: the sum of any two numbers n and m is the number of objects in the representation we get by starting with a representation of n objects and putting m distinct ones into it.

Hankel thinks the question then naturally arises as to how we can invert this operation: given a number, what number must we add to it to get a certain sum? For example: what number x , when added to 2, makes 5? Our intuitive representation can

⁷This strategy of introducing wider systems of numbers by defining new numbers as inverses originally comes from Gauss, and Hankel quotes Gauss to explain it (Hankel 1867, 5–6). Frege also employs this strategy to define, e.g., the negative numbers. As Tappenden explains, "Frege and Hankel shared an environment in which that Gauss passage was the foundation of the dominant view" (2019, 240).

⁸Frege makes fun of this proposal in *Foundations* §20. The remark is uncharitable, since it attacks a definition that Hankel himself does not endorse. And as Tappenden points out, Frege actually quotes Hankel in a misleading way, leaving out a parenthetical word (*Position*) that helps make sense of why Hankel is exploring this representation to begin with: it connects the intuitive representation of numbers with the concept of "position" or "location" that was being explored by projective geometers at the time (1997, 216–17).

answer this question: 3. Positioning three new objects in the same representation as two others yields a representation containing five objects.

But once we can ask this type of question, we can just as naturally ask questions like: what number x , when added to 5, yields 2? In this case, the limitations of our intuitive representation become apparent. The problem is that we here need a *negative* number, -3 , but there is no way to represent adding a negative number in terms of additional "puttings" of objects: putting more objects into a representation can only increase their number, but adding a negative number should decrease it. Thus, *within* the perspective of our original intuitive representation, one

cannot see how a real substance can be understood by -3 . . . and would be within his rights if he refers to -3 as a non-real, imaginary number, as a "false" one (Hankel 1867, 5).⁹

Asking for numbers that provide inverses for the operation of addition clashes with our intuitive representation. So to be able to answer this question, we need to abstract from that representation.

Hankel's idea is that we can at this point cast the intuitive representation aside, laying down the laws which define addition and stipulating that they are to hold generally. The laws he has in mind are, for example, the associative and commutative laws for addition, and the laws that define subtraction as the inverse operation to addition, such as $(a + b) - b = a$. By stipulating that these laws hold generally, we are led to recognize formal numbers which do not appear in the intuitive presentation, namely, the numbers which satisfy $x + b = c$ when $c < b$. Hankel writes that in such cases, one "adds an inverse in thought to the given

⁹Man sieht aber nicht, wie unter -3 eine reale Substanz verstanden werden kann. . . und würde im Rechte sein, wenn man -3 als eine nicht reelle, imaginäre Zahl als eine "falsche" bezeichnete.

series of objects [i.e., the natural numbers]" (Hankel 1867, 26).¹⁰ Thus we arrive at a formal definition of the negative numbers as the additive inverses for pairs of natural numbers.

How should we understand numbers that are introduced this way? What grasp do we have of these new, formal numbers in abstraction from any intuitive representation for them? Two aspects of Hankel's view come into play here which connect it with a more well-known picture of formalism.¹¹ First, Hankel holds that the laws we lay down as definitions are *up to us*, so long as they are logically consistent with each other:

How we define the rules of purely formal operations (*Verknüpfungen*), i.e., of carrying out operations (*Operationen*) with mental objects, is our arbitrary choice, except that one essential condition must be adhered to: namely that no logical contradiction may be implied in these same rules (Hankel 1867, 10).¹²

Because we lay down these laws on our own authority,¹³ not on the basis of properties that the operation has in any intuitive representation, they remain valid apart from intuitive representations. Consistent with a Kantian framework, Hankel holds that in the realm of the formal or conceptual, we are bound only by the law of non-contradiction.

¹⁰man sich zu der gegebenen Reihe von Objecten eine inverse hinzudenkt

¹¹Detlefsen (2005) provides an excellent presentation of this picture. Specifically, the two aspects of Hankel's understanding of formal numbers I discuss here align with what Detlefsen calls formalism's "creativist component" and its "advocacy of a nonrepresentational role for language in mathematical reasoning" (2005, 237).

¹²Wie wir die Regeln der rein formalen Verknüpfungen, d.h. der mit den mentalen Objecten vorzunehmenden Operationen definiren, steht in unserer Willkühr, nur muss eine Bedingung als wesentlich festgehalten werden: nämlich dass irgend welche logische Widersprüche in denselben nicht implicirt sein dürfen.

¹³Hankel's formalism shares the view that the laws of arithmetic are something we can lay down on our own authority with the formalism of Heine and Thomae, which Frege discusses in the *Basic Laws* §§86–137, though he does not mention Hankel there.

An important consequence of this constraint is what Hankel calls his "principle of permanence of formal laws" (1867, 11). Hankel invokes this principle to guide the extension of a domain with a defined operation to a wider domain which supplies that operation with a complete set of inverse elements. The principle says, in effect, that once we have laid down a formal definition of that operation, any extension to the domain must remain consistent with the original definition. This constrains how the new elements interact with the old elements and with each other under the operation and its inverse. For example, Hankel proves that his formal definition of addition and subtraction implies the law

$$(a - b) + (c - d) = (a + c) - (b + d)$$

for the natural numbers, i.e., where $(a - b)$ and $(c - d)$ are positive (1867, 26).¹⁴ The principle then tells us that this equation must remain valid once we extend the domain to include negative numbers. Hankel says we should look at the equation as *defining* what it means to add two *negative* numbers $(a - b)$ and $(c - d)$. Other results which extend the definitions of addition and subtraction to the new negative numbers will flow from the original definition of these operations on the natural numbers in the same way.

Second, Hankel holds that our symbolic representations of the laws we lay down are enough to give us a grip on their (purely conceptual) content. Hankel writes that the new, formal numbers "first appear as pure signs" (1867, 8), introduced entirely for the purpose of giving a definition of an inverse operation. For

¹⁴Actually, Hankel proves a more general result of which this equation is an instance, for a binary operation Θ and its inverse λ . Θ and λ generalize over pairs of inverse arithmetic operations, like addition and subtraction, multiplication and division, exponentiation and logarithm, and so on. This allows Hankel to apply the same result to e.g. extending the integers to the rationals by adding multiplicative inverses. Even Frege concedes that this general presentation is a valuable contribution of Hankel's formalism in *Foundations* §99.

example, when we introduce negative numbers by means of laws like $(a + b) - b = a$, these laws tell us that what we mean by any sign of the form “ $-b$ ” is just: whatever is the additive inverse of b .¹⁵ The new formal numbers are given to us simply as *that which solves a certain kind of equation*. To understand what -3 is, or what “ -3 ” means, all we need to understand is that it is the formal number which yields 2 when added to 5, and 6 when added to 9, and so on: the general laws governing addition and subtraction completely determine its behavior with respect to the operations and the other numbers.

Like other formalists, Hankel goes so far as to *identify* formal numbers with signs. He says this clearly as he summarizes his conception of a system of formal numbers:

Such a system [of signs implementing the arithmetic operations] can only be created by starting from certain elements, the units, connecting them in every possible way through certain operations, and inscribing (*signirt*) the results of these operations with new signs. These new signs will then, in accordance with the previously given rules, again be operated (*verknüpfen*) with and give rise to new signs, and so on. One goes on until one no longer reaches new signs, so the results of new operations can always be expressed through those already at hand. The thus-developed sequence of signs is called a closed system or domain, whose ordering I designate according to the number of units which have been related in its formation. . .

I call the signs of such a system numbers, and thus set their concept in a necessary context with the operations through which they are formed and pass into one another. Every change of the rules for operations brings a change of the numbers with it.¹⁶ (Hankel 1867, 35–36, emphasis added)

¹⁵I am of course combining two separate steps here: first, laying down the law that $(a + b) - b = a$ as part of a definition of subtraction as the binary operation inverse to addition, and then introducing the unary notation $-b$ as shorthand for $0 - b$.

¹⁶Ein solches System kann nur geschaffen werden, indem man von gewissen Elementen, den Einheiten ausgeht, diese auf alle mögliche Weise durch gewisse Operationen verbindet und die Resultate dieser Operationen mit

Thus, for Hankel, a formal number system is the transitive closure of a set of operations on a set of units¹⁷ (which have to be assumed). This system is determined by the initial choice of units and the definitions of the operations. For example, from the single unit ‘1’ and the general laws defining addition, we get the system of the integers under addition, by defining e.g. “2” as $1 + 1$, “ -1 ” as the number x such that $1 + x = 0$, and so on. A grasp of the conceptual definitions of these operations suffices to give us the system of formal numbers they define.

Frege of course ridicules the idea that numbers are signs; in *Foundations* §95 he criticizes Hankel as failing to distinguish signs and content, thus confusing numbers with printer’s ink. But Frege is being uncharitable here. As the passage just quoted makes clear, Hankel is thinking of “signs” as something more abstract than printed marks. Signs form completed infinite systems; so they cannot be the same as the marks we write down on paper to represent them. And in other places, Hankel clearly distinguishes between signs and their content, referring for example to the “formal meaning” (*formale Bedeutung*) of a sign, and contrasting that with any “presented meaning” the sign might

neuen Zeichen signirt. Diese neuen Zeichen werden dann nach vorstehenden Regeln wiederum zu verknüpfen sein und zu neuen Zeichen Veranlassung geben u.s.f. Fährt man so fort, bis man zu neuen Zeichen nicht mehr gelangt, also die Resultate der neuen Operationen durch die schon vorhandenen jedesmal ausgedrückt werden können, so nennt man die gebildete Zeichenreihe ein abgeschlossenes System oder Gebiet, dessen Ordnung ich nach der Zahl von Einheiten benenne, welche by seiner Bildung verwandt worden sind. . .

Die Zeichen eines solchen System nenne ich Zahlen und setze also deren Begriff in einen nothwendigen Zusammenhang mit den Operationen, durch welche sie gebildet werden und in einander übergehen. Jede Veränderung der Operationsregeln bringt eine Veränderung der Zahlen mit sich.

¹⁷In Hankel’s view, a single unit symbolized by “1” suffices to develop the real numbers. Hankel allows for multiple units because he is seeking a definition of formal numbers that can also encompass complex numbers (where an additional unit “ i ” is needed) and quaternions (which require two more).

have. Thus Hankel would say, for example, that the formal number -3 is the formal meaning of the printed mark “ -3 ”.

If Hankel distinguishes signs from their meanings, though, why does he identify formal numbers with signs, rather than with their meanings? I suggest that this identification results from a slide between two senses of “sign”, and it is not too difficult to see why Hankel makes it. A sign, as distinct from printed marks, is something that printed marks *have in common*. In the first instance this is perhaps a certain typographical shape. If we think of the sign as this common shape, then the sign is distinct from its meaning, and we can speak of it as *having* a meaning. When Hankel distinguishes signs from their presented and formal meanings, he is using “sign” in this sense, as something like a shape that printed marks have in common.

On the other hand, as far as mathematical discourse is concerned, the shape is obviously an inessential feature of the printed marks: sloppy handwriting or a change of notation do not prevent us from recognizing the same sign in different marks. What matters is that the marks are recognized to have a common meaning. So if a sign is what printed marks have in common, then in many mathematical contexts it makes sense to identify the sign with the common meaning, rather than the common shape. We rely on this understanding of signs, for example, when speaking in a logic class about the properties of “the conditional sign”: we are there talking about the properties of the common meaning of certain marks, not merely of their shape. Hankel’s formal numbers are meant to be “signs” in roughly the same sense that we speak of the conditional as a “sign”, as the common mathematical meaning of printed marks, not the printed marks themselves.

It is especially tempting for Hankel to slide into using “sign” in this second sense of a common meaning, because he thinks of the meaning of arithmetical signs in non-representational terms. Although Hankel speaks of a sign like “ -3 ” as having a formal meaning, that meaning does not consist in its representing a *fur-*

ther thing beyond the sign itself. Instead, the sign-shape is meaningful because it has a role in a rule-governed system. Hankel writes that signs “receive their formal meaning only through our determining the rules according to which they are to be operated with” (Hankel 1867, 70).¹⁸ Such formal meanings are conceptual, and have no intuitive representation; instead, our grasp of them is manifested in our ability to operate with printed marks according to the rules we lay down.

Hankel shares this non-representational understanding of formal meanings with other contemporary formalists. Thomae, for example, compares the signs of arithmetic with pieces in a game like chess:

arithmetic is a game with signs which one may well call empty, thereby conveying that (in the calculating game) they do not have any content except that which is attributed to them with respect to their behavior under certain combinatorial rules (game rules). A chess player makes use of his pieces in a similar fashion: he attributes certain properties to them that constrain their behavior in the game, and the pieces are only external signs for this behavior. (Thomae 1898, 3; translation quoted from Frege 2013, Vol. II §88)

Thomae expresses clearly here that arithmetical signs are “empty” in the sense that they do not represent something else, but they nevertheless have a non-representational kind of content, which is conferred on them by the rules governing their manipulation. This idea was a common theme in other strands of formalism, and lives on, for example, in proof-theoretic approaches to semantics. Frege’s remarks in *Foundations* §95, and his criticisms of Thomae’s view in *Basic Laws*, show us that he did not think this non-representational understanding of content was an adequate one for his purposes. But this is no reason to think that Hankel confused formal numbers with printer’s ink.

¹⁸Zeichen. . . erhalten aber ihre formale Bedeutung erst dadurch, dass wir die Regeln festsetzen, nach welchen mit ihnen zu operiren ist.

2.3. Existence of formal numbers?

Hankel's understanding of "signs" as rule-governed conceptual content has consequences for his views about mathematical existence. Hankel thinks that because formal numbers are given by a non-intuitive, purely conceptual definition, it does not make sense to ask whether they exist; we can only ask whether their definition is consistent. He makes this clear in an early passage:

If one wants to reply to the frequently put question of whether a certain number is possible or impossible, one must first get clear about the actual sense of this question. Number today is no longer a thing, a substance, which exists independently apart from the thinking subject and from the objects which give rise to it, an independent principle such as the Pythagoreans considered. The question of existence can therefore only relate to the thinking subject or the objects thought, whose relations the numbers present. The mathematician counts as impossible in the strict sense only what is logically impossible, i.e., what is self-contradictory. That numbers which are impossible in this sense cannot be admitted needs no proof. If however the numbers under consideration are logically possible, their concept clear and determinately defined for us and thus without contradiction, that question can only come to this: whether there is in the domain of the real or of the actual in intuition, of the presented (*des Actuellen*), a substrate for them; whether there are objects in which the numbers, i.e., intellectual relations of a certain sort, make their appearance (Hankel 1867, 6–7).¹⁹

¹⁹Will man die häufig gestellte Frage beantworten, ob eine gewisse Zahl möglich oder unmöglich sei, so muss man sich zunächst über den eigentlichen Sinn dieser Frage klar werden. Ein Ding, eine Substanz, die selbständig ausserhalb des denkenden Subjectes und der sie veranlassenden Objecte existirte, ein selbständiges Princip, wie etwa bei den Pythagoreern, ist die Zahl heute nicht mehr. Die Frage von der Existenz kann daher nur auf das denkende Subject oder die gedachten Objecte, deren Beziehungen die Zahlen darstellen, bezogen werden. Als unmöglich gilt dem Mathematiker streng genommen nur das, was logisch unmöglich ist, d.h. sich selbst widerspricht. Dass in diesem Sinne unmögliche Zahlen nicht zugelassen werden können, bedarf keines Beweises. Sind aber die betreffenden Zahlen logisch möglich, ihr Begriff klar und bestimmt definirt und also ohne Widerspruch, so kann jene Frage nur

Thus, for Hankel, the question of existence is directly connected with intuition, and can only be answered by giving presented numbers. There is no question of existence that properly applies to formal numbers, and thus no room for mathematical proofs that particular formal numbers do or do not exist.

Frege quotes this passage to begin his main discussion of formalism in the *Foundations*, in §92. As we will see in Section 4, the central issue in that discussion is the existence of the contents of arithmetical signs. Frege seeks to sharply distinguish his own view from Hankel's formalism on that issue, charging Hankel with postulating the existence of such contents, which is instead something that needs to be proven. The question we will face is how to understand this criticism, and what exactly it tells us about the difference between Frege and Hankel's views of the contents of arithmetical signs.

3. Against Kant: Hankel's Argument and Frege's Reception of It

Before we turn to Frege's criticisms, though, I want to examine Hankel's argument against Kant's view of arithmetic, because Hankel's influence on Frege is particularly transparent in the context of this argument. As we will see, Frege draws especially closely on Hankel's understanding of what it means for arithmetic truths to be analytic. He also follows the same strategy as Hankel for demonstrating, against Kant, that they are indeed analytic. For both authors, this strategy involves defining individual numerals recursively and using those definitions to prove basic arithmetical facts. These similarities undercut Frege's claim that his own view is completely different from Hankel's, and motivate a closer examination of what the differences really are.

darauf hinaus kommen, ob es im Gebiete des Realen oder des in der Anschauung Wirklichen, des Actuellen ein Substrat derselben, ob es Objecte gebe, an welchen die Zahlen, also die intellectuellen Beziehungen der bestimmten Art, zur Erscheinung kommen.

This is not the place to introduce Kant's view in detail. Instead, I will simply describe the basic points of Kant's view as I take Hankel to understand it. For this purpose, what is important are Kant's well-known views connecting analytic judgments with deduction and the analysis of concepts, and synthetic judgments with intuition. For Kant, analytic judgments are "based on concepts" in the sense that they can be justified just by making deductions from the definitions of the concepts involved. He often speaks of the predicate concept of an analytic judgment being "contained" in the subject concept, and of these judgments being justified purely logically, or purely in accordance with the principle of contradiction. For example, a judgment like "If this is a triangle, it has three interior angles" is analytic because the concept of having three interior angles is contained in the definition of the concept of triangle; judging a triangle *not* to have three interior angles would be self-contradictory. Synthetic judgments, by contrast, can only be justified by appealing to something beyond the definitions of the concepts involved, namely, intuition.

Kant had proposed that the truths of arithmetic are synthetic rather than analytic. He argued that the concepts which appear in them do not already contain everything needed to justify them. For Kant, an arithmetical statement like $7 + 5 = 12$ could not be proven "from the concept of a sum of seven and five in accordance with the principle of contradiction"; and in general, "twist and turn our concepts as we will, without getting help from intuition we could never find the sum by means of the mere analysis of our concepts" (Kant 1781-87 [1998], B15–B16). Because he thinks we cannot find 12 merely by analyzing our concept of the sum of 7 and 5, Kant thinks this judgment is synthetic, not analytic. Recognizing and demonstrating its truth requires taking recourse to pure intuition.

Hankel's foundational system is set up to directly respond to these views of Kant's. We have already seen that Hankel resists the idea that intuition is required in the foundations of arithmetic. Hankel's distinction between formal and presented

numbers, his claims that formal numbers are conceptual, prior to, and independent of any intuitive presentation, and his claim that such intuitive presentations present "insurmountable obstacles in the way of a general investigation of the arithmetic operations" (Hankel 1867, 36) are all part of this resistance.

Because Kant says that we must appeal to intuition to justify basic arithmetic truths, Hankel takes Kant to be committed to the view that arithmetic facts like $7 + 5 = 12$ are all *primitive* truths, which are ultimately justified not by proof from more basic truths, but by an appeal to pure intuition. Arithmetic thus contains an infinity of primitive truths. Hankel has sharp words for this view:

The view according to which the facts of addition and multiplication manifest an unlimited series of axioms, even if Kant shrinks from this name, is so inadequate and paradoxical that one hardly understands how one could content oneself with it. . . The apodictic certainty of the statements of mathematics is based on the fact that it deductively erects an infinite structure on an extremely small number of independent base truths; and here an infinite number of infinitely multifarious connected columns are supposed to carry this structure, although only one single connection needs to falter to bring the entire proud structure to the ground! (Hankel 1867, 53–54).²⁰

Frege cites this criticism approvingly in his own discussion of Kant's view in *Foundations* §5. Both Hankel and Frege see Kant's primitivism as a problematic consequence of his view that arithmetic truths can only be justified by appeal to intuition. They

²⁰Die Ansicht, nach welcher das Eins-und-eins sowie das Ein-mal-eins eine unbegrenzte Reihe von Axiomen, wenn auch Kant vor diesem Namen zurückschreckt—aufweist, ist so unangemessen und paradox, dass man kaum begreift, wie man sich bei ihr beruhigen könne. . . die apodictische Gewissheit [der] Sätze [der Mathematik] beruht darauf, dass sie auf einer äusserst kleinen Zahl von independenten Grundwahrheiten deductiv ein unendliches Gebäude errichtet; und hier soll gar eine unendliche Anzahl von unter sich unendlich mannigfach verbundenen Pfeilern das Gebäude tragen, obgleich nur ein einziges Bindeglied zu wanken braucht, um den ganzen stolzen Bau zum Umsturz zu bringen!

also propose the same remedy: to develop a formal system in which arithmetic truths can be deductively proven from a finite, surveyable set of general axioms.

Hankel points out that in a system like the one he has proposed, it is trivial to provide proofs of basic arithmetic facts. Recall that in a system of formal numbers, we lay down general laws governing the arithmetic operations, and think of the system as the transitive closure of those operations on one or more units. We assign new signs to abbreviate the numbers we reach by repeated application of the operations. The proofs exploit these features. Here is an example of such a proof for the case of $5 + 2 = 7$:

$$\begin{aligned} 5 + 2 &= 5 + (1 + 1) && \text{(definition of "2")} \\ &= (5 + 1) + 1 && \text{(associativity of addition)} \\ &= 6 + 1 && \text{(definition of "6")} \\ &= 7 && \text{(definition of "7")} \end{aligned}$$

Hankel remarks that such proofs proceed via a recursive process “without any intuition, purely mechanically” (1867, 37). So long as we have the right definitions and general laws in place, we can prove any basic arithmetic fact this way, contrary to Kant’s claim that they are indemonstrable without appeal to intuition.

Frege makes essentially the same observations in *Foundations* §6, though he attributes this style of proof originally to Leibniz.²¹ He remarks there that “I do not see how a number like 437986 could be given to us more aptly than in the way that Leibniz does it”, for

Even without having any idea of it, we get it by this means at our disposal nonetheless. Through such definitions we reduce the

²¹In addition to Leibniz, Frege also mentions Hankel and Grassmann here as sources for this strategy for proving arithmetical truths; Hankel himself is following Grassmann’s presentation. For more on the relationships of Frege and Hankel to Grassmann, see Tappenden (1995, 2008); Petsche (2009); Mancosu (2015, 2016).

whole infinite set of numbers to the number 1 and increase by 1, and every one of the numerical formulae can be proved from a few general propositions (Frege 1884 [1953], §6).

Recursively defining each of the natural numbers via “increase by 1” is a familiar method, and not unique to Hankel. What is important here, though, is that Frege is following Hankel in seeing this style of definition as a strategy to counter Kant’s primitivism: he agrees that such definitions of the numbers, together with general arithmetical laws, allows us to prove all the arithmetical facts without recourse to intuition.

The observation that all arithmetic formulae can be proven this way, from general laws and definitions of particular number signs, enables Hankel to reconceive the issue of the analyticity of arithmetic. Hankel points out that this observation reduces the question about whether *arithmetic* is analytic or synthetic to the question of whether its *axioms* are analytic or synthetic—“for there is nowhere any doubt about the possibility of analytically or deductively deriving the further mathematical theorems from these” (Hankel 1867, 51).²² In particular, if the axioms are all analytic, so are all the facts we can deduce from them, because deduction preserves analyticity.

The shift Hankel makes here, away from Kant’s talk of “containment of concepts” in analytic judgments and toward a conception of analyticity that focuses on the status of the axioms from which they are proven, is reflected in Frege’s own framing of the issue in *Foundations* §3:

Now these distinctions between a priori and a posteriori, synthetic and analytic, concern, as I see it, not the content of the judgement but the justification for making the judgement. . . The problem becomes, in fact, that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, in carrying out

²²denn über die Möglichkeit, aus diesen [Grundsätzen] analytisch oder deductiv die weiteren mathematischen Lehrsätze abzuleiten, ist überall kein Zweifel

this process, we come only on general logical laws and definitions, then the truth is an analytic one. . . . If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some special science, then the proposition is a synthetic one (Frege 1884 [1953], §3).

For a reader who is only familiar with Kant's remarks about analyticity, Frege's framing here is surprising: at first glance, it is hard to see that he is using the terms "analytic" and "synthetic" in a manner continuous with Kant's at all. Frege does not talk about analyticity in terms of containment of concepts or the principle of contradiction. Instead, he sees the issue in the way Hankel does, where the question of whether an arithmetic theorem is analytic or synthetic reduces to the question about the axioms from which it is deduced.

Frege's way of characterizing the difference between the axioms underlying analytic and synthetic truths also parallels Hankel's. Hankel goes on to argue that the axioms which justify arithmetical truths are all analytic. To make this argument, he starts by listing twelve principles from Euclidean geometry, and observing that we can distinguish two kinds of principles there. The first kind of principles "refer to relations which are essentially connected to the concept of magnitude" and includes things like *whatever are equal to one and the same thing are equal to each other* and *to add equals to equals gives equals*. The second kind "contains geometric truths" and includes principles like *all right angles are equal to each other* and *two straight lines do not enclose any space* (Hankel 1867, 51–52). Hankel clearly intends that part of what distinguishes this second group is that they are *particular* to geometry; as Frege puts it, they "belong to the sphere of some special science". This is in contrast to the more general axioms of the first group. Hankel applies Euclid's term *common notions* to the first group, partly to emphasize their generality.

The question for Hankel is whether the axioms of arithmetic are more like the "common notions" of the first group, or the particular geometric truths of the second group. Hankel, like

Frege, acknowledges that any theorem we prove by means of the second kind of axioms would be synthetic. But he argues that the axioms we need for proofs of arithmetic truths, like associativity and commutativity of addition, are more like those in the first group, and therefore analytic. Hankel offers three reasons to classify them as such:

The three given principles have indeed the character of common notions. They become *completely evident through an explication*; they are *valid for all domains of magnitudes*. . . . and can, without forfeiting their character, *be transformed into definitions*, in which one says: by the addition of magnitudes is understood an operation which satisfies these three principles (Hankel 1867, 55, emphasis added).²³

Frege quotes this passage in *Foundations* §12, as he begins his own argument that the axioms of arithmetic are analytic. Though Frege criticizes Hankel there, he still adopts Hankel's most important consideration: Hankel argues that the axioms of arithmetic are analytic by appealing to their *generality*, and in particular their validity beyond the realm of geometry. Frege, too, ties the analyticity of arithmetic to the generality of its axioms, and explicitly contrasts the axioms of arithmetic and geometry in this respect in *Foundations* §13 and 14. He argues there that arithmetic is analytic because, unlike geometry, it applies to "not only the actual, not only the intuitable, but everything thinkable", and that for this reason, if we try to deny any of the axioms of arithmetic, "even to think at all seems no longer possible" (Frege 1884 [1953], §14).²⁴

²³Diese 3 hier angeführten Grundsätze haben durchaus den Charakter der notionen communes. Sie werden durch eine Explication vollkommen evident, gelten für alle Grössengebiete. . . . und können, ohne ihre Charakter einzubüssen, in Definitionen verwandelt werden, indem man sagt: Unter der Addition von Grössen versteht man eine Operation, welche diesen 3 Sätzen genügt.

²⁴For a discussion of Frege's criticisms of Hankel in *Foundations* §12 and more about the issue of the relative generality of geometry and arithmetic, see Tappenden (2005).

Let's summarize, then, the commonalities we have seen between Frege's and Hankel's views. Hankel shares Frege's goal of carrying out a foundational program for arithmetic. Like Frege, Hankel thinks that this program must begin with an analysis of the concept of number and the other basic concepts of arithmetic, and should yield definitions of our different systems of numbers, from the natural numbers to the complex numbers. The central distinction in Hankel's foundational program, between "formal" and "presented" numbers, runs parallel to Kant's distinction between concepts and intuitions; Hankel employs this distinction because he wants, like Frege, to make room for an arithmetic which is based on concepts alone, where intuition plays no essential role. Hankel's strategy for constructing this arithmetic is to set up a formal system in which all arithmetic facts can be deduced formally or mechanically from a small number of general laws governing the arithmetic operations, plus definitions of the particular number signs in terms of how they are obtained via repeated applications of these operations. This allows Hankel to reduce the issue of analyticity to the status of these axioms in his argument against Kant. Frege's understanding of analyticity follows Hankel's, and he carries out the same strategy of axiomatization with the same goal: showing, against Kant, that arithmetic is analytic.

All of this makes it problematic to take Frege at his word when he writes at the end of his discussion of formalism that his own view "could be called formal" but is "completely different from the view criticized above under that name" (Frege 1884 [1953], §105 n 1). Frege has much more in common with Hankel than the text of the *Foundations* lets on. The goals, methods, and intended philosophical consequences of Frege's foundational program are broadly the same as Hankel's. So the question of how Frege's view differs from Hankel's requires an answer that distinguishes them in their details. I turn now to that investigation.

4. Frege's Criticism of Formalism in *Foundations*

Frege engages in an extended discussion of formalism in *Foundations* §§92–105. His argument there has two phases. In the first phase (§§92–99), he raises a problem for Hankel's formalism. Frege's central objection concerns how to understand the content of arithmetical terms like " $2 - 5$ ". He argues that, while the formalist's definitions may provide us with concepts associated with such terms, they do not suffice to prove that there are *objects* which can serve as their contents. But such existence proofs are presupposed whenever we use arithmetical terms, and need to be supplied by a foundational program. In the second phase of the argument (§§100–105), Frege considers how we might get around this problem, and argues that his own way of assigning content to natural number terms can be extended to other systems of numbers.

A surprising feature of the entire discussion is how much of the formalist view it leaves intact. On the interpretation I will offer, Frege characterizes the problem with Hankel's formalism narrowly, more as a proof-theoretical gap in his system than as a problem with its fundamental metaphysics. He also offers a solution which embraces a key formalist attitude: that we define the contents of arithmetical expressions by our own authority, and are free to do so in whatever way suits our scientific purposes. My suggestion will therefore be that the concept-object distinction, which Frege introduces expressly in order to set his view apart from formalism, should be read in this light: it is a distinction between content made available by postulation, and a kind which can only be made available via a proof from prior postulates. Since there is no room for the latter in Hankel's system, that is where the difference lies.

4.1. The problem for formalism: §§92–99

In the first phase of the argument, Frege offers a variety of criticisms of Hankel, some deeper than others. The most important criticism in this discussion, developed in §§94–96, concerns the issue of mathematical existence. Frege argues that the main problem with formalism is that it illegitimately postulates the existence of numbers, which should instead be proven. Here is how he states his conclusion:

This is the error that infects the formal theory of fractions and of negative numbers. It is made a postulate that the familiar rules of calculation shall still hold, where possible, for the newly-introduced numbers, and from this their general properties and relations are deduced. If no contradiction is anywhere encountered, the introduction of the new numbers is held to be justified, as though it were impossible for a contradiction to be lurking somewhere nevertheless, and as though freedom from contradiction amounted straight away to existence (Frege 1884 [1953], §96).

There are two parts to this criticism, which are intertwined throughout Frege's discussion. First, Frege is arguing that existence questions arise in ordinary mathematical contexts, and that they need to be answered by proof, not by postulation. Second, Frege is arguing that such existence questions cannot be answered merely by pointing to the consistency of a concept.

Frege's starting point for this criticism is Hankel's remark about the question of existence (already quoted above, with more context):

If however the numbers under consideration are logically possible, their concept clear and determinately defined for us and thus without contradiction, that question [i.e., of their existence] can only come to this: whether there is in the domain of the real or of the actual in intuition, of the presented (*des Actuellen*), a substrate for them (Hankel 1867, 6–7).

Frege reads Hankel as saying that the consistency of a concept we define, say that of natural number or integer, suffices for the

existence of such numbers. This isn't entirely fair: as explained above, what Hankel is actually saying here is that the question of existence applies to presented numbers, but not to formal numbers. Because the formal numbers are supposed to be purely conceptual, Hankel's view is that the only question we can sensibly ask of them is whether their definition is consistent.

Still, this points to the difference between the two authors: whereas Hankel sees no room for a question about the existence of formal numbers, Frege does. Frege first argues that there is room for such a question using an example from Euclid's geometry. He considers a proof in which Euclid constructs a sub-segment AD on a line AC equal to another segment AB . Frege remarks:

The proof would collapse, if there were no such point as D , and it is not enough that we discover no contradiction in the concept "point on AC whose distance from A is equal to B 's". Euclid proceeds to join BD . That there exists such a line is still another proposition on which the proof depends (Frege 1884 [1953], §94).

We can already see both parts of Frege's criticism emerging here. The validity of Euclid's proof depends on the existence of D . Thus, an existence question arises: until we have demonstrated that there *is* such a point, the proof contains a gap. Frege is further pointing out that even if we proved that no contradiction follows from *supposing* that there is such a point, this would not suffice as a proof that it does exist; for it might be that no contradiction follows from supposing that there is no such point, either. The proof thus depends on an existential presupposition which has yet to be discharged. For the proof to go through, we must not only show that a certain concept is consistent, but that there is an object falling under it.

In §95, Frege extends these points to Hankel's formal numbers, arguing that there are similar existential presuppositions underlying our use of arithmetical symbols, and that the formalist does not prove those presuppositions. He complains that

Hankel treats “ $(2 - 3)$ ” as an empty symbol, and that using it as such is “a mistake in logic. . . it is not the symbol. . . that solves the problem, but its content” (Frege 1884 [1953], §95). As explained above, on a charitable reading, Hankel *does* in fact assign a content to the symbol—a conceptual content. In this case, it is a concept like “ $x + 3 = 2$ ”, since on Hankel’s view, the formal number is introduced as whatever solves this equation (and related ones). But for Frege, what is at issue here is whether there is an *object* falling under this concept. He is saying that we presuppose there is an object falling under this concept whenever we use “ $(2 - 3)$ ” as a proper name in the context of ordinary proofs or calculations. The formalist, like the geometer in the first example, owes us a way of demonstrating that there is something falling under this concept which can serve as the content of the sign.

This criticism implicitly relies on Frege’s understanding of complex terms or definite descriptions. The two examples have in common that the object in question (the point D , the number -1) is referred to by means of a concept under which it falls, that is, by means of a description of the form “the F ”. Frege gives his criteria for when this is legitimate in a footnote to §74: “If, however, we wished to use this concept for defining an object falling under it, it would, of course, be necessary first to show two things: 1. that some object falls under this concept; 2. that only one object falls under it”. He also points out there that these criteria are independent of whether the concept contains a contradiction. Thus, in Frege’s view, using a complex term “the F ” to refer to an object always requires a corresponding proof that there is at least one F .

So Frege is arguing that existence questions arise in many ordinary mathematical contexts, whenever we pick out objects via concepts under which they fall, and that the consistency of such concepts does not settle these existence questions. The upshot of this criticism is that purely conceptual definitions do not suffice on their own to demonstrate the existence of particular objects of

arithmetic—the individual numbers. We can define any concept we like; but even if we can show that this concept is consistent, there is always a further question of whether any numbers fall under it. Frege states this clearly in his final diagnosis of the problem with Hankel’s formalism:

That this mistake is so easily made is due, of course, to the failure to distinguish clearly between objects and concepts. Nothing prevents us from using the concept “square root of -1 ”; but we are not entitled to put the definite article in front of it without more ado and take the expression “the square root of -1 ” as having a sense (Frege 1884 [1953], §97).

Frege’s fundamental principle “never to lose sight of the distinction between concept and object” clearly plays a crucial role in this diagnosis. Indeed, when Frege introduces this distinction, he motivates it by claiming that “from this it follows that a widely-held formal theory. . . is untenable” (Frege 1884 [1953], X). It seems significant that Frege introduces the concept-object distinction in the *Foundations* expressly in order to state the problem he sees with formalism. It is also a novel part of Frege’s view: Hankel has no such distinction. How then should we understand this distinction, and the role it is playing in Frege’s argument?

One might think that Frege is invoking a general metaphysical picture in his criticism, but the details of his argument tell against that interpretation. His criticisms of Hankel repeatedly emphasize the way that existence questions arise in ordinary mathematical contexts like the example from Euclid, and that they need to be answered by proof, rather than postulation.

I suggest, then, that we adopt the following interpretation of the distinction, although I cannot fully defend this interpretation here. Frege’s concept-object distinction aligns with a practical distinction in mathematics, between what we are entitled to lay down or postulate, and what we must prove. Frege agrees with Hankel that we can make a concept (or conceptual content) available to ourselves just through postulation, by laying down a

definition.²⁵ But there is always a further question whether any object falls under that concept. This question must be answered by proof from prior postulates. On this interpretation, Frege's diagnosis that formalism "fails to distinguish clearly between concepts and objects" does not say that formalism has the wrong metaphysics, but rather that formalism fails to meet a required burden of proof.

Similar readings of these passages have also been urged by other interpreters. Merrick (2020), for example, argues that Frege draws the concept-object distinction to block the inference from consistency to existence that he sees in Hankel and a variety of other formalist authors. And Tappenden writes that Frege, in his discussion of Hankel's approach to the complex numbers, "does not hold that this kind of algebraic generalisation should be despised in principle, but rather that the requisite standards of proof had not been met", because we must *prove* the consistency

²⁵An anonymous reviewer asks about the different notions of concepts in Frege, Hankel, and Kant. I must leave a thorough discussion for another place; the most important points here are as follows. Hankel does not really have a theory of concepts, and does not clearly distinguish concepts from objects. Hankel's notion of a concept, insofar as we can extract it from what I said above, is something like: whatever non-intuitive content a term contributes to a judgment. In a singular judgment like "2 is prime", Hankel associates the subject term "2" with a conceptual content, the formal number 2. He would thus perhaps be willing, like Kant, to speak of a "subject concept" in this judgment. Frege by contrast says in §88 that in singular and existential judgments, "there can simply be no question of a subject concept in Kant's sense". For Frege, the content associated with a singular term like "2" is an object and therefore cannot be a concept. So Frege's notion of concept is extensionally narrower than Kant's and Hankel's; he has re-allocated some of the role for concepts in judgments to the category of (non-intuitive) objects. Apart from this, I think there is little in Frege's understanding of concepts in the *Foundations* that Hankel could not in principle agree with. The sharp differences between them concern when and how we demonstrate the existence of objects, not claims about concepts. Although Frege does seem to have developed some of his distinctive views about concepts by the time he wrote *Foundations*—for example, that concepts are functions and essentially "unsaturated" (compare Frege 1882 [1997])—he does not express these views there, and they play no obvious role in his criticisms of Hankel.

of such a system by constructing an instance (Tappenden 1995, 338 and note 59).

Interpreting the distinction this way, as pointing us to a burden of proof that formalism does not meet, also helps make sense of a related point in this first phase of Frege's argument. In §96, Frege argues that the consistency of our postulates or definitions is *also* something that needs to be proven, and cannot simply be assumed. He repeats the point more forcefully in §102:

what we have to do first is to prove that these other postulates of ours do not contain any contradiction. Until we have done that, all rigour, strive for it as we will, is so much moonshine (Frege 1884 [1953], §102).

Rhetorically, the point has much less force against Hankel than the point that consistency does not imply existence.²⁶ So why does Frege emphasize it? Because it tells us something about *how* we can go about proving the existence of something falling under a certain concept—or rather, how we can't. The need to prove consistency means that our postulates governing any system of arithmetic must be made available *in advance*. We cannot, as it were, just add a new postulate whenever we find that an arithmetic proof depends on the existence of something we haven't yet accounted for; for any time we add a new postulate, we need to give a new consistency proof.²⁷ Thus, when we give existence proofs for objects presupposed in other proofs, we

²⁶So far as I can see, Hankel would not disagree that consistency is something that needs to be proven: as we saw above, Hankel also emphasizes the importance of consistency and of having a finite, surveyable set of axioms. He also gives a proof that a set of axioms for a higher-dimensional complex number system is *inconsistent*, which Frege cites approvingly in §94. Frege is right to point out, though, that Hankel doesn't actually give the consistency proof that would be required for the systems of formal numbers that he develops.

²⁷I don't mean to attribute an entire meta-theoretical perspective to Frege here. But he clearly does already have the idea that we can give proofs of the consistency of a concept defined by a set of postulates. So we can assume he recognizes that extending the postulates would change the definition and thus require a new consistency proof.

need to do so using axioms that have already been laid out and proven consistent. For example, in the case of the proof from Euclid, we ultimately need to prove the existence of the point D from the axioms of geometry. If those axioms are not sufficient, then the foundational project has to start over, with new axioms that imply the existence of D and a new consistency proof for them.

Frege's central complaint against formalism, then, is that it does not give us the resources to answer the existence questions that arise in ordinary mathematics. Formalism can give us general concepts. But once we recognize the distinction between concepts and objects, we see that there is a gap in the formalist foundational program: it does not give us sufficient means to prove that there are numbers falling under the concepts it defines. On this way of understanding the complaint, it is a fairly narrow one: Frege is saying that Hankel's axioms for arithmetic are not strong enough to prove the existence of individual numbers that we pick out by defining a concept that governs them, like $x + 3 = 2$. Until we have (consistent) axioms from which we can prove $\exists x(x + 3 = 2)$, we have no right to use " $(2 - 3)$ " as a numerical term. If formalism is to serve as an adequate foundation for mathematics, this gap needs to be filled in. Frege will turn to that problem in the second phase of the argument.

One final point is worth mentioning here. There is a noteworthy shift in Frege's formulation of the criticism between the beginning and the end of this first phase of the argument. From §95 on, instead of talking directly about the need to prove the existence of objects falling under concepts, Frege ascends one rung up the semantic ladder, and instead emphasizes the need to demonstrate that for each arithmetical sign, there is a corresponding content or meaning.²⁸ This might seem counterpro-

²⁸As Frege later noted himself, his semantic terminology in these sections is not yet fixed. In §95, he speaks of "content", *Inhalt*. In §98, he uses *Bedeutung*. In §97, the expression Austin translates as "having a sense" is *sinnvoll*, but Frege later said he would prefer *bedeutungsvoll* (Frege 1891 [1997], 150). We

ductive: after all, we saw above that Hankel's formalism is not one on which symbols are contentless or empty. Hankel speaks of both "presented" and "formal" meanings for number signs, and thinks of those meanings as given in intuition and via conceptual definitions, respectively. Frege's point, though, is that neither of these ways of thinking about the meaning of signs will make the right *kind* of content available for individual number terms, and this point can only be made using terminology that clearly distinguishes signs from their contents. Frege's view is that the kind of content associated with such terms, the kind of content he calls *objects* and distinguishes from concepts, can only be made available by giving existence proofs.²⁹ Formalism fails because it does not recognize the need for such proofs.

4.2. The Dilemma about content and Frege's solution: §§100–105

Having argued that we need to prove that there exists one, and only one, meaning for each numerical term, but formalism can't meet that need, Frege turns to the question of how we might meet it. His answer draws on the resources he developed earlier in the book in his own definitions of the concept of number and of individual numbers. His argument here therefore tells us something about how he sees those definitions as improving on formalism.

Over the course of §§100–103, Frege considers the example of the complex unit i , and asks how we can ensure that the symbol " i " has a determinate content. He starts by laying out a view

can thus be fairly confident that the kind of content he has in mind here is what he will later call *Bedeutung* and distinguish from *Sinn*. Since Frege had not yet drawn this distinction, I use "content" and "meaning" in the main text for the general semantic category to which both objects and concepts belong, in keeping with current scholarly practice.

²⁹Note that Frege thinks that the existence of contents for signs is something that admits of proof, as shown for example by his later practice in the *Basic Laws* (Frege 2013, Vol. I §§31–32).

on which we simply *choose* an object like the Moon, or the time interval of one second, to serve as the meaning of “*i*”. He notes that we then have to extend the laws of arithmetic to apply to the object we choose, in such a way that the rules of complex arithmetic turn out to be valid; and he grants for the sake of argument that this can be done.

The problem Frege raises is that such choices import “something foreign” into arithmetic, namely intuition. It thus threatens the analyticity of arithmetic:

Propositions proved by the aid of complex numbers would become a posteriori judgements, or rather, at any rate, synthetic, unless we could find some other sort of proof of them or some other sense for *i*. We must first make the attempt to show that all propositions of arithmetic are analytic (Frege 1884 [1953], §103).

Of course, this is exactly the problem that leads Hankel to distinguish formal from presented numbers: he needs a category of non-intuitive numbers in order to eliminate intuition from the foundations of arithmetic and demonstrate arithmetic to be analytic. Frege shares this goal, and thus shares the need to introduce non-intuitive numbers; but he has just argued that Hankel’s way of introducing them, via conceptual definitions, is inadequate.

Thus Frege—and Hankel, and anyone who shares this goal—is facing a dilemma, which Frege presents at the beginning of §104:

How are complex numbers to be given to us then, and fractions and irrational numbers? If we turn for assistance to intuition, we import something foreign into arithmetic; but if we only define the concept of such a number by giving its characteristics, if we simply require the number to have certain properties, then there is still no guarantee that anything falls under the concept and answers to our requirements, and yet it is precisely on this that proofs must be based (Frege 1884 [1953], §104).

The dilemma is this: on the one hand, in order to be able to prove the existential presuppositions of statements in arithmetic, we

must *have* an object to serve as the content of each individual number term. On the other hand, any *particular* choice of object runs the risk of making arithmetic dependent on intuition. Thus it is unclear how to satisfy the demand that comes out of the first phase of Frege’s argument, that we be able to demonstrate the existence of contents for numerical terms.

The rest of §104 contains Frege’s proposed way out of this dilemma. His proposal is to employ the same strategy for introducing other kinds of numbers as he has already proposed for the natural numbers, starting in §62: we must give definitions that determine the identity conditions for the new kinds of numbers.

In the same way with the definitions of fractions, complex numbers and the rest, everything will in the end come down to the search for a judgement-content which can be transformed into an identity whose sides precisely are the new numbers. In other words, what we must do is fix the sense of a recognition-judgement for the case of these numbers (Frege 1884 [1953], §104).

How, though, is this supposed to solve the dilemma? Why would “fixing the sense of a recognition-judgement” suffice to give us a grasp of the new objects without making that grasp dependent on intuition? After all, defining “*i*” to mean the time interval of one second fixes its identity conditions; but Frege has just complained that this definition makes complex arithmetic dependent on intuition. So how is this supposed to help with the second horn of the dilemma?

To answer this question, we have to see how Frege understands his solution to work for the case of the natural numbers. He refers in this same passage to his definitions of zero and one, and his proof that every natural number is followed by another, as examples of how a proof can be given that there is one, and only one, object corresponding to a certain kind of numerical term. Those definitions are familiar, so I will not review them in detail here. The crucial definition for all these proofs is the explicit definition in §68 of “the number which belongs to the concept *F*”

as the extension of the concept “equal (*gleichzahlig*) to the concept *F*”. With the help of this definition, Frege defines 0 as the number belonging to the concept “not self-identical”, and every other cardinal number as the number which belongs to the concept encompassing all and only its predecessors. These definitions fix the identity conditions for cardinal numbers, because every number is defined as the extension of a certain concept, and two such extensions are equal when they contain exactly the same things.

But famously, the buck stops here: Frege simply *assumes* that there are such extensions, and that they are objects (Frege 1884 [1953], §68 n 1). It is this specific assumption which solves the dilemma: because extensions are objects but not given in intuition, they can provide meanings for “0” and other number terms without threatening the analyticity of arithmetic. In other words, Frege’s solution to the dilemma seems to simply assume the existence of the objects he needs. But isn’t this exactly what he accuses Hankel of? Even if we leave aside the question of whether this solution works, how can Frege see it as improving on the formalist one?

In reply to this difficulty, I would like to suggest the following. The improvement Frege sees is not that his own view *avoids* the need to assume a class of non-intuitive objects that solves the dilemma, in contrast to the formalist view. Rather, the improvement is simply that the assumption is *explicitly formulated* in axioms which allow us to prove the existence of such objects (or will be, in the form of Axiom V, when the full system is laid out in *Basic Laws*).³⁰ So long as those axioms can still be regarded as purely logical, they are compatible with Frege’s goals.

In other words, Frege’s solution to the dilemma is that he more rigorously embraces the axiomatic approach that he shares with

³⁰Indeed, in a brief discussion of Hankel in *Basic Laws*, Frege’s complaint is precisely that Hankel fails to prove the existence of certain objects due to “a failure to formulate the assumptions in the manner of Euclid’s, paying the closest attention to making no use of any other” (Frege 2013, Vol. II §142).

Hankel. As we saw above, Hankel never quite abandons the Kantian idea that demonstrating the existence of something means presenting it in intuition; as a result, he does not acknowledge existence questions about particular formal numbers, and he does not give axioms that would always allow us to prove there exists a formal number for each arithmetical term. Frege thinks that foundational purposes demand such proofs; but he realizes that we can give them simply by defining arithmetical terms to refer to objects whose existence we presuppose. So long as we make those presuppositions explicit in our axioms, there will be no problem giving the required existence proofs; and so long as the presupposed objects are not given via intuition, they are no threat to the analyticity of arithmetic.

Notice that this way of solving the dilemma requires two things: first, that the existence of certain objects can be proven from purely logical principles, and second, that we are free to define arithmetical terms as having these objects as their meanings. Frege’s theory of extensions was his way of fulfilling the first requirement. This was an innovation that he did not share with formalists like Hankel, who saw no need for such proofs, and that Frege evidently felt driven to, perhaps by the arguments we have just looked at. The second point, though, shows that Frege’s solution to the dilemma embraces a key formalist attitude: he thinks the meanings of terms like “*i*” are given to us by means of *definitions*, and we lay down such definitions on our own authority.

Frege in fact makes several remarks in this second phase of the argument in which he affirms this attitude. In a footnote to §100 he writes, for example, that “the meaning of the square root of -1 is not something unalterably fixed before we made these choices, but is decided for the first time by and along with them”. And in §101 he adds that “perhaps it is indeed possible to assign a whole variety of different meanings to $a + bi$, and to sum and product”; what matters is that we can assign *some* definite meanings for them. In §104, Frege cites his definition of

the number 0 as a model for how to do this. When we look back at that definition, we again find him emphasizing our freedom to choose a meaning, and using that freedom for the purpose of demonstrating that arithmetic is analytic:

I could have used for the definition of nought any other concept under which no object falls. But I have made a point of choosing one which can be proved to be such on purely logical grounds (Frege 1884 [1953], §74).

Crucially, Frege never qualifies or objects to these remarks. They are written in his own voice and represent his own view, not a rhetorical foil. Frege's attitude here thus seems to be: it doesn't much matter *which* objects we take as the contents of numerical terms. So long as we can prove that each numerical term has one, and only one, such object as its content, we are free to assign those terms to whatever objects serve our purposes. Of course, choosing the moon, or the second, as the meaning of "1" won't serve the goal of showing arithmetic to be analytic; but the problem there lies in the kind of objects chosen, not in the choosing. And Frege, at least at this point, does not even regard the choice of extensions as essential (Frege 1884 [1953], §107). *Any* choice of meanings will give us the existence proofs we need, so long as the existence of the objects chosen is guaranteed by purely logical axioms.

Patricia Blanchette has recently offered an interpretation which supports this line of thought (2012, Ch. 4). On Blanchette's reading, we can explain Frege's attitude that we are free to define arithmetical terms in different ways by keeping his ultimate goal in mind: he wants to demonstrate that the truths of arithmetic are provable from purely logical principles. To achieve that goal, he needs to provide an analysis of arithmetical statements which makes it possible to derive them from the logical axioms in his formal system. According to Blanchette, Frege's practice shows that this analysis does not need to preserve either the sense of ordinary arithmetical statements or the reference of the singular

terms which appear in them. Instead, the analysis just needs to ensure that an ordinary arithmetical identity, after analysis, has the form of an identity statement which is *provably logically equivalent* to a certain purely logical statement (2012, 94). More precisely, the analysis must allow us to prove each instance of Hume's Principle: it must yield the result that the identity "the number which belongs to the concept F = the number which belongs to the concept G " is provably logically equivalent to the second order statement that says there is a bijection between the objects falling under F and G .

Getting the analysis right thus means demonstrating the proof-theoretic equivalence of certain whole sentences. But as Blanchette points out, this leaves Frege a choice at the subsentential level about how to assign a meaning to terms of the form "the number which belongs to the concept F ": different, extensionally non-equivalent definitions will yield the required equivalence, given Frege's theory of extensions. Indeed, she notes that Frege actually gives slightly different definitions in *Foundations* and in *Basic Laws*, and seems to regard the change as one of mere technical convenience, not a revision of his view about what the "right" definition is (2012, 83). She concludes that for Frege, "what's important to 'get right' in the analysis is the account of the contents of whole sentences, especially of identity-sentences, and not the referents of singular terms" (2012, 98). Because Frege's ultimate goal is to demonstrate a relationship between whole sentences, he sees himself as free to choose any meanings for arithmetical terms which will achieve that goal.

We can now see more clearly both where Frege departs from the view of formalists like Hankel, and what he has in common with them. Frege sees a gap in the formalist's foundational strategy when it comes to proving the existence of particular numbers. In order to fill that gap, Frege thinks we need to be able to prove that each arithmetical term has a definite meaning. His strategy for doing so, though, shares important aspects of the formalist approach: he thinks that we assign meanings to indi-

vidual number terms via *definitions*, that we are *free* to lay down those definitions in different ways, and that what those definitions enable us to prove matters much more than the particular meanings they assign to individual arithmetical expressions.

5. Summing up Frege's Position

On the interpretation I have offered, Frege is making a narrow break with formalism, and with Hankel's formalism in particular, over the issue of the content of arithmetical signs. The content of, say, " $2 - 3$ " is not purely conceptual, because even if we can prove that a concept associated with this sign is consistent, that will not settle whether there is a number which falls under it. So formalism needs to be supplemented with definitions that don't just assign concepts, but objects, to the individual number signs. The important feature of these definitions is that they enable us to prove, by ordinary mathematical means, that each individual number term is associated with exactly one object.

But Frege is not rejecting formalism wholesale. Frege agrees with Hankel that we can make the concepts of arithmetic available just by laying down definitions. He agrees that we have the authority to lay down whatever definitions will serve our scientific purposes. He agrees that these definitions must be consistent if they are to serve foundational purposes, though he thinks Hankel does not go far enough in *proving* the consistency of his definitions. Like Hankel, Frege thinks the important feature of these definitions is that they give us the ability to prove individual arithmetical truths from more general laws, without the aid of intuition. Thus, like Hankel, he sees a careful choice of definitions as the best strategy for showing that arithmetic is analytic.

So when Frege says, at the conclusion of his discussion of formalism, that his own view "could be called formal", he is drawing attention to genuine and important parallels between his program in the *Foundations* and Hankel's program in the

Vorlesungen; his claim that his view is "completely different" from Hankel's is overstated. Frege is more keenly aware than Hankel of the need to prove the existential presuppositions that we make use of in our proofs of arithmetical facts. In the context of a foundational program, that means those presuppositions must in all cases be provable from whatever axioms we lay down in advance. Thus Frege recognizes more clearly than Hankel a need for a kind of content whose existence cannot be postulated, and must instead be demonstrated from purely logical axioms. But apart from this issue—about which it is quite debatable whether Frege has a satisfactory solution—the two authors are in broad agreement.

This result perhaps raises more questions than it answers. I've argued that Frege's view in the *Foundations* is closer to formalism than it appears. But is the reading I've offered compatible with Frege's later criticisms of formalism, for example in the *Basic Laws* and in his engagement with Hilbert? This question turns on another: to what extent can Hankel's formalism serve as representative for these other versions of the view? I have suggested that Hankel's formalism shares the essential features of formalism more broadly: Hankel identifies numbers with "signs", he argues that they are given to us by stipulating definitions of the arithmetical operations, and so on. But the relationships between the individual viewpoints that Frege lumps under the heading of "a widely-held formal theory" should be spelled out in more detail. It will then be possible to see more clearly how Frege's engagement with formalism shaped his own views, in particular about content and its division into objects and concepts. If the above reading is on the right track, we can expect to learn that Frege's perspective on these semantic notions is closely tied to the different roles that proof and postulation play in mathematics. For Frege agrees with formalists like Hankel that arithmetical expressions are linked to their contents via definitions of our own choosing. The differences arise because Frege insists we must *prove* that.

Acknowledgements

In addition to the editors of this special issue, I am grateful to Jim Hutchinson, Daniel James, David Löwenstein, Paolo Mancosu, Tabea Rohr, Georg Schiemer, seminar audiences in Vienna, Jena, and Tübingen, and two anonymous reviewers for their comments, discussion, and support.

Richard Lawrence
Universität Tübingen
richard.lawrence@uni-tuebingen.de

References

- Blanchette, Patricia A., 2012. *Frege's Conception of Logic*. Oxford: Oxford University Press.
- Crowe, Michael J., 1972. "Hankel, Hermann." In *Dictionary of Scientific Biography*, edited by Charles Coulston Gillispie, pp. 95–96. New York, NY: Charles Scribner's Sons.
- Detlefsen, Michael, 2005. "Formalism." In *The Oxford Handbook of Philosophy of Mathematics and Logic*, edited by Stewart Shapiro, pp. 236–317. Oxford: Oxford University Press.
- Ebert, Philip A. and Marcus Rossberg, eds., 2019. *Essays on Frege's Basic Laws of Arithmetic*. Oxford: Oxford University Press.
- Frege, Gottlob, 1882 [1997]. "Letter to Marty, 29 August 1882." In [Frege \(1997\)](#), pp. 79–83.
- , 1884 [1953]. *The Foundations of Arithmetic*, Second revised edition. Oxford: Blackwell.
- , 1891 [1997]. "Letter to Husserl, 24 May 1891." In [Frege \(1997\)](#), pp. 149–50.
- , 1997. *The Frege Reader*, edited by Michael Beaney. Oxford: Blackwell.
- , 2013. *Basic Laws of Arithmetic*, edited by Philip Ebert and Marcus Rossberg. Oxford: Oxford University Press.
- Hankel, Hermann, 1867. *Vorlesungen Über Die Complexen Zahlen Und Ihre Functionen*. Leipzig: Leopold Voss.
- Kant, Immanuel, 1781-87 [1998]. *Critique of Pure Reason*, edited by Paul Guyer and Allen W. Wood. Cambridge: Cambridge University Press.
- Mancosu, Paolo, 2015. "Grundlagen, Section 64: Frege's Discussion of Definitions by Abstraction in Historical Context." *History and Philosophy of Logic* 36: 62–89.
- , 2016. *Abstraction and Infinity*. Oxford: Oxford University Press.
- Merrick, Teri, 2020. *Helmholtz, Cohen, and Frege on Progress and Fidelity: Sinning Against Science and Religion*. Cham, Switzerland: Springer.
- Petsche, Hans-Joachim, 2009. *Hermann Graßmann: Biography*. Basel: Birkhäuser.
- Tappenden, Jamie, 1995. "Extending Knowledge and Fruitful Concepts: Fregean Themes in the Foundations of Mathematics." *Noûs* 29: 427–67.
- , 1997. "Metatheory and Mathematical Practice in Frege." *Philosophical Topics* 25: 213–64.
- , 2005. "The Caesar Problem in Its Historical Context: Mathematical Background." *Dialectica* 59: 237–64.
- , 2008. "A Primer on Ernst Abbe for Frege Readers." *Canadian Journal of Philosophy Supplementary Volume* 34: 31–118.

——, 2019. “Infinitesimals, Magnitudes, and Definition in Frege.” In [Ebert and Rossberg \(2019\)](#), pp. 235–63.

Thomae, Johannes, 1898. *Elementare Theorie Der Analytischen Functionen Einer Complexen Veränderlichen*, Second edition. Halle: Verlag von Louis Nebert.

Youschkevitch, A. P., 1976. “The Concept of Function up to the Middle of the 19th Century.” *Archive for History of Exact Sciences* 16: 37–85.