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Cassirer and Dirac on the Symbolic Method in Quantum Mechanics: A Confluence of Opposites

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Determinismus und Indeterminismus in der modernen Physik (*DI*) is one of Cassirer's least known and studied works, despite his own assessment as "one of his most important achievements" (Gawronsky 1949, 29). A prominent theme locates quantum mechanics as a yet further step of the tendency within physical theory towards the purely functional theory of the concept and functional characterization of objectivity. In this respect *DI* can be considered an "update", like the earlier monograph *Zur Einsteinschen Relativitätstheorie: Erkenntnistheoretische Betrachtungen* (1921), to *Substanzbegriff und Funktionsbegriff* (1910), a seminal work considering only classical and pre-relativistic physics. But *how* does *DI* cohere with the three volumes of *The Philosophy of Symbolic Forms* (1923–29) providing a systematic survey of symbolic meanings in diverse aspects of culture, each with its own mode of "objectification" via self-created signs and images? Cassirer's "phenomenology of cognition" via distinct types of symbolic form locates Dirac's *Principles of Quantum Mechanics* (1930) as an exemplar within *physical theory* of purely symbolic thought, a realm of pure relations and their correlated meanings. In particular, Dirac's characterization of the new notion of "physical state" in quantum mechanics by a "symbolic algebra of observables" severed from particular representations points to a limiting pole of the *Bedeutungsfunktion*, the third and highest mode of symbolic formation.

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1. Introduction

The names of Ernst Cassirer and Paul Adrien Maurice (P. A. M.) Dirac are infrequently conjoined. But at least one commonality is that both remain noticeably underappreciated by contemporary philosophers of science. A recent observer correctly noted that since his death in 1945, Cassirer's reputation "has suffered a precipitous decline, especially in the Anglophone world"(Gordon 2010, xii). Recognized today as a historian of philosophy or more nebulously a "philosopher of culture", philosophers of science hardly entertain the possibility that Cassirer might have something of interest to say. Lack of philosophical attention to Dirac is more difficult to explain.[1] "*Der heilige* P. A. M." (quoting Wolfgang Pauli Jr.) is a giant of 20th century theoretical physics.[2] His role in the creation and relativistic extension of quantum mechanics is well documented by historians of physics. But *prima facie*, and supported by considerable anecdotal evidence, Dirac acquired the reputation of being perhaps the most *unphilosophical* of the great twentieth-century physicists. A case in point is the characterization of philosophical ("Class 1") problems in physics as "really not so important", a conviction that, as long as physics has not reached its final state, lack of progress on the philosophical front is "nothing to be genuinely disturbed about" (Dirac 1963, 243). Just as Richard Feynman (who in many ways emulated Dirac in this regard), the taciturn, literal-minded, and probably autistic creator of "perhaps the most magical equation of physics"[3] professed but negligible interest in philosophy.[4]

What then might Cassirer, regarded in his lifetime as "'Olympian' by appearance . . . by his wide-ranging dimensions of thought, his comprehensive problem-formulations, his cheerful face and kind openness to others, his vitality, elasticity and distinguished aristocratic bearing",[5] possibly have in common with the aloof genius Bohr deemed "the strangest man I ever met" (Farmelo 2009)? A clue lies in Cassirer's *Determinismus und Indeterminismus in der modernen Physik*, the sole detailed philosophical examination of quantum mechanics in Cassirer's *oeuvre*. Toward the end of the book's penultimate chapter a summary statement of the book's principal argument is made: the so-called "causality problem" in quantum mechanics does not implicate radical indeterminism or a surrender of the causal principle if the latter is understood "critically" (i.e., not metaphysically) as a demand for order according to law, i.e., strict functional dependence. Then a *volte-face*. The "essential problems posed by quantum mechanics for epistemology" do not at all concern the concepts of cause and effect but rather "lie at a

[1] Three notable recent exceptions are Bokulich (2004, 2008), Pashby (2012), and Wright (2016).

[2] See Pauli to Schrödinger, 9 July 1935, in von Meyenn (1985, 421). Almost from the beginning of his career, Dirac's unconventional thinking and behavior both astounded colleagues and led to caricature. Heisenberg relates that in a conversation with Pauli and Dirac one evening after dinner in October 1927 at the Fifth Solvay Conference in Brussels, Pauli responded to Dirac's characterization of religion as a jumble of false statements with the quip that Dirac too, had a religion whose guiding principle is "Es gibt keinen Gott und Dirac ist sein Prophet." See Heisenberg (1969, 122). Translations from the German are mine, unless otherwise noted.

[3] Dirac's relativistic equation of the electron implying the existence of antimatter (positrons); see Wilczek (2002).

[4] From his interview with Thomas Kuhn (1963): "I feel that philosophy will never lead to important discoveries. It's just a way of talking about discoveries which have already been made."

[5] Hamburg (1964, 210) cites a period reminiscence of Dr. Lugwig Englert.

different point", viz., they concern the transformed concept of "physical state" in quantum mechanics. Repeated citations attest to Cassirer's persuasion of this new focus of epistemological inquiry by Dirac's 1930 textbook presentation of quantum mechanics. The penultimate chapter concludes with a summary assessment of the methodological novelty of the new theory; it is a virtual transcription of the corresponding passages in Dirac's book. It will be shown that this surprising confluence is rooted in a mutual appreciation, arrived at by diverse paths, of a novel "symbolic method" of calculation that abstracts away from particular points of view, or particular "representations" corresponding to measured quantities.

To give a broad idea of the significance of Dirac's abstract symbolic approach *within* quantum mechanics, Section 2 traces its origin in Dirac's 1925 reformulation of Heisenberg's first paper on matrix mechanics. Section 3 shows that Dirac's adoption of the "symbolic method" in quantum mechanics is his response to the new abstract (*unanschauliche*) notion of "state" of a physical system that separates quantum mechanics from classical physics. An overview is given of Dirac's "symbolic algebra of states and observables", first introduced in 1930, but presented here in the updated notation of the *bra–ket* formalism that would become nearly ubiquitous in quantum texts. Section 4 reviews the main theme of *DI*—an examination of the so-called "causal problem" in quantum mechanics, turning to Cassirer's *volte-face* at the book's end, indicating Dirac's role in pinpointing the epistemological novelty of quantum mechanics as lying in a transformed notion of physical state. Section 5 finally turns to the place and significance of a pure *Bedeutungsfunktion* within the philosophy of symbolic forms, showing how Dirac's abstract symbolic calculus could be considered a paradigm instance of this mode of objectification, inverting the "striving of ontology . . . to transpose all problems of meaning into problems of pure being" (Cassirer 1929, 106; 1957, 94).

2. The Rise of Quantum Mechanics and of Dirac's Symbolic Method

The novelty of Dirac's "symbolic method" might best be appreciated by considering, as is often done in textbooks, the beginnings of quantum mechanics in terms of the two competing formalisms of matrix mechanics and wave mechanics. To tell the narrative in this way is something of a fiction, as Dirac's 1925 reformulation of Heisenberg's seminal 1925 paper appeared before Schrödinger wrote his famous papers. But doing so gives a clearer appreciation of the significance of Dirac's novel abstract approach to the new theory.

The "old quantum theory" (1900–1925) was not so much a theory but an evolving amorphous collection of computational rules, insights, analogies and approximations, "a groping without correct foundation" as Einstein put it in 1912. The very next year Bohr's model of atomic structure (and its subsequent refinements) provided the platform of an open theory on which to build, some parts of which could be considered well-established while others remained much more tentative. Yet the Bohr atom was an inconsistent mix of classical and non-classical components, an exemplar of the old quantum theory's neither this nor that character. In Bohr's assumption of the existence of "stationary states" (in which the atom does not radiate), electron orbits were described (using Fourier analysis) by classical electrodynamics; however, so-called quantum conditions fixed the allowed stationary states and the frequencies of radiation emitted in transitions between them. By 1920 or so, attempts to fashion a more consistent quantum theory were guided by a methodological stricture that the largely unknown laws of quantum physics should be constructed in some kind of correspondence to those of classical electrodynamics. According to the so-called "correspondence principle", quantum transitions of an atom closely corresponded to harmonic (Fourier) components of the electron's periodic motion considered in the sense of classical mechanics. At least for the case when the principal quantum number n is very large, light emitted in the transition $n \rightarrow n - m$ has the same frequency as the m^{th} higher harmonic of the motion of the electron in the state n, as well as the same intensity and polarization. But the correspondence is valid only for very large values of n.

The first complete formulation of quantum mechanics appeared in early July 1925 in a paper of 24 year-old Werner Heisenberg seeking "to establish a quantum-theoretical mechanics based entirely on relations between quantities that are observable in principle" (Heisenberg 1925). Seeking a mathematical formalism to characterize the phenomenon of spontaneous radiation through a classical analogy that would lead to valid results for small values of n, Heisenberg considered the hydrogen atom as a highly idealized simple periodic system ("virtual oscillator"). The guiding idea was to retain Newton's second law of motion for the atom's electron but in a "kinematical reinterpretation", replacing the time-dependent position coordinate x, classically represented as a sum of Fourier components, with a "quantum theoretic quantity". Pursuing the correspondence between classical and quantum-theoretical quantities through the appropriate terms of Fourier series (where v is the circular frequency), to each Fourier component $X(n, m)\exp\{2\pi i v(n, m)t\}$ for the n^{th} stationary state and m^{th} harmonic in the classical theory, Heisenberg posited a "transition component" $X(n, m)\exp\{2\pi i v(n, n - m)t\}$ associated with the transition $n \rightarrow n - m$. Each of the new quantities is associated with two states since, according to Bohr, this is always true of radiative transitions. Clearly the quantities are still a function of time. However, the primary significance of such quantities lay in calculating the transition components of X^0, $X^1, \ldots$ from those of X just as the Fourier components of X^0, $X^1, \ldots$ can be calculated from $X(n, m)\exp\{2\pi i v(n, m)t\}$ in the classical theory. What matters is that no other frequencies can appear in the transition components of X^0, $X^1, \ldots$ than those already existing in the transition components of X; the fact that transition components are functions of time is of secondary importance. As a result, the quantum theoretical quantity corresponding to the classical coordinate x is simply an *array* of terms of the form $X(n, m)\exp\{2\pi i v(n, n - m)t\}$. The kinematical reinterpretation then replaced the Fourier representation of X by "arrays" of these transition amplitudes, corresponding to the frequencies and intensities of emitted radiation. In this way, information from the observable hydrogen spectrum would replace kinematical variables of position and period for the unobservable electron orbit.

Recognizing that his approach required further mathematical development, Heisenberg stated a methodological intention to restrict consideration to observable quantities, letting these dictate the structure, still unknown, of a new quantum theoretical mechanics.[6] Famously Heisenberg did not realize the multiplication rule required by his arrays is equivalent to that for multiplying matrices, a rule in general non-commutative, i.e., $AB \neq BA$. Max Born, professor of theoretical physics in Göttingen and Heisenberg's postdoctoral supervisor, quickly pointed this out and in late September, together with his assistant Pascual Jordan, had cast Heisenberg's theory into the form of a "matrix mechanics". Born and Jordan, however, sought to construct "an entirely self-contained theory, without the need to invoke assistance from classical theory on the basis of the correspondence principle" (Born and Jordan 1925, 876); thus the matrix quantities constructed for the canonical quantities p and q of

[6] Heisenberg apparently believed to be following Einstein's example (in special relativity) by ridding physics of unobservable quantities (e.g., *absolute simultaneity*). Learning of Heisenberg's intent later on, Einstein is reported to have said, "a good joke shouldn't be repeated too often", allegedly a remark in reply to Phillip Frank who was in agreement with Heisenberg's method (Schaffner 1970, 362).

Hamiltonian mechanics do not represent momentum and position directly yet satisfy equations of motion identical in form to those of classical mechanics. In October matrix mechanics passed its first crucial test when Pauli used it to obtain the observed energy states of hydrogen, and in November appeared the influential "Dreimännerarbeit" of Born, Heisenberg and Jordan, a comprehensive presentation of matrix mechanics and a first attempt to extend the methods of quantum mechanics to systems with many degrees of freedom, i.e., to fields. Taken together, these papers created a new non-commutative theory of atomic physics that is set in the frame of Hamiltonian mechanics.

However, the idea of non-commutation of dynamical variables was certainly non-classical, and initially very difficult to understand physically. Moreover, in matrix form everything appears discontinuous; just as in Bohr's atom theory, there are discrete stationary states with quantum "jumps" between them. Born, Heisenberg and Jordan conceded the abstract matrix representation of relations between observable quantities to be not at all amenable to a "geometrically visualizable [*anschauliche*] interpretation" of an atomic system; indeed, they rejected any description of electron motions in terms of the concepts of space and time (1925, 558). In explicit contrast to the lack of *Anschaulichkeit* in matrix mechanics, Erwin Schrödinger in Zurich built on de Broglie's idea of associating matter particles with waves. From January to June 1926 Schrödinger completed six papers developing a theory of atomic systems in terms of a "wave mechanics", employing a mathematical tool much more familiar to physicists, a wave equation based upon a continuous "psi" function $\Psi = \Psi(x, t)$.[7] Assuming stationary states to correspond to the stationary forms of an associated wave, Schrödinger's wave mechanics gave exactly the same results as matrix mechanics for values of the quantized energy levels of the hydrogen atom. By March 1926 Schrödinger claimed to demonstrate that the new wave mechanics and matrix mechanics are "completely equivalent from the mathematical point of view" (more exact derivations were given by Dirac, Pauli, and somewhat later, von Neumann). But even in the paper claiming mathematical equivalence, Schrödinger argued that wave mechanics, unlike matrix mechanics, furnished a "guiding physical point of view" since . . .

> . . . to me it seems extraordinarily difficult to tackle problems . . . as long as we feel obliged on epistemological grounds to repress intuition [*Anschauung*] in atomic dynamics, and to operate only with such abstract ideas as transition probabilities, energy levels, etc. (Schrödinger 1926, 57, 59)

Schrödinger accordingly claimed wave mechanics could provide an "intuitive" (*anschauliche*) understanding of the emitted frequencies of atomic radiation as "beats", analogous to the fundamental frequency and harmonics of periodic waveforms. More to the point, he was able to derive accurate predictions for phenomena that had remained beyond the reach of matrix mechanics, including the behavior of the electron in a uniform magnetic or electric field (the Zeeman and Stark effects). Ultimately, however, Schrödinger had to retreat from claims of the visualizable character of wave mechanics. Although it might be plausibly argued that a single particle wave function might represent the particle propagating in ordinary 3-space, this view faced insurmountable difficulties when extended to "the *poly*-electron problem", i.e., to multi-particle systems. In such cases, Ψ is a function defined in a $3N$ dimensional configuration space, where N is the number of particles. Visualizable (intuitive) interpretation seemed possible only for the simplest of atomic systems.

On the other hand, the wave amplitude given by Ψ is a *complex*-valued function of space and time and of course observed

[7]The (time dependent) Schrödinger equation for a particle of mass m is $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$ where ∇ is the Laplacian, V is the potential energy (generally a function of both space and time) and $\hbar$ is Planck's constant h divided by 2π.

quantities are real- (indeed, rational-) valued. How then was the wave function to be physically understood? Schrödinger initially supposed a purely wave interpretation of the Ψ-function, to the exclusion of particles. This attempt came at considerable cost: it required him to deny the existence of discrete energy states and of "jumps" between them, and even to speculate that the concept of "energy" is merely a statistical generalization of the more fundamental wave concept of frequency. The issue of physical interpretation of the wave function was resolved to the satisfaction of most quantum theoreticians in a paper of Born of late June 1926. Born applied Schrödinger's new formalism to collision processes, a study that however persuaded him of the corpuscular-, not wave-, nature of electrons. Born argued what is measured is not Ψ but $|\Psi|^2$, i.e., $\Psi^*\Psi$, where Ψ^* denotes the complex conjugate to Ψ. According to what would become known as the "Born rule", $|\Psi(x,t)|^2$ is a "probability density" for a particle to be located within some small region surrounding point x at time t, while the function $\Psi(x,t)$ itself does not represent something physically real but is a mathematical tool representing a "probability current" propagating in time according to the Schrödinger equation's dynamical evolution. With Born's statistical interpretation of Ψ, wave mechanics could answer the question of its physical interpretation and it quickly proved a more pliable instrument than matrix mechanics. It soon supplanted the latter, whose methods many physicists found obscure as well as difficult to apply to actual physical problems, in particular, to the helium atom, the next simplest atomic system.

Is the electron *really* a wave packet *or* is it a point particle? The 1927 experiments on electron diffraction by crystalline solids by Clifton Davisson and Lester Germer in the USA and independently, George P. Thomson in the UK, confirmed de Broglie's hypothesis of the wave character of matter. Heisenberg, for one, did not concede the superiority of wave mechanics despite the existence of matter waves. Instead, the program of quantum mechanics (by which was meant "matrix mechanics") was viewed as requiring liberation from all "intuitive pictures" (*anschaulichen Bildern).* This called for substitution of simple relations between empirically given quantities in place of the kinematic and mechanical descriptions familiar from classical physics. As seen above, that is just what the matrix formulation was developed to do. An ensuing controversy arose concerning in what respect, if at all, quantum mechanics is or could be a "visualizable" (*anschauliche*) theory. Heisenberg correspondingly sought to redefine "visualizability" (*Anschaulichkeit*), arguing that all visualization reasonably can require of a theory is that in all simple cases the theory allow qualitative consideration of its experimental consequences. To illustrate how matrix mechanics remained "intuitive" in the revised sense of *Anschaulichkeit*, he provided, still in 1927, a simple thought-experiment example of light (ϒ-rays, i.e., light of short wavelength and correspondingly high energy) scattered by an electron (the Compton effect) then observed with a fictional "ϒ-ray microscope". Underscoring an immediate discontinuous change in the electron's state on impact by a light quantum (and, as Bohr pointed out, despite an erroneous account of the microscope's optics) Heisenberg formulated limitations or uncertainties (Δ) on the accuracy of simultaneously measured values of the electron's position and momentum ($\Delta p \Delta q \sim h$), the uncertainty relations that bear his name. In these relations, Heisenberg (1927/1981, 64) claimed a "direct physical interpretation of the equation $\boldsymbol{pq} - \boldsymbol{qp} = -i\hbar$".

Heisenberg's attempt to free quantum mechanics from classical imagery found resonance in Bohr's "complementarity", proclaimed at a conference at Lake Como in northern Italy several months later in September 1927. The philosophy of complementarity was expressly tailored to underwrite the Heisenberg uncertainties and to put to rest the dispute over *Anschaulichkeit.* The essence of quantum mechanics lay in what Bohr termed the "quantum postulate", i.e., in Planck's quantum of action h that, to Bohr, symbolized "an essential discontinuity . . . completely foreign to classical theories." On account of the "indivisibility of the quantum of action", Bohr argued for the impossibility of any sharp distinction between systems exchanging energy in an interaction, in particular, between an apparatus of measurement and the quantum system of interest. Both object system and measuring device are "appreciably disturbed" by observation with the result that an intrinsic ambiguity surrounds attribution of properties (distinct states) to individual systems. Overcoming this ambiguity required "renunciation" of a defining characteristic of classical physical theory, the description of physical phenomena through simultaneous use of both kinematical (*spatial position*, *time*) and dynamical (*momentum*, *energy*) concepts. The complete description of *quantum* phenomena required both to be used but not at the same time; employing a concept from the first group precludes simultaneous application of one from the other group and vice versa.

> The very nature of the quantum theory thus forces us to regard space-time coordination and the claim of causality, the union of which characterizes the classical theories, as complementary but exclusive features of the description, symbolizing the idealization of observation and definition respectively. (Bohr 1928/1981, 89–90)

Whereas Heisenberg argued that experimental conditions limit simultaneous (and unambiguous) use of both kinematic and dynamic concepts in the description of an atomic object, complementarity more broadly enjoined an essential limitation in application of classical concepts to the quantum domain. The separation of space-time representation from the conservation laws and causality meant that visualization in any sense familiar from ordinary perception could only be partial in quantum mechanics. One cannot say, as in classical physics, that simultaneously precise values of "complementary" kinematic and dynamic concepts are instantiated in the object. Firmly established after 1927, complementarity brought an orthodox reconciliation between "particle" and "wave" descriptions: by viewing them as jointly necessary but mutually exclusive, both concepts, though requiring distinct experimental setups for legitimate application, were required to accommodate the full range of description of quantum phenomena.

Enter Dirac

To try to understand how the youthful Dirac first fashioned his abstract approach to quantum mechanics in November 1925, it may be illuminating to briefly leave physics for geometry. Henry Frederick Baker (1866–1956) was Lowndean Professor of Astronomy and Geometry when Dirac "went up to Cambridge" as research student in 1923. Previously Dirac had studied projective geometry in Bristol with Peter Fraser, a mathematician who taught Dirac to appreciate "rigorous mathematics" and "that it was sometimes necessary to have strict logical ideas" (Dirac 1977, 113–14). Though Dirac did not attend Baker's lectures, he regularly went to the Saturday teas at Baker's house where someone, Dirac included, would give a talk on a geometrical subject. Much later, Dirac recalled the intellectual nourishment he received on these occasions:

> These tea parties did very much to stimulate my interest in the beauty of mathematics. The all-important thing there was to strive to express the relationships in a beautiful form, and they [*sic*] were very successful. (Dirac 1977, 116)

Baker was an algebraic geometer who justified the term "astronomy" in his chair's title by occasionally lecturing on periodic orbits or other mathematical topics in astronomy (O'Connor and Robertson 2003). In an obituary, one of his prize pupils, Cambridge geometer and Adams Prize winner William Hodge, wrote of Baker's continual "desire to see the algebraic significance of a geometrical theory", noting that in his early years, Baker became fascinated by the axiomatic approach to projective geometry, paying frequent visits to Felix Klein in Göttingen which "had a great influence on his subsequent work" (Hodge 1956, 51, 54). That influence is palpable in the "Introductory" to the first of the six volumes of Baker's most notable work, *Principles of Geometry* (1922–25), where he stated the philosophical justification for an abstract approach to the science of figures:

> While the view is taken that all geometrical deduction should finally be synthetic, it is also held that to exclude algebraic symbolism would be analogous to preventing a physicist from testing his theories by experiment—and to this the present volume is devoted. . . .
>
> A Science grows up from the desire to bring the results of observation, of the relations of a class of facts which appear to be connected, under as few general propositions as possible. Into these propositions it is generally found necessary, or convenient, when the science has reached a sufficient development, to introduce abstract entities, transcending actual observation, whose existence is only asserted by the postulation of their mutual relations. . . . The usefulness of the science . . . will depend on the agreement of the relations obtained for these latter entities with those which we can observe. It would seem that this process of substituting conceived entities, limited by supposed interrelations, for those which are regarded as objects of experience, belongs to every science. (Baker 1922–25, vol. 1, 1–2)

These views will be familiar as a condensed expression of downstream consequences of adopting the "postulational approach" to geometry pioneered by Pasch and Hilbert. The most important influence is surely the axiomatic analysis of elementary geometry in Hilbert's *Grundlagen der Geometrie*, the famous *Festschrift* publication of 1899. Here the subject matter of geometry is stated to lie in "three distinct systems of things" ("points", "straight lines", "planes") while a complete description of the permissible relationships between these "things" (e.g., "between", "parallel", "congruent", and so on) are given by a system of axiom groups. All fundamental geometrical notions are devoid of whatever informal sense or sensory representation is usually associated with them; they acquire meaning only through the occurrence of their respective terms in the deductive consequences of the chosen groups of axioms. For Hilbert, a geometry is then "a logical schema of concepts" (*ein logisches Fachwerk von Begriffen*) abstracted from "basic facts" (*Grundtatsachen*) presented to intuition; so long as the latter are completely described by the axioms, any collection at all, no matter how disparate it may appear, is considered merely a different instantiation of the axioms. Moreover, whereas in analytic geometry one begins with the introduction of numbers, or coordinates, and in synthetic geometry one appeals to figures presented in intuition, Hilbert stated his axiom groups as algebraic relations between symbols. To Hilbert, the result is "an analysis of intuition" rendering figures, and particular coordinates, in principle disposable even though the *Grundlagen* employed diagrams as an assist to the reader. A further aspect, with which Baker was certainly familiar, is Hilbert's demonstration of the existence of non-Pascalian geometries. Following a section (§29) showing the existence of non-Archimedean geometries corresponding to the system of non-Archimedean numbers, §§31–34 demonstrate the existence of non-Pascalian geometries corresponding to a system of non-Pascalian numbers for which the associative and distributive laws of addition and multiplication hold, but not the commutative law of multiplication.[8]

[8] Such numbers arise via the algebraic equations expressing the other axioms of geometry. These relations are defined over "number systems" (a division ring), so that a point is a triple of "coordinates" (x, y, z) (numbers belonging to the division ring), a plane is a set of triplets, satisfying an equation $ax + by + cz + d = 0$, and a line is an intersection of two planes. Hilbert pointed out that the numbers of the "system" x, a, etc. did not have to be real numbers, nor did they need to satisfy the commutative law of multiplication. For this reason, the coefficients a, b, c had to be written to the left of the coordinates. Where commutation failed, a "non-Pascalian" geometry resulted. On the relevance of Hilbertian axiomatics to Baker, see Darrigol (1992, 292–93).

In Baker's abstract symbolic approach, every algebraic manipulation of the symbols had a precise geometric meaning, with the result that algebraic proofs could be substituted for synthetic geometric proofs. In particular, Dirac may well have been familiar with Baker's algebraic proof of Pappus's theorem (Pascal's theorem is a variant) as well as the examples given in the first volume of the *Principles* "of the fact that a geometrical result obtainable without Pappus' theorem should be representable symbolically without use of the commutative law of multiplication" (Darrigol 1992, 87). In the mid-1920s, non-commutation could be taken in stride in pure geometry and in matrix algebra, and in Cambridge Hamilton's quaternions and their non-commutative multiplication were certainly known but regarded an intellectual novelty, without clear physical application. Yet according to a later reminiscence, noncommutative algebra was "so foreign to all ideas of physicists at that time" that Heisenberg "at first thought there must be something wrong with his theory and tried to correct it" (Dirac 1973, 760).

November 1925

In a paper completed at Cambridge in early November 1925, P. A. M. Dirac set out to reformulate Heisenberg's paper of July in the language of Hamiltonian mechanics. In itself this was not an unusual step to take; as many other theorists at the time, Dirac was familiar with Sommerfeld's introduction of Hamiltonian methods in the study and development of the Bohr atom. The classical equations of motion are

$$\frac{dq_i}{dt} = \dot{q}_i = \frac{\partial H}{\partial p_i}; \qquad \frac{dp_i}{dt} = \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

where the q's and p's are a set of generalized coordinates and their canonically conjugate momenta, and H is the Hamiltonian, a given function of the q's and p's equal to the energy if the time is not explicitly involved. But a straightforward generalization of these equations is blocked because they involve partial differential coefficients that in general have no meaning for the dynamical variables in the quantum theory. As he later related, while puzzling over this fact on a long walk one Sunday in October 1925, he had a dim recollection of seeing something termed a Poisson bracket (PB) in an advanced text of dynamics. He seemed to recall that the equations of motion and other equations of classical dynamics could be written in a form where partial differential coefficients occur only in the PBs. Not remembering the definition of a Poisson bracket (PB) he had to impatiently wait until Monday morning when he could look up the definition in the university library. It is:

$$[x, y] := \sum_{i=1}^{n} \left(\frac{\partial x}{\partial p_i} \frac{\partial y}{\partial q_i} - \frac{\partial x}{\partial q_i} \frac{\partial y}{\partial p_i} \right)$$

with q (position) and p (momentum) the canonical variables for the system in question (and summation over the number of degrees of freedom) while x, y are functions of these giving other dynamical quantities such as energy. Dirac then showed that quantum mechanics can be formulated as Hamiltonian dynamics though the non-commutation of a PB. In particular, it is possible to take over into the quantum theory both the classical equations of motion, as well as any other classical equations expressible in terms of PBs. One assumes that in both classical and quantum physics the PBs for the variables p, q have the same canonically conjugate form:

$$[q_r,\ q_s] = 0, \quad [p_r, p_s] = 0, \quad [q_r, p_s] = \delta_{rs}$$

where δ_{rs} is the so-called *Kronecker* function defined by $\delta_{rs} = 1$ if $r = s$; $\delta_{rs} = 0$ if $r \neq s$. The PB expression $[x, y]$ is then given a meaning in quantum theory when x and y are quantum variables by conjecturing a relation to their Heisenberg product:

We make the fundamental assumption that *the difference between the Heisenberg products of two quantum quantities is equal to* $ih/2\pi$ *times their Poisson bracket expression*. In symbols,

$$(xy - yx) \equiv \frac{ih}{2\pi}[x, y].$$

(Dirac 1925, 648)

The commutation relation $pq - qp = -i\hbar \left(\hbar = \frac{h}{2\pi}\right)$ had been previously found by Born in July but it would become the hallmark of Dirac's "symbolic algebra" of quantum observables. To set up this algebra, Dirac had first to give meaning to the differentiation of the quantum quantities, i.e., to quantities like dx/dv where v is some other quantum variable corresponding to Heisenberg's matrices. He began by rewriting the Heisenberg product in a form that shows the explicit dependence of the quantum quantities on variables for two stationary states, n, m,

$$xy(nm) = \sum_k x(nk)y(km)$$

where

$$xy(nm) \neq yx(nm)$$

He then found

$$\frac{dx}{dv}(nm) = \sum_k [x(nk)a(km) - a(nk)x(km)]$$

the coefficients a representing another quantum variable. The most general form of quantum differentiation could be written

$$\frac{dx}{dv} = xa - ax$$

where the new variable in components is $a(nm)$. Thus the differential of a quantum variable x with respect to another quantum parameter v is expressed as the Heisenberg product of the quantum variables x and a. Moreover, the equations of motion of the classical theory can be written in terms of PBs with H the Hamiltonian of the system as

$$\dot{p}_k = [p_k, H] \text{ and } \dot{q}_k = [q_k, H],$$

Dirac then showed that the equations of motion of quantum theory have the form

$$\dot{x} = [x, H]$$

for any *quantum* variable x. In consequence, the differential equations of classical mechanics could be replaced by algebraic equations with addition and multiplication of quantum variables. Dirac could thus bring the full apparatus of classical Hamiltonian dynamics to bear on quantum mechanics. In this way he was able to show that Heisenberg's results could be obtained through a generalization of the Poisson algebra of canonically commuting variables of classical Hamiltonian mechanics.[9] An abstract correspondence is then established between quantum and classical physics:

The correspondence between the quantum and classical theories lies not so much in the limiting agreement when $h \to 0$ as in the fact that the mathematical operations on the two theories obey in many cases the same laws. (Dirac 1925/1995, 73–74)

Rather than follow the somewhat imprecise relation between quantum and classical physics of Bohr's correspondence principle ($h \to 0$), already in 1925, Dirac identified a deep structural correspondence emphasizing the continuity between the two theories. The connection between classical and quantum was accordingly far more profound in Dirac than in Heisenberg's kinematical transposition of the Newtonian equations of motion. This will prove highly important to Cassirer.

[9]See the concise discussion in Longair (2013, 245–54).

The significance of Dirac's novel abstract approach to the new quantum mechanics should now be clear. Whereas Heisenberg used the correspondence principle to set up an analogy between classical and quantum mechanical representations, Dirac related the two theories through a *structural correspondence* of the algebras of their respective abstract Hamiltonian formulation in terms of PBs. He could thus show that quantum theory primarily differed from the laws of classical electrodynamics in just one respect, a slight modification of the algebra of PBs. However, this meant that the relation between the two theories could be demonstrated at a purely abstract level, where the fundamental equations of the new quantum theory are formulated in a manner independent of any reference frame. Moreover, by embedding classical mechanics into a generalized quantum Hamiltonian dynamics, Dirac introduced his own method of quantizing a classical theory. The idea is to construct a suitable quantum analog to the Poisson bracket relations for classical dynamical variables.

Dirac's more abstract approach, formulated in 1926 in terms of distinct algebras of "q-numbers" and "c-numbers", enabled him in 1927 to show that matrix mechanics and wave mechanics, suitably recast as schemes of matrices, could be connected to one another via "transformation theory".[10] Here Dirac showed how to make coordinate-free calculations in terms of "q-numbers" and then to interpret the results via "c-numbers" that could be matched with experimental measurements. From abstract heights of transformation theory the two methods differed only in their choice of dynamical variables; matrix mechanics favored a matrix scheme in which the Hamiltonian or energy function is represented by a diagonal matrix, wave mechanics preferred a matrix scheme in which the configuration variables $q_1, q_2, \ldots, q_n$ are the chosen quantities.[11]

3. Symbolic Algebra of States and Observables

Dirac took no apparent interest in, and so did not participate in, the polemics among the founders of quantum mechanics regarding the respective superiority of matrix mechanics or wave mechanics, a controversy issuing, as seen above, in Heisenberg's 1927 uncertainty relations paper. The idea of writing a textbook on quantum mechanics was not his (Kragh 2013). Rather a representative of Oxford University Press approached him to publish such a work, to appear as the first volume of a monograph series still in existence some 90 years on. The fact that the new series was to be edited by Cambridge physicists well-known to Dirac, R. H. Fowler and P. Kapitza, surely played a role in his agreeing to undertake the task. He began writing in 1928; the book was published towards the end of summer 1930. It may only be a coincidence that Dirac's title echoes that of Baker's volumes on geometry, but the "principles" involved naturally are different. For Dirac, these are above all the "Principle of Superposition of States" and the "Principle of Indeterminacy", discussion of which is taken up already in §3. Many translations followed and further editions appeared in 1935, 1947, and 1958; the latter remains in print. But even beyond its longevity, the book's influence is far more significant. Particularly after the widely adopted "*bra*" and "*ket*" notation introduced in the third (1947) edition, Dirac's *Principles* effectively established a nearly universal symbolic language for physicists writing on quantum mechanics.

After an initial chapter on the quantum principles of superposition and indeterminacy, the work is divided into two parts, the

[10]See Dirac (1926/1995, 223): "*The eigenfunctions of Schrödinger's equation are just the transformation functions . . . that enable one to transform from the (q) scheme of matrix representation* [i.e., wave mechanics] *to a scheme in which the Hamiltonian is a diagonal matrix* [i.e., matrix mechanics]."

[11]For a mathematically clear reconstruction of the details, see Zalamea (2016, chap. 1).

first developing the general theory of quantum mechanics in abstract algebraic form, entirely without regard to applications, the topic of the second part. This mode of procedure distinguished the book from all other period presentations; presumably Einstein (1931, 73) was not alone in judging the work "the most perfect exposition, logically, of [quantum mechanics]." Vindicating Buffon's maxim, *le style est l'homme même*, Dirac's laconic, sparse but pristinely clear syntax, as well as the use of the abstract symbolic method he pioneered, sets the book apart from nearly every other scientific text even today. In a 2003 colloquium at Cornell celebrating the centenary of Dirac's birth the previous August, particle physicist Kurt Gottfried (2003, 1) affirmed that Dirac's *Principles* "belongs to the great literature of the 20th century", even stating "it reminds me of Kafka".

The "Preface", dated 29 May 1930, laid out a philosophy in pursuing and developing an abstract symbolic approach to quantum mechanics. In classical physics "one could form a mental picture in space and time of the whole scheme" but in quantum physics it has been become clear that "nature works on a different plan". Quantum laws are fundamental, whereas classical laws are not; however:

> [Nature's] fundamental laws do not govern the world as it appears in our mental picture in any very direct way, but instead . . . control a substratum of which we cannot form a mental picture without introducing irrelevancies. (Dirac 1930, v)

Mental pictures, the inconclusive debate over *Anschaulichkeit*, unsuccessful attempts to subordinate "particle" concepts to those of "waves" or vice versa, all brought in their train "irrelevancies" that pertain not to nature but to the limits of human imagination or to the representational conventions of classical physics. Though superficially similar to Bohr's Como declaration (1928/1981, 88) of the need for "a renunciation as regards the causal space-time co-ordination of atomic processes", Dirac was not a supporter of complementarity, apparently for several reasons.[12] Indeed, there is considerable evidence that Dirac, at least later on in his career, held out hope for a deterministic theory underlying quantum mechanics, much in the manner of Einstein.[13]

In place of complementarity, Dirac identified two methods used to present the mathematical form of the new theory. The first is the customary one, "the method of coordinates or representations, which deals with sets of numbers corresponding to these quantities." Dirac noted that with one exception, this first method was used in extant presentations of quantum mechanics. Of course, from the standpoint of transformation theory, wave mechanics and matrix mechanics are simply different points of view regarding the same physical phenomena, the point of view depending of a preferred choice of variables. This is in accord with the great methodological lesson taught by relativity theory, that the "growth of the use of transformation theory . . . is the essence of the new method in theoretical physics". From the perspective of transformation theory,

> The important things in the world appear as the invariants (or more generally, the nearly invariants, or quantities with simple transformation properties) of these transformations. (Dirac 1930, v)

As Dirac observed, the new mathematical method was employed previously only in Hermann Weyl's 1928 book *Gruppentheorie und Quantenmechanik.* Weyl's structural characterization of quantum kinematics (for kinematic variables p, q) on the

[12]On the one hand (Bokulich 2004, 386–87), Dirac felt complementarity was "rather indefinite", on the other, as he wrote to Bohr on 9 December 1929, "I am afraid I do not completely agree with your views. . . . I believe that quantum mechanics has its limitations and will ultimately be replaced by something better, . . . I cannot see any reason for thinking that quantum mechanics has already reached the limit of its development."

[13]See Bokulich (2008, 103–14). Bokulich also argues against Kragh's appraisal (1990, 80) that "[b]y and large, Dirac shared the positivist and instrumentalist attitude of the Copenhagen-Göttingen camp".

basis of group theory[14] as well as Dirac's own transformation theory are seen as examples of "the symbolic method, which deals directly in an abstract way with the quantities of fundamental importance" (the invariants or near-invariants of the transformations). While the customary method of coordinates has the advantage of mathematical familiarity, Dirac stated that he will use the symbolic method, as it "seems to go more deeply into the nature of things". This is because the new theory is "built up from physical concepts which cannot be explained in terms of things previously known to the student, which cannot even be explained in words at all". In virtue of its use of abstract symbols, it sidesteps the entire debate over *Anschaulichkeit* to deal directly with the new notion of physical state, cutting through the ambiguities and irrelevancies of particular representations that, in any event, are related by transformation theory. The symbolic method thus does not attempt to portray or represent microphysical processes but considers that "the only object of theoretical physics is to calculate results that can be compared to experience" (1930, 7).

For the first time in Dirac's publications on transformation theory, *Principles* employs Greek symbols ψ, φ, etc. to abstractly represent the so-called quantum state, the symbol standing indifferently for the particular wave functions pertaining to quantities such as position $\psi(x)$ or its Fourier transform in the momentum representation $\hat{\psi}(p)$, etc. As wave functions form a linear space (adding wave functions produces another wave function, as does multiplying a wave function by a complex number) the abstract symbol is known as a "state vector", a vector whose orientation in the state space according to quantum orthodoxy contains all possible information about the state of a quantum system at a given time.[15] The state vector is defined in a linear space with an inner product over the field of complex numbers $\mathbb{C}$ but its nature is not further specified. Presumably, according to remarks on the dust jacket,[16] Dirac almost surely followed the example of Weyl's 1928 book, which introduced the idea that every quantum state could be represented as a vector (of modulus 1) in a "system space". Also in Göttingen at roughly the same time, John von Neumann had taken the now-standard further step of specifying the vector space as an abstract Hilbert space; however, only after von Neumann's book (1932) was this widely adopted. Indeed in his book, von Neumann pointed out that on account of several mathematical "fictions", in particular the so-called δ-function that Dirac admitted was not a classical function, difficulties arose in finding a mathematical justification for some of Dirac's abstract calculations. Nonetheless, as was subsequently shown, the formal machinery of Dirac's abstract symbolic calculus can be rigorously justified in the setting of abstract Hilbert space by combining Hilbert space with the theory of distributions of Laurent Schwartz, giving rise to the notion of a "rigged Hilbert space", introduced by I. Gelfand and A. Vilenkin much later. (See de la Madrid 2005.)

Dirac's symbolic calculus of states and observables became widely, though not universally, adopted following his introduction of the "bra"–"ket" notation in 1939, incorporated into the 3rd (1947) and subsequent editions of *Principles*. Figure 1 shows the "translation table" provided in Dirac's 1939 paper.

[14] For discussion of Weyl's early group theoretic approach, see Scholz (2008).

[15] To be sure, in the first 1930 edition *Principles* employs a relativistic notion of "state": "We must regard the state of a system as referring to its condition throughout an indefinite period of time and not to its condition at a particular time, which would make the state a function of time. Thus a state refers to a region of 4-dimensional space-time and not a region of 3-dimensional space." In the 2nd (1935) and all later editions, Dirac simplifies the presentation using a non-relativistic notion of state.

[16] From the dust jacket of the 1930 edition: "one is rightly tempted to ... survey the existing situation and attempt to put what is already known into more symbolic form. This is the sure way to progress in the *understanding* of the new theory which is perhaps for many workers the *outstanding* need of the present time. One great attempt by Weyl is already well-known. The present book by Dr. Dirac is another, written as it is from an abstract standpoint, but much more physical in outlook than that by Weyl."

The development of the new notation to include linear operators and observables can be effected without difficulty. Below is a list of the various types of quantity involving a linear operator α, written on the left in the old notation and on the right in the new.

$\alpha\psi$	$\alpha\rangle$
$\alpha\psi_a$	$\alpha\|a\rangle$
$\phi\alpha$	$\langle\alpha$
$\phi_a\alpha$	$\langle a\|\alpha$
$\phi_a\alpha\psi$	$\langle a\|\alpha\rangle$ or $\langle a\|\alpha\|\rangle$
$\phi\alpha\psi_a$	$\langle\alpha\|a\rangle$ or $\langle\|\alpha\|a\rangle$
$\phi_a\alpha\psi_b$	$\langle a\|\alpha\|b\rangle$
$\phi(q')\alpha\psi(q'')$ or $(q'\|\alpha\|q'')$	$\langle q'\|\alpha\|q''\rangle$.

(Dirac 1939, 416)

Figure 1

The presentation here will follow this widespread practice, though it should be emphasized that doing so involves only a notational improvement over the first appearance of the symbolic calculus in the first edition of the *Principles*. The quantum states are now described by symbols ψ, φ, α, β etc. enclosed within a "ket" symbol "$|\ \rangle$", i.e., $|\psi\rangle$. *Kets* are vectors in Hilbert space and in accordance with the principle of superposition may be added and multiplied by scalars, generally complex numbers, e.g.,

$$|\psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle$$

where ψ is a superposition of two distinct quantum states ϕ_1 and ϕ_2; the complex coefficients essentially state the relative "weights" of the combining terms in composing the sum. In general, given an orthonormal set of *basis kets* $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$, every *ket* in the space can be expressed in the form

$$|\psi\rangle = c_1|e_1\rangle + c_2|e_2\rangle + \cdots + c_n|e_n\rangle$$

where the c_i are the *expansion coefficients* of $|\psi\rangle$ in the $|e_i\rangle$ basis. Vector addition and scalar multiplication obey the usual algebraic rules (commutativity, associativity):

$$c_1|\phi_1\rangle + c_2|\phi_2\rangle = c_2|\phi_2\rangle + c_1|\phi_1\rangle,$$
$$(c_1|\phi_1\rangle + c_2|\phi_2\rangle) + c_3|\phi_3\rangle = c_1|\phi_1\rangle + (c_2|\phi_2\rangle + c_3|\phi_3\rangle)$$

To see how the notation works, recall that the customary integral for normalized wave functions of the form

$$\int_{-\infty}^{+\infty} \psi^*\psi\, dx = 1$$

states that a wave function extending over space exists with probability one. The term $\psi^*\psi = |\psi|^2$ is an inner product in vector notation. Dirac "invented" a vector called a *bra* designated $\langle\psi|$; it is dual to the *ket* vector $|\psi\rangle$ and so $|\psi\rangle + \langle\psi|$ has no meaning, since the vectors "live" in different spaces. But *bra* notation enables representing the inner product as $\langle\psi|\psi\rangle$. Thus the *bra* represents the complex conjugate ψ^* to the real part of the wave function. Every *ket* has a corresponding *bra* formed by complex conjugation of the coefficients of the *ket*. E.g., a *ket* of the form

$$c_1|\phi_1\rangle + c_2|\phi_2\rangle$$

has a corresponding dual *bra*,

$$c_1|\phi_1\rangle + c_2|\phi_2\rangle \Leftrightarrow c_1^*\langle\phi_1| + c_2^*\langle\phi_2|$$

where c_j is of the form $x + iy$ and c_j^* of the form $x - iy$. As the example shows, *bras* are not really vectors but linear functionals in 1–1 correspondence with *kets* that map *kets* into the field of complex numbers $\mathbb{C}$. More generally, $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^* \neq \langle\beta|\alpha\rangle$ represents the complex number that is the *inner product* of the *ket* $|\alpha\rangle$ and the *ket* $|\beta\rangle$.

Another kind of product can be formed in the *bra–ket* scheme; it proves essential to the fact that observables in quantum mechanics are represented by Hermitian (or self-adjoint) linear operators. This is the so-called *outer product*,

$$|\alpha\rangle\langle\beta|$$

the meaning of which becomes transparent by multiplying the above expression on the right by an arbitrary *ket*,

$$(|\alpha\rangle\langle\beta|)|\gamma\rangle$$

Assuming associativity of multiplication, $|\alpha\rangle\langle\beta|$ can be interpreted as an operator $\hat{O}$ acting *from the left* that maps the *ket* $|\gamma\rangle$ into another *ket* $|\gamma'\rangle$

$$(|\alpha\rangle\langle\beta|)|\gamma\rangle = \hat{O}|\gamma\rangle = |\gamma'\rangle.$$

Outer product is also defined for *bras*,

$$\langle\gamma|(|\alpha\rangle\langle\beta|) = \langle\gamma|\hat{O} = \langle\gamma'|$$

While $|\gamma\rangle\hat{O}$ is not defined in the scheme, $\langle\gamma|\,\hat{O}$ is but in general, the dual to the *ket* $\hat{O}|\gamma\rangle$ is not $\langle\gamma|\hat{O}$ but the *bra* formed with the *Hermitian conjugate* of the operator $\hat{O}$, $\langle\gamma|\hat{O}^\dagger$.[17] Linear operators can be added and multiplied together,

$$(\hat{O}_1 + \hat{O}_2)|\gamma\rangle = \hat{O}_1|\gamma\rangle + \hat{O}_2|\gamma\rangle,$$
$$(\hat{O}_1\hat{O}_2)|\gamma\rangle = \hat{O}_1(\hat{O}_2|\gamma\rangle = \hat{O}_1|\gamma'\rangle = \gamma''\rangle$$

But it is possible that $(\hat{O}_1\hat{O}_2)|\gamma\rangle \neq (\hat{O}_2\hat{O}_1)|\gamma\rangle$, and so in general (there *are* exceptions), multiplication of operators is noncommutative,

$$\hat{O}_1\hat{O}_2 \neq \hat{O}_2\hat{O}_1$$

[17] If $\hat{O} = |\alpha\rangle\langle\beta|$, then $\hat{O}^\dagger = \big|\beta\rangle\langle\alpha\big|$.

It should be stressed that up to this point nothing has been said directly of the physical interpretation of the symbolic calculus; the nature of the symbols is specified only through the algebraic rules that they obey. Only in Chapter IV of *Principles* did Dirac turn to consider *representations* of these abstract symbols, sets of numbers (matrices) with properties completely corresponding to those of the symbols they represent. Dirac's symbolic calculus gives a purely abstract characterization of the formal machinery linking the dynamical equations of quantum mechanics to observation. It is a complete algebraic scheme involving three kinds of quantities, *bra* vectors, *ket* vectors, and (linear) operators. They can be combined in the ways prescribed above. The associative and distributive laws of multiplication hold, but in general the commutative law of multiplication is not valid. Physical interpretation for a given observable (operator) requires specification of a set of *basis* vectors, so that if it is known what an operator does to this set of basis vectors, it can be known what the operator will do to any other vector in the space. Furthermore, nothing has been said about probability or measurement, in accordance with Dirac's view that probability considerations enter into quantum mechanics only in the process of measurement.[18] While Dirac's abstract symbolic method perhaps reveals the influence of H. F. Baker's algebraic treatment of geometry, his distinction between abstract symbols and their representations precisely mirrors that in §22 of Weyl's *Gruppentheorie und Quantenmechanik* (1928) between an *abstract group* and its various representations via linear transformations.[19]

[18] See Dirac (1930, 4): "One may therefore, as has been pointed out by Bohr, ascribe the lack of determinacy in the [measurement] result to the uncertainty in the disturbance with the observation necessarily makes, although one cannot inquire too closely into how it comes about. The apparent failure of causality is from this point of view due to a theoretically necessarily clumsiness in the means of observation."

[19] Cassirer cites Weyl's 1928 book in one of his last lectures, for providing "new confirmation of my general conviction that the concept of group is of

4. *Determinism and Indeterminism in Modern Physics*: Overview

According to Toni Cassirer's memoir (1981, 189), while still in Hamburg, Cassirer began work on *Determinismus und Indeterminismus in der modernen Physik* (*DI*). After the Cassirers left Germany from Hamburg on 2 May 1933, most of the monograph was written in exile at Oxford in 1934 and 1935. It is quite possible that during these Oxford years, Cassirer became acquainted with Dirac's *Principles*, though it had been translated into German in 1931. From Cassirer's letter of 11 September 1936 to the Warburg Institute's Fritz Saxl (see note 19), it appears that Schrödinger, also present in Oxford from October 1933 to summer 1936, read parts of *DI* in manuscript; Schrödinger certainly would appreciate the innovative character of Dirac's book. In any event, *DI* appeared obscurely in the original German in November 1936 in the *Göteborgs Högskolas Årsskrift*, a journal understandably little known outside of Sweden; a *separatum* by *Elanders Boktryckeri Aktiebolag* was published in Göteborg early in 1937. The "Foreword" dated December 1936 states that the manuscript was completed in April at Göteborg where the previous August Cassirer had taken up a University position. Prior to publication Cassirer had solicited reactions of leading atomic physicists, including Schrödinger and Bohr.[20] After the monograph appeared Cassirer sent copies to many leading physicists, including Einstein, Max von Laue, H. A. Kramers, Heisenberg, Schrödinger, and Max Born.[21] In the Preface to the English translation, Yale physicist-philosopher Henry Margenau reported that Cassirer approached him with the idea of an English translation of an updated version just a few months before the latter's sudden death by heart attack (13 April 1945) at the age of 70, the day after the death of President Franklin D. Roosevelt. For reasons described by Margenau the English translation of the original text was considerably delayed, appearing only in 1956.

A close reading of *DI*'s "Foreward" obliquely signals a shift of epistemological attitude. On the one hand Cassirer affirmed continuity with his earlier works of *Erkenntniskritik* and the characteristic Marburg postulate of the *fact* of physical science (*Faktum der Wissenschaft*). The viewpoint of *DI* is largely the one laid out twenty-five years earlier in *Substanzbegriff und Funktionsbegriff*:

> The fundamental viewpoint, in accordance with which I have dealt with these problems ["certain basic questions of the new physics" of quantum mechanics], does not differ essentially from that of my *Substance and Function*. This viewpoint is, I believe, still justifiable. Indeed, I think I can now justify it better and formulate it more precisely on the basis of the development of modern physics than was the case earlier. (Cassirer 2004, 5; 1956, xxxiii)

These remarks lend support to a reading of *DI* as a continuation, and update, of *Substanzbegriff und Funktionsbegriff* (*SF*) and *Zur Einsteins'chen Relativitätstheorie* (*ERT*), extending to quantum mechanics Cassirer's neo-Kantian epistemological analysis identifying a methodological trajectory within physical theory that is a progressive transformation of substance concepts into relational concepts, of "thing concepts" into "concepts of function".[22] Much of the book can and has been read in just this way

universal applicability and extends over the whole field of human knowledge" (1945, 290).

[20] Cassirer's letter to Fritz Saxl, 11 September 1936, states his wish to show the manuscript "once again" (*noch einmal*) to Schrödinger who, however, had left Oxford and was not yet at his new position in Graz (Cassirer 2009, 152). Krois (2011, 9) cites a February 1937 letter to Bohr, thanking Bohr for a discussion in Copenhagen, and a 1936 letter to (Elof) Åkesson in which Cassirer revealed that following this "thorough" (*eingehende*) conversation with Bohr, the book could be published.

[21] See the various letters acknowledging receipt in Cassirer (2009). A letter from Schrödinger dated 9 May 1937, presumably acknowledging receipt, is not in this volume but in the restricted collection of the Max-Planck-Institut für Physik (Werner-Heisenberg-Institut), Munich.

[22] E.g., see Mormann (2015, 32):

for the relational character of the basic concepts of quantum mechanics is a perceptible theme throughout *DI*. On the other hand, Cassirer admonished that epistemology must always be prepared to revise its presuppositions in step with the advance of science, and therefore . . .

> . . . there is certainly a great deal in the earlier investigations that I would not maintain today in the same sense or that I at least would justify differently. (Cassirer 2004, 5–6; 1956, xxiii)

Cassirer had already alluded to continuing controversies over quantum foundations by citing Eddington's jocular remark that a warning to "prying philosophers" might be posted over the entrance gate to the new physics reading, "Structural Alterations in Progress—No Admittance Except on Business". Unlike the previous changes and additions to classical physics brought by the special and general theories of relativity, even the creators of the new quantum mechanics were not in agreement over the interpretation of the theory, nor what it implied regarding the doctrine of causality. Cassirer affirmed that "boundaries" between philosophy and physics should not become "barriers". However, *DI* should be seen not as the attempt of a philosopher to resolve the foundational controversies concerning "questions that are generally agreed to be as yet far from their ultimate solution". Rather, and more modestly, the aim is only that of trying "to prepare the ground for a common inquiry". Just what ground is this? A tendency to focus on *DI*'s continuity with *SF* and *ERT* has led readers to ignore or downplay the extent to which Cassirer now viewed his growing distance from the founders of the Marburg school, Hermann Cohen and Paul Natorp:

> Thus my connection with the founders of the 'Marburg School' is not loosened and my debt of gratitude to them is not diminished when it turns out in the following investigations that I have been led, in the epistemological interpretation of the basic concepts of modern physical science, to essentially different results than those in Cohen's *Logik der reinen Erkenntnis* (1902) and Natorp's *Die logischen Grundlagen der exakten Wissenschaften* (1910). (Cassirer 2004, 6–7; 1956, xxiv)

Cassirer does not explicitly name these different results but a reasonable conjecture is that they have to do with the reorientation of *Erkenntniskritik* by the philosophy of symbolic forms, expanding its scope into an encompassing theory of culture. This will be discussed in Section 5.

DI is principally concerned to establish two related theses, the first more general, the other, quite specific. The broader thesis forms a backdrop, claiming a methodological and architectonic continuity between classical and quantum physics once the causal principle is understood "critically", that is, as the demand for determination according to law. The second and narrower thesis identified the principal epistemological innovation of quantum mechanics not as the failure of the causal concept but as the transformation of the concept of physical state. Both theses are developed in tandem with a knowledgeable presentation of the historical routes to, and development of, quantum theory from Planck's radiation law and discovery of the quantum of action (1900), continuing through the Bohr atom's explanation of the Balmer series of the spectrum of hydrogen in 1913 (the first real success of what would become known as the "old quantum theory"), and culminating in the contributions of Born, Heisenberg, Jordan, Schrödinger and Dirac in 1925–27.

Four of the five parts of *DI* (up to Chapter 12) have the broader aim of demonstrating methodological and architectonic conti-

To put it bluntly, in *DI* Cassirer was engaged in interpreting quantum mechanics in the same neo-Kantian frame that he used more than fifteen years earlier in *ERT* to make philosophical sense of Einstein's relativity theory. Even more, in *DI* he put forward the thesis that quantum mechanics provided a further proof of the relational character of the concepts of modern physics. This entailed that the relational (or functional) *Ansatz* of his philosophy of science—first elaborated in *SF*—remained unaffected by the scientific revolutions of the 20th century.

nuity between classical and quantum physics. Of course the then, and still prevailing, opinion most familiar today from the writings of Thomas Kuhn, is of a complete epistemological "rupture" or "paradigm shift". The overall argument of this broader aim can be summarized in a superficially simple way. It starts (2004, 148; 1956, 122–23) with a critique of Heisenberg's famous statement (in the 1927 uncertainty relations paper) that the "law of causality" in its "sharp formulation" ("when we know the present precisely, we can predict the future") is inapplicable in quantum mechanics because the antecedent cannot be satisfied: it is impossible to know the present with sufficient accuracy (i.e., by violating the uncertainty relations). Cassirer points out, first, that if the law of causality is a material conditional, the statement may still be true when the antecedent is false. More importantly, if the requirement of causality is merely the general requirement of conformity to law, as Helmholtz in particular maintained for classical physics (see below), then the ensuing "crisis of causality" in quantum mechanics is better termed a "crisis of visualization" (*Krise der Anschauung*) for it does not pertain to the concept of cause itself (2004, 196ff.; 1956, 164ff.). Rather the so-called "crisis" stems from the fact that the causal concept cannot be combined with space-time description. Here indeed is a difference with classical physics as well as an apparent incompatibility (at which Cassirer barely hints) with the *Critique of Pure Reason* or at least with the chapter on "the schematism of the concepts of the pure understanding" (A137–47/B176–87) in the Analytic of Principles. There the schema of cause-and-effect of a thing in general is the *a priori* time determination according to the rule that wherever something is posited, something else always follows. However, unlike Bohr who proclaimed the fissure between causal and space-time descriptions to be the distinguishing feature of quantum mechanics, Cassirer traced it back to the foundations of the differential calculus beginning with Leibniz and culminating in Weierstrass's demonstration that continuity (space-time description) and differentiability (the basis of causal description via differential equations) need not coincide. The further development of physical theory by quantum mechanics only exacerbated this fissure, showing that former assumptions of "uniformity" and "equiformity" in nature must be abandoned. But it is important to Cassirer's argument to establish that these trends were already underway within classical physics.

This selective epistemological reading of Kant is informed by critical analysis of conceptual transformations within modern physics, i.e., relativity and quantum theory. In place of focusing on the *a priori* Kantian category of causality and an alleged necessity to structure experience in terms of the relation of cause and effect, Cassirer proceeded from the standpoint of "a scientific clarification of causality" by pointing to the "general laws" of physics, e.g., of gravitation, hydrostatics and hydrodynamics, etc. These are "the proper components of the assumed causal relations". But unlike the concept of cause, all too easily and dogmatically taken to be applicable to "things", the mathematized laws of physics possess their "own symbolic language, which is far removed from the language of 'things'" (2004, 31; 1956, 22). Distancing critical philosophy from the metaphysical trappings inherent in the very concept of cause, Cassirer drew from the Transcendental Dialectic and its treatment of the role of reason in the construction of science, employing a purely methodological notion of causal *principle*. And he showed that within classical physics, Helmholtz in particular, clearly underscored that the operative significance of the causal concept is found within the regulative principle of a "general conformity to law" (2004, 75–78; 1956, 61–64). To motivate this otherwise surprising claim, Cassirer has already argued in the book's first chapter (2004, 9–18; 1956, 3–10) that the causal concept of classical physics is in any case not at all adequately captured by a Laplacian ideal of causal determination that, everyone agreed, utterly fails in quantum physics. Rather, and contrary to the usual assumption, the Laplacian ideal within classical physics itself is largely an inapplicable metaphysical fiction, not an empirically attestable notion. To the extent that it finds limited application there, it is seen to pertain to systems that can be treated as highly idealized point masses or perfectly rigid bodies; however, it is neither required nor implementable in systems that cannot, i.e., in many classical physical theories. Hence the causal concept applicable in both classical and quantum physics is better expressed by the demand for order according to law, in accordance with the regulative principle of unity of knowledge of nature.

Architectonic isomorphism of statements of classical and of quantum physics

The methodological reformulation of the causal principle shifts its significance from things and events to a requirement of adequacy for cognition in natural science. Thus the "general principle of causality" expresses a regulative demand for order according to law. As a proposition, it occupies the innermost core of a structure of physical statements common to both classical and quantum physics (2004, 37–87; 1956, 29–70). For both the structure is "not a pyramid" but a Parmenidean "well-rounded sphere" (2004, 45; 1956, 35) of onion-like concentric layers in which distinct types of physical statement emanate outward. The spherical image, while spatial, is to be interpreted as standing for the functional relations between the different kinds of statements. The general causal principle at the center contains the postulate of "comprehensibility of nature", mandating the formulation and description of observable events in mathematical language (2004, 226; 1956, 188). This is also a requirement to search for more and more general laws governing the phenomena; it does this by regulating dynamical transitions or "jumps" ("μεταβασις εις αλλο γενος") (2004, 71; 1956, 57) between the other layers, whereby statements in one layer are informed by, revised by, or derived from, those in another.

Naturally, the particular statements of classical and quantum physics in their respective layers differ in detail. Outermost in both are statements of measurement and observation; in quantum mechanics these are statistical statements of possible measurement outcomes informed by the Born rule. Working inward, the next layer in classical physics is comprised of particular force laws (Newton, Maxwell-Lorentz), the statistical laws of gas theory (e.g., the Maxwell-Boltzmann distribution law) and the second law of thermodynamics, while in quantum mechanics the corresponding layer contains both particular statistical laws (Planck's radiation law for blackbodies, the laws of radioactive decay for different elements) and the dynamical law (Schrödinger wave equation) for quantum systems. Within each layer of particular laws lies a further layer of meta-laws or physical principles; these are "basically nothing other than means of orientation: means of surveying [*Umschau*] and gaining perspective [*Überschau*]" (2004, 67; 1956, 53). Initially physical principles have only hypothetical validity, either in enabling derivation of particular force laws (in classical physics: the variational principle of Hamilton, the principle of conservation of energy, a prohibition of perpetual motion machines) or by informing the interpretation of other statements (in quantum mechanics: the Heisenberg uncertainty relations, the Born rule, unitarity, the Pauli "Verbot" (exclusion principle), and "the Archimedean point", the Einstein-de Broglie relation $E = h\nu$ of proportionality between energy and frequency). But as these principles inherit the ever-increasing confirmation of the particular statements they inform or govern, they become more and more entrenched, effectively rendering them presuppositions of further inquiry. One is understandably struck by the close similarity to the more familiar Quinean "web of belief": each layer (except the innermost) is revisable from either direction. However, at the center of the whole, the "general principle of causality", understood methodologically as above, remains an *a priori* invariant, not because it is grounded in our mental organization, but as a "postulate of empirical thought":

> What it demands and what it axiomatically presupposes, is only this: that the completion can and must *be sought*, that the phenomena of nature are not such as to elude or to withstand in principle the possibility of being ordered by the indicated process [dynamical structure of physical statements, TR]. (Cassirer 2004, 74; 1956, 60)

Each layer, as well as the transitions between layers, presupposes a methodological understanding of the causal principle. And so at the center of both onion-like structures lies the regulative directive to seek lawfulness (*Gesetzlichkeit*) in the connection of experiences. In this way, the structure of law statements of classical physics can easily accommodate both the dynamical laws of mechanics and electrodynamics and the statistical laws of gas theory according to Maxwell and Boltzmann, whereas that of quantum mechanics permits irreducibly statistical laws such as the Born probabilities for measurement outcomes. According to the purely methodological significance of the causal law, period claims of a "crisis of the causal concept" could be shown to be metaphysical statements hardly consistent with the overtly non-metaphysical philosophy of nearly all quantum physicists.

Yet the principal epistemological departure from classical physics required by the quantum theory is a far more radical change than that resulting from the "critical" reformulation of the causal concept discussed above. As noted above, in Cassirer's perspective the so-called "crisis of the causal concept" has far more to do with a "crisis of visualization", itself rooted in the failure of Schrödinger and others to interpret the wave function Ψ as pertaining to waves or other physical objects propagating in physical space and time. The root of the difficulty lies in the fundamental transformation of the very concept of physical state itself. While the quantum mechanical concept of physical state can be viewed in terms of an already underway transformation of substance-concepts into functional concepts within classical physics (say, from objects in space and time to events in space-time), it cannot be seen merely as a further stage of that process. It is something fundamentally new, and not a phase in the continued development of the classical concept of physical state. After all, thing-attribute logic, criticized already in *Substance and Function*, underlies the notion of physical state in classical physics. That notion presupposes an "axiom" of classical logic that "the state of a thing in a given moment is completely determined in every way and with respect to all possible predicates" (2004, 226–27; 1956, 188–89).

The reader is reminded that Kant's "ideal of pure reason" (A574/B602) rests upon the metaphysical "principle of thoroughgoing determination" (A571/B579) essentially identifying the two concepts of "reality" and "complete determination". Elaborating Cassirer's exposition a bit, recall that the "object" of classical physics is a bearer of determinate properties, and that in classical physics it is taken for granted that properties refer to a definite class of physical quantities, with values lying in specified ranges. The epistemological question concerning knowledge of properties of an object is answered by measurement, a particular type of interaction designed to display the value of a specific physical quantity that, as a property of the object, is intrinsically attached to the object (at a given time) independently of the measurement interaction. ("Perfect" measurements in classical physics are an allowable ideal.)[23] Some properties of an object are not inherent but "change in time" in a *deterministic* manner, i.e., knowledge of the laws and a *sufficiently complete* set of properties of an object at one time t_1 suffices to predict with certainty the values of the properties at any later

[23] The separation of *observer* and *system* has no fundamental significance in classical physics; both are considered parts of a single, objectively existing world, potentially describable by the same laws.

(or earlier) time t_2. Given deterministic laws, failure to predict the behavior of the object with certainty at t_2 is accorded as due to an incomplete knowledge of the object's properties at t_1. In broader compass, the concepts of *physical quantity* and *property* are encompassed in the notion of the space S of *states* of a system of one or more objects, with the understanding that, at a given time, a unique state $s \in S$ can be assigned to the system. A state assignment must satisfy the condition that the specification of a state s at any time suffices to determine the values of *all* physical quantities belonging to the system, and that the state at any time t_2 is determined uniquely by the state at any earlier time t_1.

On account of the principle of superposition as well as due to the limitations on simultaneous assignment of precise values of conjugate observables formulated by the Heisenberg uncertainty relations, the notion of physical state in quantum mechanics has been radically transformed. In orthodox views of quantum mechanics, a quantum object does not possess the form of spatiotemporal connection taken over from macroscopic objects characterizing the classical notion of physical state, of occupying a definite point of space at a definite moment of time. According to the principle of superposition, a quantum object is in general not in a determinate (*eigen-*) state at a given time, unless it has been put into a definite state by a preparation process (e.g., passing through a Stern-Gerlach device) or as an outcome of measurement. The superposition principle also entails that the part-whole relations of quantum systems are completely different from those of classical systems; e.g., knowledge of the separate states of the component quantum systems Ψ_1 and Ψ_2 at time t_1 does not determine the state of the joint system Ψ_{12} that, in general, is a superposition at t_2. The notion of physical state is further constrained by the Heisenberg uncertainty relations setting bounds on what can be simultaneously known about an object's canonically conjugate quantities with obvious implications for what can be predicted about the object's state at a later time. Within quantum mechanics only statistical predictions of outcomes of measurements are possible. In fact the very notion of an "individual" object has been transformed and the different families of quantum objects (fermions, bosons) require the distinct non-classical methods of quantum statistics, of determining "what is to be counted as 'one'" (2004, 225; 1956, 187). All of this naturally demonstrated to Cassirer that quantum mechanics indeed has taken a decisive further step in the functionalization of the concept of a physical state. But this transformation can also be regarded from the standpoint of symbolic forms where its radical nature is given full recognition. From this perspective there occurs a decisive *qualitative* step, a "μεταβασις εις αλλο γενος", the transition to a different mode of symbolism. It is here that the expanded outlook of the philosophy of symbolic forms signals a departure from Marburg *Erkenntniskritik* and it is here that a confluence of Cassirer and Dirac occurs. To see this, it will be necessary to briefly elaborate motivating factors of the philosophy of symbolic forms.

5. Philosophy of Symbolic Forms

In the final chapter of the 1921 monograph on Einstein's theory of relativity, after considering the "methodological meaning" of Minkowski's postulate of an "absolute world" of space-time events, the discussion takes a sudden turn. Cassirer's topic shifts to a distinction between "theoretical scientific knowledge" and "other form- and meaning-connections of independent type and independent lawfulness" (1921/2001, 112; 1953, 446). Whereas *Substanzbegriff und Funktionsbegriff* and the relativity monograph assumed the characteristically narrow focus of *Erkenntniskritik* on exact science, Cassirer now vastly expands its scope. A "truly general *Erkenntniskritik*" will have to investigate, besides theoretical knowledge, other *forms* of understanding of the world, ethical, aesthetic and so on, each incommensurable with any other, and each insufficient in itself to completely grasp and bring to adequate expression "actuality" (*Wirklichkeit*) as such. The study of the transformation in exact science from concepts of substance to those of function, i.e., to relations, and operations on relations, becomes the study of various and complex systems of symbols, the symbolic forms through which a synthetic view of the world is constructed, in myth, religion, language, art and science. Yet the notion of a determinate totality of forms of understanding still exists as a regulative idea, opening the door to the possibility of a systematic philosophy that will grasp and elucidate the "totality of symbolic forms" (*das Ganze der symbolischen Formen*) and their possible interrelations (1921/2001, 113; 1953, 447). The development of such a systematic philosophy of culture became the project that occupied most of the remainder of Cassirer's life.

More closely considered, the mandate of this generalized *Erkenntniskritik* is to investigate the symbolic forms constructed by the processes of mind (*Geist*) across the entire spectrum of culture. The philosophy of symbolic forms seeks to identify the formative patterns by which "the energy of mind" (*Energie des Geistes*) attaches specific meaning-content to signs and symbols in language, the mythical-religious world, the arts, as well as in the human and natural sciences. In every domain of culture, mental constructive activity reveals that "consciousness is not satisfied to receive an impression from outside, but rather that it permeates and connects each impression with a free activity of expression [*mit einer freien Tätigkeit des Ausdrucks*]."[24] The overarching assumption of the philosophy of symbolic forms is that the objective reality of things (as normally considered) will be seen to be a world of meanings, of self-created signs and images. Thus this mode of inquiry takes a recognizable Copernican turn, reversing the traditional "striving of ontology to transpose problems of meaning into problems of pure being".

The first volume of *The Philosophy of Symbolic Forms* (*PSF*), subtitled "Language", appeared in 1923. The "Introduction" restates the problem of the new *Erkenntniskritik* to be that of surveying the special sciences, including the cultural sciences, with the aim of discovering whether the "intellectual symbols" by which specialized disciplines "consider and describe reality" are merely autonomous, existing side by side, or whether they are "diverse manifestations of one and the same basic human function". If the latter is confirmed, the task becomes that of "setting out the general conditions of this symbolic function and illuminating the principle" governing their concrete diversity.[25] It is arguable that Cassirer did not find an adequate solution to these questions until later, and that midway through writing the second and third volumes of *PSF* (the latter completed in 1927, published in 1929), his thought underwent a further "symbolic turn". Quite possibly as the result of his affiliation from 1922 to 1925 with art historian Aby Warburg's *Kulturwissenschaftliche Bibliothek* in Hamburg, Cassirer came to realize that "language could not be taken as the prototype and model for a philosophy of symbolism" (Krois 2011, 11). By at least 1925 Cassirer had

[24] See Cassirer (1923a, 79):

> Unter einer „symbolischen Form" soll jede Energie des Geistes verstanden werden, durch welche ein geistiger Bedeutungsgehalt an ein konkretes sinnliches Zeichen geknüpft und diesem Zeichen innerlich zugeeignet wird. In diesem Sinne tritt uns die Sprache, tritt uns die mythisch-religiöse Welt und die Kunst als je eine besondere symbolische Form entgegen. Denn in ihnen allen prägt sich das Grundphänomen aus, daß unser Bewußtsein sich nicht damit begnügt, den Eindruck des Äußeren zu empfangen, sondern daß es jeden Eindruck mit einer freien Tätigkeit des Ausdrucks verknüpft und durchdringt.

[25] See Cassirer (1923b, 6; compare 1955, 77):

> [Die philosophische Kritik der Erkenntnis] muß die Frage stellen, ob die intellektuellen Symbole, under denen die besonderen Disziplinen die Wirklichkeit betrachten und beschreiben, als ein einfaches Nebeneinander zu denken sind, oder ob sie sich als verschiedene Äußerungen ein und derselben geistigen Grundfunktion verstehen lassen. Und wenn diese letztere Voraussetzung sich bewähren sollte, so entsteht weiter die Aufgabe, die allgemeinen Bedingungen dieser Funktion aufzustellen und das Prinzip, von dem sie beherrscht wird, klarzulegen.

come to regard myth rather than language as the primordial basis of human thought, realizing that the second volume of *PSF* (1925), subtitled "Mythic Thought", should have initiated the series.

There is evidence, however, that the sought-for "principle" relating the diverse forms of understanding, and hence unifying the three published volumes of *PSF*, was not clearly formulated until Cassirer (1927), a paper on the position of the symbol problem in philosophy. Here for the first time Cassirer outlined a kind of general parameter space (*ein allgemeinstes gedankliches Bezugssytem*) intended to encompass all possible symbolic forms. The space is constructed along three orthogonal axes of meaning function, expression (*Ausdrucksfunktion*), representation (*Darstellungsfunktion*), and signification (*Bedeutungsfunktion*). Each particular symbolic form is to be referred to this space with the objective of fully describing and determining its "orientation" (*Orientierung*) (1927, 259). Also for the first time Cassirer described how the third and highest sphere of meaning function, one of "pure meaning" (*reine Bedeutung*), is distinguished from the sphere of representation on the ground of its complete independence of any intuitive shaping (*Gestaltung*), its symbolic force attained "so to speak (by) swimming in the free aether of pure thought". Signs possessing such a meaning function neither express nor stand for anything; rather they are "signs in the sense of mere abstract coordination [*Zuordnung*]"; not surprisingly, Hilbert's axiomatization of geometry is mentioned as the essential paradigm of this mode of meaning function (1927, 261).

The third volume of *Philosophy of Symbolic Forms*, the last to appear in Cassirer's lifetime, is subtitled "Phenomenology of Knowledge"; the intended sense of "phenomenology" corresponds roughly to that employed by Hegel in outlining the different "shapes" of *Geist*. Of course, Cassirer's target is not consciousness *per se* but its manifestations partitioned by the three modes of symbol formation mentioned above. The "shapes" of symbol formation are the corresponding trends of objectification, dynamically ranging from the primary subjective sources of meaning in perceptive consciousness, through intuitive consciousness and the representation of things in space and time, up to objective theoretical-scientific knowledge and the realm of pure meanings in modern axiomatic mathematics. Beginning with expressive meaning, the primitive form of symbolic meaning, each level serves as the precondition for the next higher level. Expressive meaning invests experience with affective or emotional meaning, i.e., desire, fear, wonder, pleasure, and is the meaning function underlying mythic and primitive religious belief. Natural language is the principal vehicle for *Darstellungsfunktion*, the representative function of thought. It is the medium for understanding the everyday world of things in space and time. As well, it gives rise to the logic of propositions and the propositional copula by which properties are attributed to objects, enabling reference to objects outside or beyond the speaker's location and so transmission of information. The final, and highest, form of symbolic meaning is that of *Bedeutungsfunktion* or significative meaning, the subject of the last third of the volume.

The Pythagorean concept of number is viewed as the crucial initial step in the development of the *Bedeutungsfunktion* since it yields a number concept no longer based on any resemblance to physical objects yet allegedly pertaining to true laws governing the world of experience. For the first time a form of knowledge arises firmly separated from perception and intuition. In fact, significative meaning is presupposed in all scientific concept formation but it is exhibited in its purely structural form only within the "pure category of relation" (i.e., logic of relations). Citing Russell's facetious definition of mathematics as "the subject in which we never know what we are talking about, nor whether what we are saying is true" (Russell 1901, 366), Cassirer recognized the *Bedeutungsfunktion* as yielding in the limit a fundamentally distinctive type of symbolic functioning, possessing a specific meaning yet neither requiring nor allowing an intuitive substrate or intuitive object. The illustrious precursor of this ideal limiting form of the *Bedeutungsfunktion* is Leibniz, "one of the most consistent exponents of the rigorously formalist point of view" who recognized that "'intuitive' and 'symbolic' cognition were not separate but indissolubly connected" (1929, 444; 1957, 386). His ideal *Characteristica Universalis* aspired to be a calculus allowing the contentual composition of any concepts to be expressed by purely abstract symbols that are signs for primitive concepts where algebraic "operations with 'signs' replace operations with 'ideas'" (1929, 445; 1957, 386). A mathematically rigorous proof in this calculus would convey the power of conviction by replacing a *succession* of distinct acts of thought (*Denkschritte*) with the pure *simultaneity* of overview (*Überblicks*), the ideal limiting pole of understanding and comprehension. Leibniz's "proof theory" (*Beweistheorie*) provides the ideal of such an achievement, one can be attained only by *symbolic* thinking, an ideal that . . .

> . . . by its very nature . . . does not operate with the thought contents themselves, but associates a definite sign to each thought content and by virtue of this coordination achieves a compression [*Verdichtung*] through which it becomes possible to concentrate all the terms of a complex chain of proof into a single formula, and embrace them in *one* glance as an articulated totality. (Cassirer 1929, 447–48; 1957, 389)

Recent statements of Hilbert resurrect, and moreover provide acute expression of, Leibniz's fundamental idea. Recalling Hilbert's famous statement "*in the beginning . . . is the sign*" (*am Anfang . . . ist das Zeichen*), Cassirer observed that Hilbert's "Beweistheorie", properly so-called, the formalization of logical and mathematical reasoning for the purpose of attaining consistency proofs of mathematical theories, similarly shifts the process of "verification" from specific content to "symbolic" thinking (*Der Prozeß der „Bewährung" ist von der Seite des inhaltlichen Denkens nach der des „symbolischen" Denkens verschoben*) (1929, 437; 1957, 379). Signs are given first in sensuous intuition yet by their form and rules of combination they make possible the clear display of objects, in all their parts, concerning which inferences are made. In this way signs of pure symbolic thinking emancipate thought from "the dangers and ambiguities of mere *reproduction*" (1929, 448; 1957, 389).

The new type of symbolic meaning comes into play when the meaning of a term is bestowed solely by its occurrence in a set of axioms and their deductive consequences. The constructive nature and activity of mind is best exemplified in the formation and use of abstract symbols, defined within an axiomatic system or symbolic calculus, precisely because the abstract relations reveal an objectivity towards which theoretical knowledge progresses and aims but can never be conclusively determined, establishing the physical object only in the form of a "limiting idea" (*Grenzidee*) (1929, 552; 1957, 475). While implicit definition in pure mathematics can be seen as the first instance of the limiting pole of "pure meaning" of the *Bedeutungsfunktion*, Cassirer noted already in 1927 that when the symbols of an abstract-formal doctrine of relations pertain to knowledge of actual things (and not to the ideal objects of mathematics), a new methodological ideal is formed that alters the very meaning of natural scientific cognition.[26] So theoretical knowledge in natural science can also be characterized as manifesting the same tendency to renounce representation and all mediation of intuition, and to construct a realm of meanings no longer bounded by the horizons of sen-

[26]See Cassirer (1927, 15): "[S]ondern wo [einer abstract-formalen Beziehungslehre] auf die *Wirklichkeits*erkenntnis übergreift und diese dem neuen Ideal gemäß bestimmt. Man kann sagen, daß es eben diese methodische Neubestimmung, diese veränderte Grundansicht vom Sinn des Naturerkennens und von den Mitteln, deren es sich zu bedienen hat, gewesen ist, die die Krisis in der modernen mathematischen Physik herbeigeführt hat."

suous experience or intuition. The symbolic spaces of axiomatic mathematics and mathematical (theoretical) physics then can be considered functional spaces of "pure meanings".

Cassirer and Dirac

Within the expanded outlook of the philosophy of symbolic forms, Cassirer allowed in the "Foreword" of *DI* that in its "fundamental tendency", his former position is retained, except that the tendency is now expressed "more in the form of a general question than in particular answers". Further explicit reference to the "general question" as such is absent from the text of *DI*. Yet the "general question" does emerge in Cassirer's admission in the book's penultimate chapter that "the essential problems posed by quantum mechanics for epistemology" concern not "the category of cause and effect" but that of physical state, i.e., "thing and attribute, of substance and accident" (2004, 226; 1956, 188). Recalling the discussion above, the *Darstellungsfunktion* is the mode of symbolic function underlying the notion of "state" in classical physics, the predication of spatial, temporal and other definite properties to an object. In classical physics, as well as in both empiricist and rationalist epistemologies, talk of causal determination, and even of what a "thing" *is*, presuppose that at a given time specified properties or attributes definitely do, or do not, inhere in an object. In the new sense of "state" of quantum mechanics, this kind of *absolute determination* has to be abandoned. With the transformation of the concept of "state" of a system by quantum mechanics, the general question then is "with what justification can we presuppose such a 'state' when knowledge lacks every access to it" (2004, 232; 1956, 193). This new notion of "state" remains a determination by law, but it is a relative determination of being; nothing is "in itself" that is not "for us", i.e., can be expressed as physical knowledge in some sense. By satisfying the specified quantum conditions, the new mode of determination reverses the relation between "thing" and "attribute":

> This type of *determination* dictates limits to the "being" [*Sein*] that we can attribute to things in nature; it is not the in itself determined being [*Sein*] that sets permanent limits to knowledge and remains impenetrable in its absolute intrinsic nature [*absoluten Wesenheit*]. (Cassirer 2004, 232–33; 1956, 194)

Both Dirac and Cassirer appreciated that the more abstract approach to the notion of state had been initiated in the special theory of relativity by the use of Lorentz transformations between inertial frames, and then greatly extended by the requirement of general covariance in general relativity.[27] Just as the *same* intrinsic relations between space-time events are independent of particular coordinate designations, so the canonical transformations of Dirac's quantum mechanics are transformations from one representation of observables to another representation of the *same* abstract state. In referring to the new notion of "state" in quantum mechanics, each recognized the further significance of the "symbolic method". On account of the Heisenberg uncertainty relations and the principle of superposition, the notion of "state" must not and cannot be associated with cognate classical concepts, nor with intuitive content; rather, it is best considered only in terms of abstract symbols. As noted above, Dirac's book *exposited* quantum mechanics from the standpoint of an abstract "symbolic method" on the ground that the theory's concepts "cannot . . . be explained adequately in words at all" (Dirac 1930, v). Strong evidence that Cassirer was impressed by Dirac's use of the symbolic method is found in *DI's* concluding statement

[27] In general relativity, the mathematician naturally uses coordinate-free geometrical methods but the physicist learns the theory by solving problems in a convenient preferred choice of coordinate system (e.g., the spherical coordinates of the Schwarzschild solution), knowing that other coordinate systems, though often impractical, are in principle equally valid. For a clear discussion of the issues here, see Zalaletdinov, Tavakol and Ellis (1996).

of the methodological tendency in quantum mechanics towards abstract symbolism; Cassirer's text is a virtual paraphrase of the corresponding passage in *Principles of Quantum Mechanics*.[28]

At this point it will be apparent how Dirac's textbook presentation of quantum mechanics, in particular, Dirac's emphasis on the novel transformation of the notion of physical state and on the epistemic value of using abstract symbolic methods to express this new state of affairs, resonates with an overarching theme of Cassirer's philosophy of symbolic forms. For this reason, presumably, *DI* praised Dirac for placing the principle of superposition at the center of his exposition of quantum mechanics, thus underscoring the fundamental transformation of the concept of physical state. The new notion shows that quantum physics has abandoned the old mode of absolute determination, connected as it is with the symbolic form of classical physical theory (*Darstellungsfunktion*), and in its place adopted the principle of relative determination, one that manifests the abstract level of symbolic functioning of the *Bedeutungsfunktion*, bounded as it is by the Heisenberg uncertainty relations and dependent on the mode of observation employed. To be sure, light or matter can be "pictured" as a wave or a particle, but neither can be represented as a "thing" in the classical sense, something absolutely determined in itself independently of the instruments of observation. Nonetheless the relative mode of determination of quantum mechanics remains the "highest degree of relative determination of which physical knowledge is capable". With reference to §§3–4 of Dirac's book, Cassirer elaborates:

> For if for the definition of a physical system we allow only such elements of determination as satisfy the conditions expressed in the uncertainty relations, if we are satisfied with "maximal observations", i.e., with the greatest number of independently compatible observations, then we can bring these into a sharply defined relationship with each other. We can then establish the theorem [*Satz*] that when a maximum observation of a physical system is made, its subsequent state is completely determined by the result of this observation—and this theorem can be employed as the axiom to express what we regard as the 'state of a system' in the sense of atomic physics. (Cassirer 2004, 230; 1956, 191–92)

As an ensuing quotation from Dirac (1930, 9) makes clear, the "complete determination" referred to above is in general probabilistic, where in special cases the probability may be unity. In any case, the relative determination of the quantum state by measurement results from the Heisenberg uncertainty relations' placement of epistemic "conditions of 'accessibility' [*Zugänglichkeit*]" on any attribution of physical properties to an

[28]See Cassirer (2004, 234–35; 1956, 195):

> Modern quantum mechanics thus tended more and more to begin by not positing definite realities, which are subsequently brought into relation with each other, but . . . starts out with the establishment of certain symbols expressing the state and the dynamic variables . . . From these, on the basis of definite axiomatic presuppositions, other equations are derived, and physical consequences drawn from them. At first it is not necessary to dwell on the exact significance of the symbols in a particular case. Only at a later stage of consideration are the representations of the abstract symbols examined—in other words things and attributes are examined which satisfy the rules valid for the interrelationship of the symbols.

Compare Dirac (1930, 18):

> We introduce certain symbols which we say denote physical things such as states of a system or dynamical variables. These symbols we shall use in algebraic analysis in accordance with certain axioms which will be laid down. . . . A typical calculation in quantum mechanics will now be run as follows: one is given that a system is in a certain state in which certain dynamical variables have certain values. This information is expressed by equations involving the symbols that denote the state and the dynamical variables. From these equations other equations are then deduced in accordance with the axioms governing the symbols and from the new equations physical conclusions are drawn. One does not anywhere specify the exact nature of the symbols employed, nor is such specification at all necessary. They are used all the time in an abstract way, the algebraic axioms that they satisfy and the connexion between equations involving them and physical conditions being all that is required. The axioms, together with this connexion, contain a number of physical laws, which cannot conveniently be analyzed or even stated in any other way.

object. Abiding the transcendental formula, "the conditions of the possibility of experience are the conditions of the possibility of the object of knowledge", in Cassirer's epistemological examination of quantum mechanics, the "conditions of accessibility" are "conditions of the *objects* of experience" (2004, 214; 1956, 178–79). By formulating "conditions of accessibility" restricting physical knowledge to attestable phenomena rather than metaphysically latent properties, the Heisenberg uncertainty relations acquire a "purely critical meaning" (*rein kritischen Sinn*) in place of the skeptical message they appear to have from the standpoint of the classical concept of physical state (2004, 233; 1956, 194).

6. Conclusion

The philosophies of both Cassirer and Dirac have a common influence; both were strongly influenced by Hilbertian abstract axiomatic pure mathematics. Yet for each, symbolic methods acquire heightened significance in the context of physical theory, encompassing "actual things" of physics rather than ideal objects. From the perspective of the philosophy of symbolic forms, Dirac's *Principles of Quantum Mechanics* exemplifies the *Bedeutungsfunktion*'s fundamental impetus even as symbols pertain not to ideal objects, but to objects of experience. In particular, this impetus is to renounce any concept of meaning tied to sensuous presentation, intuition or spatial-temporal representation, and to create in its stead a new domain of "pure meaning" that "swims in the free aether of pure thought". To Dirac, the "symbolic algebra of states and observables" is particularly suited to quantum mechanics because it treats of matters that "cannot . . . be explained adequately in words at all." For Cassirer, Dirac's symbolic method provides palpable evidence that the problem of objectivity in contemporary physical theory can no longer be viewed in terms of the representation of objects, but itself has become a "pure problem of meaning". What is termed the object of knowledge "is no longer a schematizable, intuitively realizable 'something' with definite spatial and temporal predicates (but) a point of unity . . . a mere '*X*' in relation to which representations have synthetic unity" (Cassirer 1929, 549; 1957, 473). This development has been made possible through the use of abstract symbolic methods, a *terminus ad quem* or limiting case of "pure meaning" that oversteps the bounds and limitations of intuition and representation.

Cassirer's reorientation of *Erkenntniskritik* through the philosophy of symbolic forms sought to bring epistemological investigations of the exact sciences into the vastly broader orbit of an attempt to theoretically grasp how signs, invested with specific meaning through their cultural function, are created, expressed and interrelated. Mathematics and theoretical physics are no longer an exclusive focus; rather they are but aspects in a phenomenology of knowledge, although they remain "the highest and most characteristic attainment of human culture" (Cassirer 1944, 207). The widened perspective does not, and should not, imply that epistemology take no further interest in particular problems within the special sciences. But the parameters of such investigations have been enormously expanded. They are no longer to be considered solely in autonomous isolation, insulated from other domains of culture where other, and culturally earlier, modes of symbolic expression are exclusively found. While not losing sight of the specific scientific details, epistemological investigations in physical science are also to be viewed in relation to these other cultural manifestations of symbolic representation, receiving illumination from, and in turn illuminating, other symbolic forms. It is from this perspective that Cassirer could recognize the traditional "striving of ontology to transpose problems of meaning into problems of pure being" and attempt to reverse it.

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