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Cassirer and the Structural Turn in Modern Geometry

Georg Schiemer

The paper investigates Ernst Cassirer's structuralist account of geometrical knowledge developed in his *Substanzbegriff und Funktionsbegriff* (1910). The aim here is twofold. First, to give a closer study of several developments in projective geometry that form the direct background for Cassirer's philosophical remarks on geometrical concept formation. Specifically, the paper will survey different attempts to justify the *principle of duality* in projective geometry as well as Felix Klein's generalization of the use of geometrical transformations in his Erlangen program. The second aim is to analyze the specific character of Cassirer's geometrical structuralism formulated in 1910 as well as in subsequent writings. As will be argued, his account of modern geometry is best described as a "methodological structuralism", that is, as a view mainly concerned with the role of structural methods in modern mathematical practice.

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1. Introduction

Ernst Cassirer's *Substanzbegriff und Funktionsbegriff* (1910) presents a central contribution to neo-Kantian philosophy of science. The book also contains a detailed and historically informed analysis of several methodological developments in nineteenth-century mathematics. In recent years, increased scholarly attention has been drawn to Cassirer's philosophy of mathematics. A specific focus has been put here on his analysis of Dedekind's foundational work in algebra and arithmetic (see, in particular, Heis 2011, Biagioli 2016, Yap 2017, and Reck and Keller forthcoming). It is argued that Dedekind's influence led Cassirer to formulate an early version of structuralism that is comparable in several respects to modern debates in philosophy of mathematics.

A second line of recent scholarship is concerned with Cassirer's neo-Kantian understanding of modern geometry. This includes the study of Cassirer's philosophical reflections on projective geometry, for instance, on the principle of continuity in work by Poncelet (see, in particular, Heis 2011, 2007). It also concerns Cassirer's detailed discussion of group-theoretic methods in geometry, in particular Felix Klein's *Erlangen Program* (Ihmig 1997, 1999; Biagioli 2016). As was first shown by Ihmig, Klein's study of different geometries in terms of their corresponding transformation groups first outlined in Klein (1893) exercised a significant influence on Cassirer's understanding of modern mathematics and science more generally.

In this paper, we aim to further connect these two lines of research. In particular, the paper will investigate Cassirer's structuralist account of mathematical knowledge developed in response to several methodological developments in nineteenth-century geometry. We want to defend the hypothesis here that complementary to the *axiomatic* structuralism inspired by Dedekind, Cassirer also articulated a version of geometrical structuralism that is directly motivated by the systematic use of transformations and invariants in projective geometry and in Klein's program. The aim here will be twofold. First, to give a closer study of several developments in projective geometry that form the direct background for Cassirer's philosophical remarks on geometrical concept formation in Cassirer (1910) as well as in subsequent writings. More specifically, the paper will survey two related developments: (i) different attempts in work by Poncelet, Chasles, Gergonne, and Pasch to justify the *principle of duality* in projective geometry; (ii) Klein's generalization of the use of transformations in his Erlangen program.

The second aim in the paper will be to analyze the specific character of Cassirer's geometrical structuralism formulated in Cassirer (1910). As will be shown, his account of modern geometry is best described as a "methodological structuralism", that is, as a view mainly concerned with structural methods in modern mathematical practice (see Reck 2003). In particular, in the context of modern geometry, Cassirer identifies two general methods governing how the structural content of a geometrical theory can be specified. This is the focus on *invariants* of geometrical configurations specified relative to certain structure-preserving transformations on the one hand and the use of formal *axiomatic* definitions of geometrical notions on the other hand. The paper will address how these "structural" methods in geometry, expressed most generally in Klein's program and in Hilbert's *Grundlagen der Geometrie* of 1899, are analyzed in Cassirer's book. Concerning modern axiomatics, it will be argued that Cassirer's structuralist understanding of axiomatic geometry clearly reflects the transition from a purely syntactic to a genuinely semantic or model-theoretic approach in Hilbert's work. Moreover, it will be shown how the central notion of "transfer" ("*Übertragung*") of relational content between different geometrical configurations is described by Cassirer in the context of both the axiomatic and the mapping based accounts of geometry.

The article is organized as follows. Section 2 contains a brief survey of some of the relevant geometrical background of Cassirer (1910). The focus will lie here on nineteenth-century projective geometry, specifically on the principle of duality. Section 3 will then give a closer study of Cassirer's philosophical reflections of modern geometry in the book. Subsection 3.1 first discusses his general remarks on the logic of concept formation in mathematics. In Subsection 3.2, we then focus on Cassirer's remarks on two particular developments, namely the use of structure transfers in projective geometry and Klein's group-theoretic approach. Subsection 3.3 discusses Cassirer's account of formal axiomatics and the structural conception of theories implied by this approach. In Section 4 we will then turn to a more general assessment of Cassirer's structuralism concerning geometrical knowledge. Specifically, in Subsections 4.1 and 4.2, we will discuss various points of contact between his position and recent versions of mathematical structuralism. Section 5 gives a brief summary of our findings as well as some suggestions for future research.

2. Geometrical Background

Modern projective geometry plays a central background for Cassirer's work on mathematical concept formation in Cassirer (1910).[1] In particular, as we will see in the next section, a significant part of his book is dedicated to a discussion of several conceptual and methodological developments in nineteenth-century projective geometry that begin with the work of Jean-Victor Poncelet. In the present section, we will give a brief overview of these geometrical developments. In particular, our focus will be on two distinct methods developed in this field. The first one is the systematic use of structure-preserving transformations in projective geometry, in particular, the use of projective transformations in the study of certain general properties of geometrical figures. The second method is the axiomatic specification of projective space first developed in work by Moritz Pasch. In order to illustrate both methods and their genuinely structural character, we will focus here on a particular issue in projective geometry where both methods were fruitfully applied. This is the so-called *principle of duality*, that is, the principle that any theorem in plane and solid projective geometry can be translated into another theorem about figures with a *dual* (or *reciprocal*) structure.

2.1. Projective geometry

Projective geometry became an independent research field in the nineteenth century through work by a number of eminent geometers such as Jean-Victor Poncelet, Joseph Gergonne, Karl von Staudt, Julius Plücker, and Moritz Pasch, among many others.[2] Roughly put, the central idea underlying this branch of geometry was to study those properties of geometrical figures or configurations that are preserved under certain projective transformations. Poncelet (1788–1867) is generally considered to be one of the founders of modern projective geometry. The ideas and methods developed in his monumental book *Traité des propriétés projectives des figures* of 1822 were in several ways formative for the subsequent development of the field.

[1]See, in particular, Heis (2007) for a detailed study of Cassirer's discussion of different developments in nineteenth-century projective geometry.

[2]See Gray (2007) for a detailed historical study of the rise of projective geometry in the nineteenth century. The present section closely follows a discussion of this historical development given in Schiemer and Eder (2018).

Poncelet's approach was motivated by a simple question which also played a central role in Cassirer's discussion of modern geometry: Why does analytic geometry yield results that are more general in character than those of synthetic geometry practiced in the spirit of Euclid? Poncelet's answer to this was that the analytic presentation of geometry and algebra more generally is characterized by the use of variables or "abstract signs". It is this use of abstract signs ranging over *both* real and imaginary magnitudes that makes the analytic or algebraic approach in geometry more fruitful than classical synthetic geometry. Reasoning in classical geometry, Poncelet argued, usually relies on the use of concrete diagrams. As such, it is not possible here to yield geometrical results that extend the properties of concrete geometrical configurations to such imaginary magnitudes. Compare Poncelet on this point:

> One always reasons upon the magnitude themselves which are always real and existing, and one never draws conclusions which do not hold for the objects of sense, whether conceived in imagination or presented to sight. (Quoted from Nagel 1939, 153)

A important case in geometry where such abstract considerations turn out as fruitful is the case of *ideal elements*. The use of such elements in geometrical reasoning goes back to work of Girard Desargues (1591–1661). In particular, in his book *Brouillon projet d'une atteinte aux événements des rencontres du cône avec un plan* (1639), Desargues first introduced so-called *points at infinity*. The general idea here was simple: to every straight line exactly one "point at infinity" can be added. Similarly, to each plane a corresponding "line at infinity" can be added. This ensures that, in contrast to Euclidean geometry, any two straight lines including parallel ones have an intersection point. In the case of parallel lines, the intersection point is simply the point at infinity shared by them. Given this move, classical Euclidean space can be extended by these ideal points and lines.

A central motivation for the introduction of such ideal elements in nineteenth-century projective geometry was the fact that this has a strong unifying effect in geometrical reasoning: points and lines at infinity allow one to state theorems in greater generality. Consider the well-known Desargues theorem to illustrate this fact:

Theorem 1 *For any two triangles ΔABC and $\Delta A'B'C'$: if the lines determined by corresponding points of the triangles meet in a point O, then the corresponding sides of the triangles meet in three distinct points J, K, and I that all lie on a line (see Fig. 1).*

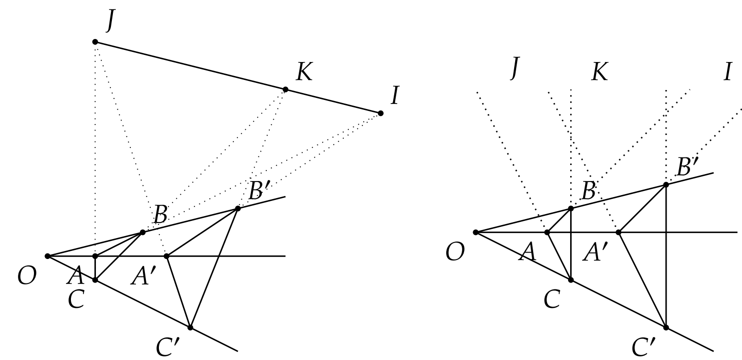

Figure 1: Two variations of Desargues's Theorem.

In Euclidean geometry, this theorem does not generally hold since certain pairs of lines in the configuration might be parallel. In the second diagram of Fig. 1, for instance, the lines AB and $A'B'$ and CB and $C'B'$ are parallel and hence do not have intersection points. However, by stipulating the existence of points and lines at infinity, Desargues's result becomes a theorem in projective geometry. In particular, if parallel lines are taken to "meet at infinity" and the points of infinity are taken as points on a "line at infinity", the diagram on the right-hand side also becomes a valid representation of the general theorem. Thus, the extension of the domains of concrete lines and points by ideal lines and points allows the geometer to get more general results concerning the projective properties of figures. It also allows her to unify proofs by effectively abstracting from a case-to-case reasoning based on particular diagrams. Thus, assuming a projective setting, the two figures in Fig. 1 are in fact representations of the same geometrical fact and Desargues's theorem applies to each one of them.[3]

Poncelet was arguably the first to emphasize the generality gained by the introduction of ideal elements in projective geometry. His attempt to justify the reference to points and lines at infinity is based on what he took to be a general geometrical principle, namely the "*principle of permanence*" or "*continuity*":

> The principle of continuity, considered simply from the point of view of geometry, consists in this, that if we suppose a given figure to change its position by having its points undergo a continuous motion without violating the conditions initially assumed to hold between them, the . . . properties which hold for the first position of the figure still hold in a generalized form for all the derived figures. (Quoted from Nagel 1939, 151)

The principle of continuity expressed here can be understood as a geometrical transfer principle. It roughly states that the projective properties of a given figure are preserved under continuous transformations of the figure. Thus, given two "general" figures and a continuous transformation between them, any incidence relation between the points and lines of the first figure also applies to the second figure. This holds also in cases where the real lines and points of the original figure are mapped to points and lines of infinity in the second configuration. Consider again the diagram on the left-hand side of Fig. 1 illustrating Desargues's theorem: if the intersection points are moved in certain directions, we eventually get the figure on the right-hand side where the corresponding sides of the triangles are parallel. The central assumption expressed in Poncelet's principle is that if one can derive a second figure from a given figure in terms of such a continuous transformation, then "any property of the first figure can be asserted at once for the second figure."[4]

2.2. Principle of duality

A second geometrical principle first discussed in Poncelet's work concerns the issue of projective *duality*. Roughly put, this is the fact that every theorem concerning the projective properties of a figure can be translated into another theorem by *interchanging* the words "point" and "line" in planar geometry and the words "point" and "plane" in solid geometry. Consider Desargues's theorem to illustrate this fact. We saw that the theorem states that if the lines going through corresponding vertices of the two triangles meet in a single point, then the intersection points of the corresponding lines of the triangles all lie on a single line. Desargues's theorem can be shown to be *self-dual*. Thus, it can be demonstrated that the dualized version of the statement is also generally valid in a projective setting. As mentioned above, dualization effectively means a syntactic translation here that is based on the substitution of the term "point" in the formulation of the theorem by the term "line" and of the predicate "lying on a line" by the predicate "meeting in a point". Given this dual translation, it turns out that the converse statement of Desargues's theorem is also a theorem in projective geometry. Thus, if the three points determined by corresponding sides of any two triangles all lie on the same line, then the lines determined by corresponding points of the triangles all meet in the same point.

A second, also well-known example of a pair of dual theorems in projective geometry are the theorems of Pascal and Brianchon. Pascal's theorem is this:

[3]Compare again Schiemer and Eder (2018) for a more detailed discussion of this example. I would like to thank Günther Eder for his permission to use the diagrams in Fig. 1 and Fig. 2 also in the present paper.

[4]A second point relevant in the discussion of the principle of continuity are so-called "imaginary elements", that is, points and lines which are usually represented by complex coordinates in analytic geometry. See again Schiemer and Eder (2018) for a closer discussion of this.

Theorem 2 *Let A, B, C, D, E, F be six points on a conic section that form a hexagon. Then the intersection points of the pairs of opposite sides $\overline{AB}$ and $\overline{DE}$, $\overline{FA}$ and $\overline{CD}$ and $\overline{BC}$ and $\overline{EF}$ of the hexagon will always lie on a line. (See Fig. 2, left diagram.)*

Now consider Brianchon's theorem, first formulated more than a hundred years after Pascal's result:

Theorem 3 *Let a, b, c, d, e, f be six lines that form a hexagon which circumscribe a conic. Then the principal diagonals $a \cap b$ and $e \cap d$, $a \cap f$ and $c \cap d$, and $e \cap f$ and $b \cap c$ of the hexagon meet in a single point. (See Fig. 2, right diagram.)*

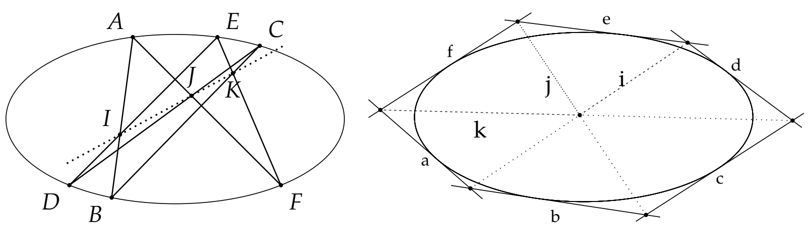

Figure 2: Pascal's and Brianchon's theorem.

A moment's reflection shows that Brianchon's and Pascal's theorems are in fact dual: one can get the one from the other via the method of dualization, that is by substituting the words "point" for "line" (and vice versa) as well as all the concepts defined in terms of them in their formulations.

Poncelet and other geometers in the nineteenth century realized that this and many other examples of dual theorems exist due to a general metatheoretical principle valid in plane and solid projective geometry, namely the *principle of duality*. The practical importance of this principle lies in the fact that it expands knowledge about geometrical figures and their projective properties. In particular, with each proof of a theorem in projective geometry, one immediately gets a proof of another dual statement of a symmetrical nature. A second important feature of duality, also first stressed by Poncelet, is that the principle has drawn attention away from particular geometrical configurations and towards their general *invariant form* or *structure*. Given that theorems about a configuration can always be transformed into new theorems via the method of dualization, it turns out that the primitive geometrical elements (such as points, lines, and planes) turn out to be interchangeable. Thus, similarly to Poncelet's principle of continuity, duality can be viewed as a "structural" method that allows one to abstract from the concrete spatial elements and to focus on the general form of certain configurations.

To see how this general projective form of figures was usually characterized in nineteenth-century geometry, it is instructive to study how the principle of duality was *justified* in the nineteenth century. At least two different approaches were developed at the time, namely the justification (i) in terms of certain "dual" transformations between figures that preserve their projective properties and (ii) in terms of the axiomatic specification of these projective properties.[5] The transformation-based approach is first specified in Poncelet's *Traité*. The principle of duality is introduced here in the context of Poncelet's new theory of poles and polars. Poles and polars are concepts of the geometry of conic sections: polars are lines that can be assigned to given points in the plane, relative to some conic section and based on a uniform method of construction. Poles are points that can be assigned to any line of the plane, again relative to a conic and based on a given construction method. A polar transformation

[5] A third approach to explain duality, first introduced by Julius Plücker, and based on the analytic presentation of geometrical concepts, will not be discussed here. See Gray (2007) and Nagel (1939) for a closer discussion of the analytic approach. Compare Klein's *Vorlesungen über Nicht-Euklidische Geometrie* (1928) and Nagel (1939) for more detailed discussions of these three approaches. See Schiemer and Eder (2018) for a more detailed discussion of the different approaches to duality.

is a transformation or mapping between figures that assigns to each point its polar line and to each line its corresponding pole.[6] The key feature of such a transformation is that it preserves the incidence relations between points and lines: if a point P lies on line l, then the polar line l' of P goes through the pole P' of l. Put in modern terminology, a polar transformation between points and lines preserves the incidence structure of a geometrical configuration.

Given the general theory of poles and polars, Poncelet successfully showed in his book how one can construct new figures from existing ones based on the use of such dual transformations. This method of construction secures that all of the projective properties of the original figure are transformed into "similar" dual properties of the second figure based on the mapping of points to lines and vice versa. In Poncelet's *Traité*, this fact is taken to explain the general validity of the principle of duality in projective geometry. More specifically, the phenomenon that projective theorems can be dualized is viewed as a consequence of the fact that, relative to a given conic section, one can always construct a dual mapping between geometrical configurations in the plane in which their incidence properties are preserved.[7]

The second approach to provide a mathematical justification of the principle of duality is based on the axiomatic presentation of geometry. An early and formative articulation of this view can be found in the work of Joseph Diez Gergonne (1771–1859), in particular, in a number of articles from the 1820s in which the duality of projective theorems is explicitly discussed. Even though Gergonne does not provide a general explanation of duality here, there exists some textual evidence that he intends to justify the principle based on a (proto-)axiomatic presentation of the general laws of projective geometry.[8] In particular, it is first emphasized in these works that duality is ultimately a result of the symmetrical nature of the deductive proofs of dual theorems. Thus, according to Gergonne, the duality phenomena in projective geometry should be understood syntactically, that is, in terms of a "correspondence" between the primitive laws and proofs, and not object-theoretically, in terms of dual mappings between geometrical figures.

A more detailed treatment of the axiomatic approach to duality is present in Moritz Pasch's *Vorlesungen über Neuere Geometrie* (1882). Pasch's book contains the first systematic axiomatization of projective geometry.[9] Moreover, Section 12 of the book also explicitly addresses the issue of *reciprocity* (that is, of duality) as a property of statements about solid projective geometry. Duality is described here again purely syntactically as a translation between theorems that result from the substitution of the primitive constants "point" and "plane" (as well as of the geometrical concepts defined in terms of them). His argument for the general validity of the principle is based on two central premises. The first one concerns the fact that the axiom system presented in the book is symmetrical in the sense that the respective axioms of the system always come in pairs: for every axiom there exists a dual axiom of the same logical form:[10]

- P1. It is always possible to lay a line through two points.
- P2. Every line is determined by any two of its points.
- P3. Every line that contains a proper point is a proper line.
- P4. It is always possible to lay a plane through three points.
- P5. Every plane is determined by three of its points that do not lie in a straight line.

[6] See Coxeter (1974/1987) for a detailed study of polar theory.

[7] This explanation of projective duality in terms of polar transformations was significantly generalized in Michel Chasles' *Aperçu historique sur l'origine et le développement des méthodes en géométrie* (1837). See Nagel (1939) for detailed discussion of Chasles' work on duality.

[8] For a detailed discussion of Gergonne's views on duality and references compare Nagel (1939) and Gray (2007).

[9] See, in particular, Schlimm (2010) for a detailed survey of Pasch's work.

[10] See, in particular, Pasch (1882, §§7–8). The author would like to thank Dirk Schlimm for providing him with translations of Pasch's axioms.

P6. Every plane that contains a proper point is a proper plane.
P7. A line that has two points in common with a plane lies completely in it.
P8. It is always possible to lay a plane through a line and a point.
P9. Every plane is determined, if one knows one of its lines and a point that is outside of the line.
P10. It is always possible to lay a plane through two lines that have a point in common.
P11. Every plane is determined by any two of its lines.
P12. Two lines in a plane have always a point in common.
P13. A line and a plane have always a point in common.
P14. Two planes have always a line in common.
P15. Three planes always have a point or a line in common. (Pasch 1882, §8)

The second premise used in the proof of the duality principle concerns what Pasch calls the "rigorous deductive method" in geometry. This is the claim that all theorems of projective geometry are deductively derivable from the axiom system specified in the book. Thus, no information other than that specified in the axioms is needed to prove the general truths in projective geometry. In particular, Pasch was the first to show that no reference to diagrams is needed in the demonstration of a theorem. Given these two assumptions, duality is "verified" by Pasch as follows:

> The law of reciprocity can be verified first for the graphical sentences of §§7, 8, 9, since the reciprocal sentence of every sentence also belongs to this group. Every other sentence to be considered here is a consequence from these sentences. In its articulation and proof only graphical concepts are used. One can restrict oneself here to the base concepts; the other concepts are deduced from the base concepts or can be given with the help of the relevant definitions. Every theorem is thus the result of a consideration in which only graphical base concepts are mentioned and in which one only refers to the graphical sentences mentioned above. If one substitutes systematically the word 'point' by 'plane', 'plane' by 'point' and the used theorems by its reciprocals in this approach, then its correctness remains untouched; but as a result one finds 'point' and 'plane' interchanged, i.e. one has proved the reciprocal theorem. (Pasch 1882, 96)

This justification of the general principle of duality thus runs as follows: the axioms of projective geometry are symmetrical in the sense specified above. Each theorem provable from this set of axioms contains only the primitive constants (or "base concepts") specified in the axioms or defined constants. It follows from this that the dual translation of every theorem must therefore also be a theorem since it must be provable from the dual axioms of the axioms used in the derivation of the original theorem. Notice that this justification of duality is strictly syntactic in character. Pasch's focus is, as we saw, on the translation between geometrical statements and, most importantly, on the purely *formal* character of deductive geometrical proofs. This formalism is described in a well-known passage in his book:

> In fact, if geometry is genuinely deductive, the process of deducing must be in all respects independent of the sense of the geometrical concepts, just as it must be independent of figures; only the *relations* set out between the geometrical concepts used in the propositions (respectively definitions) concerned ought to be taken into account. (Pasch 1882, 98)

Thus, what is relevant in the justification of the principle of duality is, according to Pasch, the purely relational properties specified by the axioms of projective geometry.

Given this brief overview over certain developments in projective geometry, two points of commentary are in order here. Notice first that all of the newly developed methods outlined here gave rise to a structural conception of geometrical objects. This is evident, in particular, in both principles of projective geometry discussed here, namely the principles of continuity and of duality. As we saw, the duality of theorems effectively shows that the geometrical content of a configuration is in fact restricted to its projective properties, for instance, properties concerning the incidence relations between points and lines in them. Moreover, two particular figures such as the diagrams representing Brianchon's and Pascal's theorems can be treated as structurally equivalent from a projective point of view since they share all relevant properties up to duality. Secondly, our brief survey of the debate on the proper justification of duality shows that two complementary approaches on how to capture this abstract geometrical content or the notion of sameness of such content have been devised in nineteenth-century geometry. This is, on the one hand, the notion of invariance under certain mappings or transformations. On the other hand, structural content is explained in terms of formal axiomatic conditions. In the following section, we will survey how these structural ideas in modern geometry were addressed in Cassirer's work.

3. Cassirer on Modern Geometry

3.1. Concept formation in mathematics

Cassirer's philosophical analysis of modern geometry is based on a more general study of concept formation in the exact sciences. The first chapter of Cassirer (1910) is concerned with the "logic of concept formation", more specifically, with the question of how abstract concepts are introduced in modern mathematics.[11] As Cassirer points out, the new philosophical understanding of concepts outlined in the book, in particular the shift from "*substance* concepts" to "*function* concepts", is directly motivated by modern mathematics. What is characteristic about mathematical concept formation according to him? A central feature that Cassirer is concerned with has to do with the specific relation between abstract concepts and their concrete instances. In the case of geometry, this is the relation between abstract concepts of certain curves or spaces and concrete objects, e.g., particular intuitive figures exemplifying them.

This relation between mathematical concepts and concrete instances is elucidated by Cassirer based on a distinction between two "schemata of concept formation". The first one is the traditional philosophical theory of substance concepts based on an abstractionist account of concept formation. Very roughly, this is the view that general concepts are formed by abstraction from particulars in the sense that specific properties of the instances are neglected except those shared by all individuals. Heis describes this "abstractionism about conceptual formation" in the following way: "concepts are formed by noticing similarities or differences among particulars and abstracting the concept, as the common element, from these similarities or differences" (Heis 2014, 248). Notice that this approach presents a *bottom-up* view of concept formation. Thus, by constructing general concepts from particular instances via abstraction, the concepts turn out to be functionally dependent on their instances. In a sense, knowledge of the instances is presupposed in the construction of general concepts.[12]

Now, Cassirer is critical of the traditional logic of concept formation via abstraction. He contrasts this account with a second, in his view preferable, scheme which is also paradigmatic for modern mathematics. The construction of so-called "functional concepts" is not based on the elimination of particular properties of instances. Instead, the specific nature of the particular objects falling under a functional concept is preserved or implicitly contained in a general concept. Concept formation in modern mathematics is therefore not based on Aristotelian abstraction,

[11] Compare Heis (2014) and Ihmig (1997) for detailed studies of Cassirer's account of mathematical concept formation and his intellectual background, in particular the Marburg school of neo-Kantianism.

[12] See again Heis (2014) for a more detailed survey of Cassirer's presentation of the Aristotelian view.

but rather on a different type of generalization. How does Cassirer understand this new form of conceptual generalization? Let us see how the logic of functional concepts is illustrated by him in the context of analytic geometry, more specifically, in the case of the analytic presentation of second-order curves:

> The genuine [mathematical] concept does not disregard the peculiarities and particularities which it holds under it, but seeks to show the necessity of the occurrence and connection of just these particularities. What it gives is a universal rule for the connection of the particulars themselves. Thus we can proceed from a general mathematical formula, for example, from the formula of a curve of the second order, to the special geometrical forms of the circle, the ellipse, etc., by considering a certain parameter which occurs in them and permitting it to vary through a contiguous series of magnitudes. Here the more universal concept shows itself also the more rich in content; whoever has it can deduce from it all the mathematical relations which concern the special problems, while, on the other hand, he takes these problems not as isolated but as in continuous connection with each other, thus in their deeper systematic connections. The individual case is not excluded from consideration, but is fixed and retained as a perfectly determinate step in a general process of change. (Cassirer 1923, 19–20)

The central idea expressed here concerns the notion of a "rule of variation" between individuals. According to Cassirer, a mathematical concept such as the concept of a second-order curve specifies a general rule or law on how the individual objects—that is, particular curves in a given space—are interrelated. Moreover, this example also makes clear how the method of "deducing" special instances from a general concept is understood here. Concepts such as second-order curves are often represented analytically in modern geometry, that is, by functions expressed in terms of algebraic equations. The link to concrete instances of such concepts is given by the fact that the variables occurring in the corresponding equation can be assigned different numerical values and thus different interpretations in the underlying coordinate system. It is in this sense that knowledge about special geometrical figures is retained in the general concept.

As we will see in the following two sections, Cassirer identifies several types of mathematical concept formation in his discussion of nineteenth-century geometry. Before turning to a closer analysis of these methods, two further points of commentary about the general logic of functional concepts should be made here. (Both points will be relevant for our philosophical assessments of Cassirer's geometrical structuralism given in Section 4.) The first issue concerns the way in which the relation between general concepts and concrete instances is viewed by Cassirer. As we saw above, a central feature of the Aristotelian account of conceptual abstraction is what Heis has termed the "primacy of particulars": given the fact that concepts are formed via abstraction from particular instances, it follows that our knowledge of these particulars has to be presupposed in the construction of the concepts. In the case of functional concepts, the dependency relation between concepts and objects falling under them is reversed. Cassirer explicitly holds that the construction of mathematical concepts is independent of the mathematical objects instantiating them. Compare, for instance, the following passage on the new account of functional abstraction in mathematics:

> What lends the theory of abstraction support is merely the circumstance that it does not presuppose the contents, out of which the concept is to develop, as disconnected particularities, but that it tacitly thinks them in the form of an ordered manifold from the first. The concept, however, is not deduced thereby, but presupposed; for when we ascribe to a manifold an order and connection of elements, we have already presupposed the concept, if not in its complete form, yet in its fundamental function. (Cassirer 1923, 17)[13]

[13]Compare also the following, closely related passage:

Thus, in Cassirer's view, concept formation in mathematics is effectively a *top-down* approach: functional concepts are constructed independently of their instantiations and are thus "logically prior" to them.[14]

The second issue to be addressed here concerns the connection between Cassirer's novel account of concept formation and mathematical structuralism. As will be shown in closer detail in Section 4, Cassirer's understanding of function concepts closely reflects a structuralist conception of mathematics. Notice, in particular, how Cassirer describes the notion of "serial form" as the characteristic content of functional concepts:

> The serial form $F(a, b, c \ldots)$ which connects the members of a manifold obviously cannot be thought after the fashion of an individual a or b or c, without thereby losing its peculiar character. Its "being" consists exclusively in the logical determination by which it is clearly differentiated from other possible serial forms $\Phi, \Psi \ldots$; and this determination can only be expressed by a synthetic act of definition, and not by a simple sensuous intuition. (Cassirer 1923, 26)

The passage is crucial for the understanding of Cassirer's general conception of mathematical knowledge: mathematical theories have as their true subject matter a "serial form" which is determined by the relations between elements of a given domain or manifold. This form is abstract and specified through a "synthetic act of definition".[15] Paraphrased in modern terminology, one can say that a "serial form" presents an abstract structure defined by a mathematical theory that can be instantiated by the elements of any concrete system. In the following two subsections, we will turn to a closer discussion of Cassirer's philosophical analysis of nineteenth-century geometry in order to see how this general understanding of mathematical concepts is illustrated in this field.

3.2. Transformations and invariance

Cassirer's main focus in the third chapter of Cassirer (1910) is on nineteenth-century geometry, spanning the period between Poncelet's *Traité* of 1822 to Felix Klein's *Erlangen program* first outlined in his monumental paper "Vergleichende Betrachtungen über neuere geometrische Forschungen" of 1872 (see Klein 1893). A general topic in Cassirer's survey concerns the question how mathematicians like Poncelet and Klein conceived of the role of concrete figures or diagrams in geometrical proofs. According to Cassirer, modern projective geometry is primarily characterized by an abstraction from diagrams in geometrical reasoning which allows one to reach a level of generality that is comparable to the results in analytic geometry:

> We saw how the progress of geometrical thought tends more and more to allow the subordination of the particular intuitive figures in the proof. The real object of geometrical interest is seen to be only the relational connection between the elements as such, and not the individual properties of these elements. Manifolds, which are absolutely dissimilar for intuition, can be brought to unity in so far as they offer examples and expressions of the same rules of connection. (Cassirer 1923, 251)

Cassirer identifies Poncelet's work as a starting point in this

> . . . for instead of the community of "marks" the unification of elements in a concept is decided by their "connection by implication." And this criterion, here only introduced by way of supplement and as a secondary aspect, proves on closer analysis to be the real logical prius; for we have already seen that "abstraction" remains aimless and unmeaning if it does not consider the elements from which it takes the concept to be from the first arranged and connected by a certain relation. (Cassirer 1923, 24)

[14] See, in particular, Heis (2007, 2014) for a rich survey of the (neo-)Kantian background of Cassirer's top-down approach to mathematical concept formation.

[15] It should be mentioned here that Cassirer's account of mathematical knowledge was influenced by other contributions within neo-Kantian philosophy, in particular Natorp's *Die logischen Grundlagen der exakten Wissenschaften* of 1910. We would like to thank an anonymous reviewer for stressing this point.

gradual development towards formal geometrical reasoning.[16] The central method employed by Poncelet and by subsequent mathematicians in the demonstration of general results is based on the use of projective transformations. More specifically, it is to study those geometrical properties of configurations that are preserved by such transformations. Cassirer describes the general idea underlying this approach in the following way:

> [The universality of the synthetic approach is secured], as soon as we regard the particular form we are studying not as itself the concrete object of investigation but merely as a starting-point, from which to deduce by a certain rule of variation a whole system of possible forms. The fundamental relations, which characterize this system, and which must be equally satisfied in each particular form, constitute in their totality the true geometrical object. What the geometrician considers is not so much the properties of a given figure as the network of correlations in which it stands with other allied structures. (Cassirer 1923, 80)

This "rule of variation" which determines the relevant "fundamental relations" between concrete figures, is specified here in terms of the notion of continuous transformations:

> We say that a definite spatial form is correlative to another when it is deducible from the latter by a continuous transformation of one or more of its elements of position: yet in which the assumption holds that certain fundamental spatial relations, which are to be regarded as the general conditions of the system, remain invariant. The force and conclusiveness of geometrical proof always rests then in the invariants of the system, not in what is peculiar to the individual members as such. (Cassirer 1923, 80)

Given Cassirer's general remarks on this new style of reasoning in projective geometry, several points should be emphasized here. Notice first that, given this focus on the invariant properties of geometrical figures preserved by continuous transformations, the particular nature of figures becomes irrelevant in geometrical proofs. The use of transformations can thus be viewed as a way to generalize over individual figures in order to grasp to the real subject matter of a geometry, namely the "fundamental" relations between the spatial forms of a given manifold. Thus, as Cassirer points out, the true geometrical objects are invariant forms of figures which are induced by a certain type of correlations.

Secondly, Cassirer's discussion of projective geometry closely mirrors his general remarks on mathematical concepts in the introductory chapter of Cassirer (1910). In particular, we can view his remarks on the use of projective transformations and the resulting study of invariants in work by Poncelet as an illustration of the general logic of concept formation outlined there. Recall that, according to Cassirer, what is specific about "functional" concepts in modern mathematics is that knowledge about the individual cases is not lost as in the case of "substance" concepts. Instead, the objects and their specific properties are, so to speak, retained in the general concepts and can be "deduced" from them. A very similar picture emerges from Cassirer's discussion of Poncelet's projective method:

> [In Poncelet's geometry] between the 'universal' and 'particular' there subsists the relation which characterizes all true mathematical concept formation; the general case does not absolutely neglect the particular determinations, but it reveals the capacity to evolve particulars in their concrete totality from a principle. (Cassirer 1923, 82)

Thus, by focusing only on the incidence properties as well as on those metrical properties of a geometrical figure that are preserved under projective transformations, one does not neglect its specific nature. Rather, Poncelet's approach allows one to consider a figure as an exemplification of an abstract geometrical form that is possibly shared by many other figures.

[16] Compare, for instance, an interesting related remark on Poncelet in one of Cassirer's later writings: "[Poncelet] had to emancipate geometrical thought from all connection with 'elements' that can be given in intuition, and to consider the *relations* between these elements as the proper and only subject-matter of geometrical knowledge" (Cassirer 1944, 23–24).

The third point to mention here is that Cassirer's discussion of the new style of reasoning in projective geometry clearly sounds structuralist to the modern reader. This concerns, in particular, his remarks on the purely "relational" character of geometrical knowledge. In shifting attention away from the particular diagrams and towards their invariant properties, geometrical theories turn out to have a new and abstract subject-matter, namely the "network of correlations" between particular figures. Compare again Cassirer on this point:

> We saw how the progress of geometrical thought tends more and more to allow the subordination of the particular intuitive figures in the proof. The real object of geometrical interest is seen to be only the relational connection between the elements as such, and not the individual properties of these elements. Manifolds, which are absolutely dissimilar for intuition, can be brought to unity in so far as they offer examples and expressions of the same rules of connection. (Cassirer 1923, 251)

Cassirer's structuralist view is articulated most explicitly in his discussion of Poncelet's principle of *continuity*, already mentioned in Section 2. Recall from above that this principle was introduced in the *Traité* as a way of generalizing demonstrations about projective properties of spatial configurations. Cassirer describes the general significance of the principle in the following way:

> It is this interpretation, which Poncelet characterizes philosophically by the expression *principle of continuity*, and which he formulates more precisely as the *principle of the permanence of mathematical relations*. The only postulate that is involved can be formulated by saying that it is possible to maintain the validity of certain relations, defined once for all, in spite of a change in the content of the particular terms, i.e., of the particular relata. We thus begin by considering the figure in a general connection [*Lage*], and do not analyse it in the beginning into all its individual parts, but permit changes of them within a certain sphere defined by the conditions of the system. If these changes proceed continuously from a definite starting-point, the systematic properties we have discovered in a figure will be transferable to each successive "phase," so that finally determinations, which are found in an individual case, can be progressively extended to all the successive members. (Cassirer 1923, 80–81)

Given this description, the central mathematical idea underlying Poncelet's principle is indeed a structuralist one: the principle is formulated based on the distinction between general (or "systematic") relations and concrete elements (or figures) as their possible relata. These general properties are characterized by the fact that they can be mapped from one particular figure to another one in terms of a suitable continuous transformation. The principle of continuity thus effectively allows one to abstract from the concrete relata of a given relation in order to yield a general relational form.

The central method underlying this abstraction from concrete figures it is what Cassirer terms the "*operation of transfer of relations*" ("*Verfahren der Relations-Übertragung*"), i.e., the method of transferring a certain relational structure between different geometrical configurations. In the context of Poncelet's work, the relevant relational structure consists of those invariants of a given figure preserved under the projective correlations. As we saw in the previous section, these are mainly properties concerning the *incidence* relations holding between the points, lines, and planes in a geometrical configuration. For instance, in the example of Desargues's theorem, we saw the relevant projective properties of the co-linearity of points and the concurrency of lines.[17]

A second example of a projective transfer principle that is also explicitly discussed in Cassirer's work is the principle of duality. Recall from the previous section that Poncelet explained duality

[17] Projective properties are not limited to incidence relations, but also concern a number of metrical invariants, for instance, the cross-ratio of points on a projective line. See, for instance, Coxeter (1974/1987) for a detailed study of the projective invariants.

phenomena in projective geometry in terms of dual transformations. Such "*indirect* correlations" between figures, that is, transformations that reverse the "order" of composition of the parts of a figure, are also briefly mentioned in Cassirer (1910, 108). The relevant feature of such mappings is that they also preserve the projective properties of a figure, at least up to dual equivalence. For instance, polar transformations are so construed that any incidence relation between lines and points in the original figure can be translated into a reciprocal incidence relation that holds between the corresponding poles and polars.

Now, similar to simple projections, dual transformations can also be viewed as a way to "transfer" the projective content of one configuration to another one. The existence of such a mapping between two figures shows that they share the same structural properties and are thus structurally indiscernible, at least in a projective setting. Dual transformations are in fact more general than simple projections, since they show that the particular nature of the basic elements is irrelevant in the specification of the geometrical structure of a figure: different configurations such as the illustrations of Pascal's and Brianchon's theorems in Figure 2 can nevertheless share the same abstract incidence structure up to duality. Thus, in Cassirer's terms, what matters is not the "content of the singular elements of a relation" but rather what is preserved under the change of such elements under such a transformation. Compare Cassirer's related discussion of dual transformations in his unpublished text "Einheit der Wissenschaft", written in 1931:

> It is, for instance, characteristic of "dual transformations" which play a decisive role in the construction of the modern projective geometry that, based on them, configurations of different levels can be transformed into each other. A statement about points and lines is not subject to change if we interchange the words 'point' and 'line' in conformity with the principle of duality. It thus conforms with the viewpoint of modern geometrical concept formation that two dualistically opposing figures are not viewed as two distinct, but as essentially the same figures. (Cassirer 2010, 167)

The most general and systematic expression of this notion of a transfer of relational structure can be found, according to Cassirer, in Felix Klein's group-theoretic approach in geometry. As is well known, Klein's programmatic paper "Vergleichende Betrachtungen" of 1872 offered a radically new method for the study of different geometries studied at the time, for instance projective, Euclidean, hyperbolic, and conformal geometry (among others).[18] This was the use of groups of transformations to characterize the relevant properties of configurations of a given space. More specifically, Klein's principal idea was to identify each geometry with a space or, more formally, a manifold M and a group G of transformations acting on M that leave the relevant geometrical properties invariant. A manifold of n dimensions for Klein was simply a set of n-ary tuples of real or complex numbers. The transformations of a manifold were usually understood analytically, that is as functions expressed by certain algebraic equations.[19]

Klein's important insight was that certain classes of spatial transformations—equipped with a suitable composition function—form a group in the algebraic sense of the term that, in a way, encodes the abstract content of a given geometry.[20] Compare Klein on this point:

> The most essential idea required in the following discussion is that of a group of space-transformations. The combination of any number of transformations of space is always equivalent to a single transformation. If now a given system of transformations has the property that any transformation obtained by combining any

[18]Compare, e.g., Gray (2007), Hawkins (2000), and Birkhoff and Bennett (1988) for detailed historical studies of Klein's program, its mathematical background, and its influence on subsequent geometrical research.

[19]This analytic presentation of transformations of a space is not explicitly discussed in Klein (1893). However, in subsequent writings on the Erlangen program, the relevant transformations are usually characterized by Klein in this way. See, e.g., Klein (1926).

[20]See Wussing (1984/2007, 186) for a closer discussion of Klein's concept of a transformation group.

transformations of the system belongs to that system, it shall be called a group of transformations. (Klein 1893, 217)

Given this set-up, Klein was able to characterize different geometries in terms of their corresponding transformation groups, or more precisely, in terms of the properties of configurations that are preserved by the transformations of the respective group. For instance, Euclidian geometry is characterized in terms of what Klein called the "*principal group*" of spatial transformations. This is the group of isometries, viz., the distance-preserving transformations including reflections, rotations, and translations. Projective geometry, in turn, is characterized by the group of all projective transformations of a manifold, and so on.

This novel group-theoretic approach in geometry brought with it a new conception of the subject matter of a geometrical theory.[21] Briefly put, geometry turns into a form of invariant theory, i.e., into a study of those properties of figures that are preserved under certain transformations. Compare Klein's well-known description on this new account of geometry:

> Given a manifold and a group of transformations of the same; to investigate the configurations belonging to the manifold with regard to such properties as are not altered by the transformations of the group . . . to develop the theory of invariants relating to that group. (Klein 1893, 218–19)

Now, the central mathematical motivation for this focus on transformation groups and their invariants was to provide a uniform method for the comparison of different geometries: this is achieved by the fact that the groups corresponding to several geometries can be ordered in terms of the group-theoretic notions of subgroup and group extension. Presented schematically, think of two groups of transformations A, B such that A forms a subgroup of B. Then all invariant properties of configurations in a manifold M relative to B also turn out to be invariant relative to A (but not vice versa). Given this, one can say that for two geometries $\mathsf{A} = \langle M, A\rangle$ and $\mathsf{B} = \langle M, B\rangle$, geometry A is a *subgeometry* of B if the transformation group A is a *subgroup* of B. Given the fact the transformation groups corresponding to several geometries present subgroups of this form, Klein was able to present a hierarchy of geometries studied at the time. For instance, he was able to give a precise account of the relation between Euclidean and projective geometry. Since the group of all isometric transformations characteristic for Euclidean geometry forms a subgroup of the projective transformations, it follows that Euclidean geometry is a subgeometry of projective geometry. This means that all of the projective invariants (such as the cross-ratio of four collinear points) are also invariants in Euclidean geometry. However, it is not the case that invariant properties studied in Euclidean geometry are also preserved under projective transformations. This concerns, in particular, simple metrical properties concerning the sameness of lengths or angles.[22]

Cassirer, in his *Substanzbegriff und Funktionsbegriff* of 1910 as well as in later works, viewed this group-theoretic approach in geometry as a direct generalization of the transformation-based approach in projective geometry. In fact, Klein's program is described by him as a culmination point in the development towards a structural conception of geometry outlined above.[23] More specifically, Cassirer holds that given the plurality of different geometrical methods, there is "a uniform basic form of geometrical concept formation". This is the use of transformations of geometrical objects for the specification of their invariant form. As he points out, the most systematic expression of this basic form is to be found in Klein's approach:

[21]See, in particular, Rowe (1985) and Marquis (2009) on this point.

[22]Compare again Wussing (1984/2007) for a detailed account of Klein's classification of geometries in terms of their groups.

[23]Compare in particular Ihmig (1997) for a detailed study of Cassirer's reception of Klein's program and of the significant influence the group-theoretic approach in geometry exercised on Cassirer's general philosophy.

> This development reaches its systematic conclusion in the theory of groups; for here change is recognized as a fundamental concept, while, on the other hand, fixed logical limits are given to it. . . . Geometry, as the theory of invariants, treats of certain unchangeable relations; but this unchangeableness cannot be defined unless we understand, as its ideal background, certain fundamental changes in opposition to which it gains its validity.
>
> The unchanging geometrical properties are not such in and for themselves, but only in relation to a system of possible transformations that we implicitly assume. Constancy and change thus appear as thoroughly correlative moments, definable only through each other. (Cassirer 1923, 90–91)

This and related remarks suggest that Cassirer takes Klein's group-theoretic approach to be a specification of how "functional" concepts—expressing an abstract "network of correlations" between concrete objects—can be constructed in geometry. Specifically, he makes two important philosophical observations regarding the abstract character of Klein's focus on transformations of space.

The first observation concerns the notion of "geometrical properties". What Klein's group-theoretical approach shares with the preceding developments in projective geometry is that attention is gradually shifted away from particular figures in space and towards their relational properties. Cassirer emphasizes at several places that Klein first introduced a systematic method on how to specify the relevant structural properties of a given geometry. Recall that for Klein, a geometry consists of a manifold M and a group G of transformations $f : M \to M$ acting on M. If we think of configurations as subsets of the manifold, i.e., $F_1, F_2 \subseteq M$, then a property P of configurations in M can be called a *G-property* if is it invariant relative to G, i.e. for any $F_1 \subseteq M$: if $P(F_1)$ then for all $f \in G$: $P(f(F_1))$. Notice that, in this account, what counts as a geometrical property is clearly dependent on the choice of a particular group of transformations. Thus, by changing the relevant group of transformations, for instance by adding certain types of transformations, what counts as a geometrical property will also change. Compare Cassirer on this relative character of geometrical properties:

> "Geometry is distinguished from topography by the fact that only such properties of space are called geometrical as remain unchanged in a certain group of operations." If we adhere to this explanation, we gain a view of very diverse possibilities for the construction of geometrical systems, all equally justified logically. For as we are not bound in the choice of the group of transformations, which we take as the basis of our investigation, but can rather broaden this group by the addition of new conditions, a way is opened by which we can go from one form of geometry to another structure [*Struktur*] by changing the fundamental system to which all assertions are related. (Cassirer 1923, 89)

Notice the explicit use of the notion of "structure" in this context: the structure of a given geometry is determined by its group of transformations. Modifying this group will lead to a different, but "logically equal" geometrical structure. The comparison of projective and Euclidean geometry mentioned above illustrates this point. The respective groups of projective and of Euclidean transformations determine different sets of geometrical invariants. In Cassirer's terms, they induce different geometrical "structures". Nevertheless, these geometries are on par with each other in the sense that none is preferable to the other, at least if both are conceived as pure geometrical theories.

Cassirer's second important observation is related to the idea of the transfer of relational structure already discussed above. Recall that, according to him, the method of transfer in Poncelet's work on projective geometry is based on the use of correlations that preserve the projective properties of a figure. Moreover, we saw that two distinct figures connected by such a transformation are usually considered to be equivalent or structurally indiscernible, at least from a projective point of view. As Cassirer notes, this notion of structural equivalence is again generalized in Klein's group-theoretic approach. In particular, Klein explicitly points out in 1872 that one can use the transformations of a given type to transfer certain properties of a given figure to another one. For instance, he argues here that "[e]very space-transformation not belonging to the principal group can be used to transfer the properties of known configurations to new ones" (Klein 1893).

Moreover, Klein also points out that if a figure can be transformed into another one in this sense, then the two figures are congruent relative to the underlying group of transformations. More generally, given a geometry consisting again of a manifold M and a group G of transformations on M, we can say that two figures $F_1, F_2 \subseteq M$ are *G-congruent* if there exists at least one transformation $f \in G$ such that $f(F_1) = F_2$. Given the structural character of Klein's approach, one could also say that *G-congruence* presents a relative notion of structural equivalence between the configurations in a given manifold. Thus, two figures in a given space are structurally equivalent with respect to a given geometry if there exists at least one transformation that transfers all relevant properties from one to the other.[24]

Precisely this understanding of the geometrical congruence of figures as a kind of structural equivalence is also highlighted in Cassirer's analysis of Klein's approach. Compare, for instance, the following passage in his later article "The Concept of Group and the Theory of Perception" of 1944:

> From this definition of "geometrical properties" the conditions become immediately apparent under which two spatial concepts/configurations are "equivalent" to each other, i.e. are but different expressions of one and the same geometrical "essence". The "essence" of a triangle is not altered, the logical *assertions* about it are not invalidated, when we change its individuality in certain ways, e.g., displace it in space or make the absolute lengths of the side increase or decrease. We may say quite generally that two series of expressions which are transformed in this manner must be considered as geometrically equivalent, i.e., defining identical geometrical figures. (Cassirer 1944, 6–7)

Notice again that what counts as structurally equivalent in a given space is again dependent on the particular choice of a group of transformations acting on the space.[25] For instance, in Euclidean geometry, one usually distinguishes between different types of conic sections—namely hyperbola, parabola, and ellipses—given the fact that they have different metrical properties. In contrast, in projective geometry, these three types of conics are treated as the same geometrical configuration since they are projectively equivalent.

3.3. Formal axiomatics

The second methodological innovation in modern geometry discussed in detail in Cassirer (1910) concerns the development of formal axiomatics. As was shown in Section 2, the axiomatic method became of central importance in projective geometry in the nineteenth-century. In particular, it was Pasch who presented the first systematic axiomatization of projective geometry in his *Vorlesungen über neuere Geometrie* (1882). As we saw, Pasch's axiom system was taken to be descriptively complete in the sense that all theorems of solid projective geometry can be deduced from the axioms alone. Moreover, he was first to emphasize the importance of formal and rigorous deductive reasoning and the fact that demonstrations of geometrical theorems should not depend in any way on empirical intuition or on the use of diagrams.

[24] Klein, in his writings, was explicit about the fact that this understanding of geometrical equivalence is rooted in modern projective geometry, for instance, in the principle of duality. Compare the following remark: "From the modern point of view two reciprocal figures are not to be regarded as two distinct figures, but as essentially one and the same" (Klein 1893, 221).

[25] Compare again Cassirer on this relativity of structural identity in Klein's account: "Thus, what in the geometrical sense must be taken as 'identical' and what as 'different' is by no means predetermined at the outset. On the contrary, it is decided by the nature of the geometrical investigation, viz., by the choice of a determinate group of transformations" (Cassirer 1944, 7).

This abstraction from geometrical intuition in axiomatic proofs also marks the transition from the classical to the modern understanding of axiomatic geometry in Cassirer's view. While, as in Pasch's account, intuition may still provide a relevant source for the specification of the primitive geometrical concepts–that is, of points, lines, and planes in space—the relations of incidence or concruence between them are now to be "deduced conceptually" from the axioms of a theory. Compare Cassirer's description of what is essentially Pasch's *empiricist* approach to axiomatic geometry, illustrated in terms of the notion of "betweenness":[26]

> We can still take the elementary contents of geometry: the point, the straight line and the plane, from intuition; but all that refers to the connection of these contents must be deduced and understood conceptually. In this sense, modern geometry seeks to free a relation, such as the general relation of "between," which at first seems to possess an irreducible sensuous existence, from this restriction and to raise it to free logical application. The meaning of this relation must be determined by definite axioms of connection in abstraction from the changing sensuous material of its presentation; for from these axioms alone is gained the meaning in which it enters into mathematical deduction. By this extension, we can make the concept of "between" independent of its original perceptual content and apply it to series in which the relation of "between" possesses no immediate intuitive correlate. (Cassirer 1923, 91–92)

Thus, intuition may still be used to grasp the domain of basic elements of a geometrical theory. However, all of their relevant properties have to be specified independently of it. A geometrical notion such as the "betweenness" between points is understood here as a "general relation" whose meaning is to be determined through general axiomatic definitions and thus without reference to any concrete or intuitive relata to which it may apply.

It is interesting to see here that Cassirer takes this new focus on axiomatic definitions and rigorous deductive proofs to be closely connected to the developments in projective geometry outlined in Section 2. In particular, his article "Kant und die moderne Mathematik" (1907) contains an explicit discussion of Pasch's axiomatic justification of the principle of duality:

> As is generally known, the law of duality is the fact that every projective statement remains true if one interchanges the words "point" and "plane" in it whereas one leaves unchanged the straight lines together with all those properties they share with points and planes. The proper logical basis of this reciprocity lies in the fact that, in the geometrical theory present in front of us, the concepts of "points" and "lines" were assumed as undefinable such that their content cannot be relevant for the truth of the theory; this truth thus has to remain valid if one assigns a different meaning to these entities; given the condition that one ascribes to them only and precisely those relations which they possessed before. (Cassirer 1907, 28)

This account of the "logical basis" of duality corresponds precisely to Pasch's justification of the principle based on his account of formal geometrical proofs that preserve only logical structure and are independent of the concrete meaning of the primitive geometrical terms. Moreover, Cassirer takes Pasch's account to be characteristic for a general tendency in geometry to neglect "intuitive elements" in geometrical proofs.

Even though Pasch is mentioned briefly in this context, Cassirer's main attention is dedicated to David Hilbert's work, in particular his *Grundlagen der Geometrie* of 1899. Hilbert's axiomatization of Euclidean geometry is described by Cassirer as a "pure science of relations" ("Beziehungslehre") in which all ties to geometrical intuition are given up. Whereas in Pasch's account geometrical intuition still plays a role in the process of axiom choice, Hilbert's axioms are viewed here as free-standing

[26] See, again, Schlimm (2010) for a detailed study of Pasch's empiricist account of axiom choice.

conditions that implicitly define the primitive terms of the theory.[27] Before turning to a closer discussion of Cassirer's remarks on Hilbert's new method, let us briefly outline the basic approach in *Grundlagen*.

As is well known, Hilbert starts his treatment of Euclidean geometry by mentioning the three types of primitive objects (namely points, lines, and planes) and by presenting a number of axioms which specify the relations between these objects. His axiom system is divided into five axiom groups: the *axioms of connection*, the *axioms of order*, the *axiom of parallels*, the *axioms of congruence*, and the *axiom of continuity* (in particular, Archimedes' axiom and the axiom of completeness added in the second edition of the book). This classification of the axioms into different groups is based on the particular kinds of geometrical properties they specify: The axioms of connection form the "projective basis" of his system (that is, the *incidence axioms* in the modern sense).

The axioms of order specify the basic properties of the ordering of points on a straight line and thus define the notion of "betweeness" already mentioned in Cassirer's above remark. The third group contains only the well-known axiom of parallels. The axioms of congruence, in turn, determine the notion of congruence of line segments and angles. Finally, the two completeness axioms are introduced by Hilbert in order to get a complete and, in modern terms, categorical axiomatization of Euclidean space. The *Archimedes axiom* roughly states that, if sufficiently often repeated on a line, every line segment exceeds the length of any previously given line segment. The axiom of completeness was added in the second edition of the book and has a different, genuinely metatheoretic status. It states that the system of points, lines and planes satisfying the base axiom system cannot be extended without violating one of the remaining axioms.

The central innovation in Hilbert's *Grundlagen* does not lie in the particular formulation of these axioms or their classification into groups, but rather on his new understanding of an axiomatic theory.[28] In particular, the axioms of the five groups are no longer viewed as true descriptions of an intuitively accessible domain, but rather as implicit definitions of the primitive terms as well as of the primitive relations of incidence, parallelity, congruence, and so on. Consider, for instance, Hilbert's "projective" basis of Euclidean geometry, that is, the *axioms of connection*:

> 1.1 Two distinct points A and B always completely determine a straight line a. We write $AB = a$ or $BA = a$.
> 1.2 Any two distinct points of a straight line completely determine that line; that is, if $AB = a$ and $AC = a$, where $B \neq C$, then is also $BC = a$.
> 1.3 Three points A, B, C not situated in the same straight line always completely determine a plane α. We write $ABC = \alpha$.
> 1.4 Any three points A, B, C of a plane α, which do not lie in the same straight line, completely determine that plane.
> 1.5 If two points A, B of a straight line a lie in a plane α, then every point of a lies in α.
> 1.6 If two planes α, β have a point A in common, then they have at least a second point B in common.
> 1.7 Upon every straight line there exist at least two points, in every plane at least three points not lying in the same straight line, and in space there exist at least four points not lying in a plane. (Hilbert 1899)

These axioms are taken to express general conditions for the incidence relations between points, lines, and planes which need

[27] It should be noted here Pasch's axiomatic work in projective geometry exercised a significant influence on Hilbert's axiomatic approach. See Schlimm (2010) and Toepell (1986) for a closer discussion of this line of influence.

[28] Compare, in particular, Torretti (1978), Nagel (1939), and Gray (2008) on Hilbert's conception of formal axiomatics.

to be met by any system of Euclidean geometry. Hilbert's new understanding of axioms as implicit definitions brought with it a fundamental change in how geometrical theories are conceived. Roughly put, axiomatic theories are no longer about a particular geometrical space, but are now understood as *schematic* or *formal* in the modern sense of the term. As such, they can be (re-)interpreted in different systems that satisfy the abstract conditions specified in the axioms.

Hilbert's approach in *Grundlagen* is described as another culmination point in the development of pure geometry in Cassirer's book. What is particularly interesting here is that Cassirer was likely the first philosopher to see a close conceptual connection between Hilbert's contributions to structural axiomatics and the group-theoretic approach in geometry introduced by Klein three decades earlier.[29] In his view, both accounts present endpoints to two different developments in nineteenth-century geometry that eventually led to a "structural turn" in the field. In Klein's case, this was the systematic use of transformations first introduced in work by Poncelet and Chasles. In Hilbert's case, this was the axiomatic tradition starting with work by mathematicians such as Gergonne and Pasch.

How are these two geometrical methods related according to Cassirer? Two points should be emphasized here. The first concerns his understanding of the subject matter of geometrical theories. In Cassirer's view, in both Klein's and Hilbert's accounts, geometrical theories are effectively about relational structures. We saw that in Klein's algebraic approach of studying geometries in terms of their characteristic transformation groups, geometry becomes the study of invariants. As Cassirer pointed out, a group of spatial transformations can be considered as an abstract concept that represents, in his own words, the "structure" of a particular geometry. In turn, in Hilbert's *Grundlagen*, Euclidean space is determined axiomatically in the following sense: the axioms of the five groups specify different structural properties—or, in Cassirer's terms, "characteristic conditions"—of the primitive relations of the theory.

Given this approach, it seems natural to say that what an axiomatic theory is really about is also an abstract structure implicitly defined by it. Such a view is expressed by Cassirer in Cassirer (1910) as well as in his related writings. Consider, for instance, the following remark:

> Wherever a definite form of connection is given, which we can express in certain rules and axioms, there an identical "object" is defined in the mathematical sense. The relational structure as such, not the absolute property of the elements, constitutes the real object of mathematical investigation. (Cassirer 1910, 92–93)

Notice again Cassirer's explicit use of the notion of "relational structure", now in the context of modern axiomatic geometry. In axiomatic theories such as Hilbert's axiom system for Euclidean geometry, an abstract structure functions as the proper object of investigation. The nature of particular geometrical objects such as points, lines, and planes is determined solely through their role in such a structure.[30]

This and related passages show that Cassirer proposes an structuralist interpretation of Hilbert's work which is similar in character to his analysis of Klein's group-theoretic approach. In fact, the two methods are described here as alternative types of concept formation in geometry that allow for the specification of the same geometrical concepts by different means. For instance,

[29] This fact was first stressed in Ihmig's work on Cassirer. See, in particular, Ihmig (1997, 327).

[30] Compare Cassirer on this purely relational conception of geometrical objects in Hilbert's axiomatic approach: "The point and the straight line signify nothing but structures which stand in certain relations with others of their kind, as these relations are defined by certain groups of axioms. Only this systematic 'complexion' of the elements, and not their particular characters, is taken here as the expression of their essence. In this sense, Hilbert's geometry has been correctly called a pure theory of relations" (Cassirer 1910, 93–94).

just as Klein's focus on the *principal group* of isometric transformations can be thought of as a way to specify the abstract concept of Euclidean space, Hilbert's axiomatic conditions also specify an abstract or higher-level concept, namely the structure of Euclidean space that can be instantiated by different concrete systems.[31]

The second point to be mentioned here again concerns the notion of "transfer" of relations. In Cassirer's understanding, this method plays a central role both in projective geometry and in Klein's group-theoretic approach. Generally speaking, the relevant transfers in these contexts are effected in terms of "correlations" that preserve certain geometrical properties. For instance, in the case of duality, we saw that one relevant type of transformations concerns Poncelet's polar transformations which allow one to identify geometrical configurations with reciprocal incidence properties. Interestingly, in Klein's *Vergleichende Untersuchungen* of 1872, the topic of transfer principles is also discussed in a more general sense, namely on the level of geometrical theories. As was mentioned before, Klein's general motivation for his group-theoretic approach was not primarily to study particular geometries such as Euclidean or projective geometry in isolation, but rather to compare different theories in terms of their corresponding transformation groups. In Section 4 of the paper—titled "Transfer by Representation" ("Übertragung durch Abbildung")—he introduces a new account of "transfer principles" to show the equivalence of different geometries that share similar transformation groups. Very roughly, the idea sketched here is that two geometries, conceived again as groups of transformations acting on a given manifold, can be taken to be "essentially the same" if there exists a transfer between the two manifolds which induces an isomorphism between the corresponding transformation groups. Compare Klein on this method of "transfer by mapping":

> Suppose a manifoldness A has been investigated with reference to a group B. If, by any transformation whatever, A be then converted into a second manifoldness A', the group B of transformations, which transformed A into itself, will become a group B', whose transformations are performed upon A'. It is then a self-evident principle that the method of treating A with reference to B at once furnishes the method of treating A' with reference to B', i.e., every property of a configuration contained in A obtained by means of the group B furnishes a property of the corresponding configuration in A' to be obtained by the group B'. (Klein 1893, 223)

Paraphrased in modern terms, the idea expressed here is that two geometries—understood as tuples of the form (A, B) and (A', B')—are essentially similar if there exists a bijective mapping $F : A \to A'$ that induces an isomorphism between the corresponding groups $\alpha : B \to B'$ that preserves the group actions of B and B' on A and A' respectively.[32]

Now, this general method of transfer by mappings on the level of geometries is clearly structural in character. A central consequence of Klein's approach is that it allows one to identify the abstract content of geometries that describe spatial elements of a very different kind. This indifference to the nature of the basic elements of a geometry is explicitly addressed in Section 5 of Klein's paper titled "On the Arbitrariness in the Choice of the Space-Element":

> As element of the straight line, of the plane, of space, or of any manifoldness to be investigated, we may use instead of the point any configuration contained in the manifoldness, a group of points, a curve or surface, etc. . . . But so long as we base our geometrical

[31] The view of an axiom system as an explicit definition of a higher-level mathematical concept goes back to Frege's critical discussion of Hilbert's axiomatics. Compare also Carnap's discussion of "explicit concepts" in mathematics that are defined by an axiom system. See, in particular, Carnap (1929).

[32] Compare Rowe (1985) and Marquis (2009) for a closer discussion of Klein's transfer principles.

investigation on the same group of transformations, the geometrical content [*Inhalt der Geometrie*] remains unchanged. That is, every theorem resulting from one choice of space element will also be a theorem under any other choice; only the arrangement and correlation of the theorems will be changed. The essential thing is thus the group of transformations; the number of dimensions to be assigned to a manifold is only of secondary importance. (Klein 1893, 224)

Thus, in the study of a geometry, the particular nature of the basic elements of space is not relevant. The real "content of a geometry" is the abstract structure encoded in the group of spatial transformations. Consequently, two geometries that describe manifolds with distinct basic spatial elements can nevertheless be identified in terms of their abstract content if their corresponding groups of transformations are "similar" (that is, in modern terminology, isomorphic) with each other.[33]

Returning to Cassirer's *Substanzbegriff und Funktionsbegriff*, it is somewhat surprising that Klein's treatment of transfer principles and of the structural equivalence of geometries is not addressed here (nor, to the best of our knowledge, in any of his later writings on the group-theoretic approach in geometry). However, Cassirer refers to a closely related notion of equivalence of geometries in his discussion of modern axiomatics. Compare the following passage in which Cassirer gives an illustration of this kind of equivalence between axiomatic theories:

> Two complexes of judgments, of which the one deals with straight lines and planes, the other with the circles and spheres of a certain group of spheres, are regarded as equivalent to each other on this view, in so far as they include in themselves the same content of conceptual dependencies along with a mere change of the intuitive "subjects," of which the dependencies are predicated. In this sense, the "points" with which ordinary Euclidean geometry deals can be changed into spheres and circles, into inverse point-pairs of a hyperbolic or elliptical group of spheres, or into mere number-trios without specific geometrical meaning, without any change being produced in the deductive connection of the individual propositions, which we have evolved for these points. This deductive connection constitutes a distinct formal determination, which can be separated from its material foundation and established for itself its systematic character. The particular elements in this mathematical construction are not viewed according to what they are in and for themselves, but simply as examples of a certain universal form of order and connection; mathematics at least recognizes in them no other "being" than that belonging to them by participation in this form. For it is only this being that enters into proof, into the process of inference, and is thus accessible to the full certainty, that mathematics gives its objects. (Cassirer 1923, 93)

This passage again highlights Cassirer's structuralist conception of geometrical theories. Two theories can describe systems of basic spatial objects of different sorts, for instance lines and planes in one case and circles and spheres in the other case. Nevertheless, they can be said to have the same content if there exists a systematic replacement between the elements of the two systems such that the "deductive connection" between axioms and theorems is preserved.

In view of Cassirer's geometrical background, there exist at least two ways to understand this method of *replacement* in the present axiomatic context. One is syntactic, the other model-theoretic in character. Recall from Section 2 that one way to justify the principle of duality in projective geometry was based on the axiomatic specification of projective space. In particular, in Pasch's account in 1882, the principle is expressed as a purely syntactic result: given the symmetric character of his axiom system as well as a notion of formal geometrical proof, it follows that any theorem about the projective properties of a configuration can be translated into a dual statement which is also deducible from the axiom system in question. Given Cassirer's above characterization of the equivalence between two

[33]Klein discusses a number of geometries in his paper whose equivalence can be shown based on the existence of a suitable transfer principle. See, in particular, Klein (1893, §4).

"complexes of judgments", i.e., sets of statements, in terms of the preservation of the "deductive connection" between them, it seems natural to interpret this kind of transfer in direct analogy to the axiomatic justification of duality, namely as a purely syntactic procedure.

That said, we mentioned above that Cassirer's main background in his discussion of modern axiomatics is not Pasch but Hilbert's *Grundlagen*. As was shown, Pasch's purely syntactic view is complemented here by a genuinely semantic or model-theoretic conception of axiomatic theories. Specifically, Hilbert's metatheoretic consistency and independence results are presented in the *Grundlagen* in terms of analytic model constructions and based on the fact that the axioms of his theory can be reinterpreted relative to these models.[34] Hilbert was explicit about this model-theoretic conception of axiomatic theories and also the use of model-theoretic methods, in particular of structure preserving mappings between models, in the proof of his metatheoretic results. The new style of reasoning is expressed most clearly by Hilbert in his famous correspondence with Frege. In particular, in a letter from 29 December in 1899, he writes:

> But surely it is self-evident that every theory is merely a framework or schema of concepts together with their necessary relations to one another, and that the basic elements can be construed as one pleases . . . each and every theory can always be applied to infinitely many systems of basic elements. For one has to apply a univocal and reversible one-to-one transformation and stipulate that the axioms are the same also for the transformed things. Indeed, this is frequently applied, for example in the principle of duality, etc.; I also apply it in my independence-proofs. (Frege 1980, 40)

Notice Hilbert's reference to projective duality in this passage. As we saw in Section 2, a common way to justify the principle of duality in projective geometry was based on the notion of structure-preserving mappings. A similar idea of a transfer of structure in terms of transformations is also discussed here in the context of Hilbert's model-theoretic approach to formal theories.[35] Given this background of Cassirer's remarks on modern axiomatics, it is thus plausible to interpret his above discussion of the equivalence of axiomatic theories in a genuinely semantic way. The replacement of the geometrical objects of one domain by those of another domain is then not understood in terms of the syntactic translation of the primitive vocabulary of a geometry, but rather in terms of the semantic reinterpretation of its statements. The kind of semantic transfer of relations is thus induced by mappings between the two systems that satisfy the axiomatically defined relational structure.

This interpretation of Cassirer's understanding of transfers in the context of axiomatic theories receives further confirmation if one looks at the central reference in his discussion of this topic, namely Weber's and Wellstein's *Enzyklopädie der Elementar-Mathematik* of 1903. Cassirer mentions this book for its "very instructive examples and elucidations" in a footnote attached to the above passage. In Volume 2 of the textbook, Weber gives a detailed discussion of Hilbert's axiomatization of Euclidean geometry and the idea of such semantic transfers. More specifically, it is shown that Hilbert's axiom system can be reinterpreted in a spherical geometry if the primitive terms are reinterpreted in a new geometrical system of bundles of spheres. The fact that all of Hilbert's theorems are also true in this new geometry is justified in terms of the existence of a "mapping procedure", that is, a mapping between the two domains that preserves all of the axiomatically defined properties. Compare Weber on this point:

[34] Compare Sieg (2014) and Hallett (2008) for detailed discussion of Hilbert's metatheoretic approach.

[35] In particular, the second sentence in this passage can be read as an early formulation of the model-theoretic isomorphism lemma: relativized to Hilbert's categorical theory of Euclidean geometry: any two systems (possibly with domains of different objects) are identical *as models* of the theory if there exists a bijective mapping between them that preserves the structural properties expressed in the axioms.

. . . [in the study] of the primitive elements "point", "line", "plane", "space" and of the primitive concepts "between", "distance", "angle", "congruence", one has to strictly distinguish between the properties which can be transferred from conventional space to any other linear and three-dimensional manifold and those properties which apply to these concepts individually. Transferable are, for instance, the properties of connection and order, of continuity, and congruence, in so far as they are collected in the (Hilbertian) axioms. . . . The transferable properties concern the relations of the primitive concepts to each other, the individual properties concern the relations to our sensuality. (Weber and Wellstein 1903, 109–10)

Notice, in particular, the explicit use of the term "transferable properties" in this discussion of Hilbert's work. It is precisely this semantic notion of transfer of structure that Cassirer was also referring to in his account of modern axiomatics.

4. A Geometrical Structuralism

Cassirer's philosophical reflections on modern geometry in Cassirer (1910) as well as in related writings present an attempt to describe the general "structural turn" in the field. More generally, it is justified to say that Cassirer formulated an early version of structuralism concerning mathematical knowledge.[36] This fact has already been emphasized in recent scholarly work. In particular, Yap, Heis, and Reck have surveyed the formative influence of Dedekind's work on the foundations of arithmetic on Cassirer's philosophy of mathematics. It is argued there that Dedekind's proto-axiomatic presentation of arithmetic presented a general paradigm for Cassirer's understanding of structural mathematics. In the present section, we want to take up this debate and analyze how Cassirer's discussion of the structural methods in nineteenth-century geometry relates to this general Dedekind-style structuralism about arithmetic and to contemporary philosophy of mathematics more generally.

4.1. Methodological structuralism

On first glance, Cassirer's general discussion of the structural nature of mathematical knowledge seems closely connected to modern structuralism. Consider again how the subject matter of mathematical theories is characterized by him in 1910:

> The relational structure as such, not the absolute property of the elements, constitutes the real object of mathematical investigation. . . . The particular elements in this mathematical construction are not viewed according to what they are in and for themselves, but simply as examples of a certain universal form of order and connection; mathematics at least recognizes in them no other "being" than that belonging to them by participation in this form. For it is only this being that enters into proof, into the process of inference, and is thus accessible to the full certainty, that mathematics gives its objects. (Cassirer 1923, 93)

This reads as a variant of modern non-eliminative structuralism, as developed in work by Shapiro, Resnik, and Parsons (among others).[37] All of the central ingredients of their accounts are present here: mathematical theories are taken to study abstract structures or patterns as their subject matters. Mathematical objects, in turn, are merely positions in such structures, specified in terms of their interrelations with the other objects. These objects have no intrinsic nature or properties outside their structure. In Cassirer's terms, their very mathematical "being" is determined by the fact that they instantiate the abstract relational structure in question.

That said, it should be emphasized that Cassirer's account also differs in crucial respects from modern versions of structuralism. First of all, Shapiro's (and to a lesser degree also Resnik's)

[36] This point has first been emphasized in Ihmig (1997). See also Heis (2011) and Biagioli (2016).

[37] See, in particular, Shapiro (1997), Parsons (1990), and Resnik (1997).

theories are primarily theories about a proper structuralist ontology of mathematics. Both philosophers take structures and the positions in them to be entities that (i) exist in some abstract realm and (ii) whose metaphysical nature needs to be specified in some form, for instance in terms of Platonic universals in Shapiro's case. In contrast, in Cassirer's work, one finds little interest in such metaphysical speculations concerning the nature of structural objects. Rather, as we saw in Section 3.1, his main focus is on the status of mathematical concepts and on the general logic of concept formation. Put differently, Cassirer's philosophy of mathematics is less concerned with mathematical ontology or our epistemological access to mathematical objects, than it is with the study of different methods of constructing abstract concepts.

Cassirer's position is thus best characterized as an early version of "methodological structuralism".[38] In particular, looking at the extensive discussion of modern geometry in the third chapter of *Substanzbegriff und Funktionsbegriff*, we saw that he identified two "structural methods", namely formal axiomatics and the transformations-based approach expressed most systematically in Klein's Erlangen program. A central insight of Cassirer was that these two methods lead to a similar structuralist conception of the subject matter of geometrical theories. Moreover, he takes both approaches to characterize a notion of structural transfer between different mathematical domains.

[38]Compare Reck on a closer specification of this position: "As the term 'methodological structuralism' suggests, this first position has primarily to do with mathematical method, rather than with semantic and metaphysical issues as the others do. . . . [M]ethodological structuralism consists then of such a general, largely conceptual approach (as opposed to more computational and particularist approaches)" (Reck 2003, 371).

4.2. A top-down view

Whereas Cassirer was clearly less concerned with the ontology of mathematical objects, there is a number of interesting points of contact with the modern debate. These concern, in particular, his understanding of the relation between mathematical concepts and concrete instances, for instance, between abstract geometrical concepts of a given space or curve and concrete systems or configurations of this form. As we want to show here, Cassirer's view of this relationship anticipates a kind of *top-down* structuralism similar in spirit to Shapiro's *ante rem* structuralism (see Shapiro 1997). Moreover, viewed in this way, it can also be shown how Cassirer's account of geometry connects with the general "Dedekind-style structuralism" highlighted in recent scholarly work.

Let us look at the two structural methods described in Cassirer (1910) in turn. In Cassirer's discussion of Hilbert's structural axiomatics, the connection to Dedekind's approach is evident. Dedekind, in his *Was sind und was sollen die Zahlen?* (1888), gave an axiomatic presentation of arithmetic that is, from a methodological point of view, very similar to Hilbert's axiomatic treatment of Euclidean geometry. As has recently been pointed out by Reck, Heis, and Yap, Cassirer describes Dedekind's approach as a structuralist one, using more or less that same terminology as in his discussion of Hilbert:

> What is here [in Dedekind's work] expressed is just this: that there is a system of ideal objects whose content is exhausted in their mutual relations. The 'essence' of the numbers is completely expressed in their positions. (Cassirer 1923, 39)

Thus, given the axiomatic definition of the abstract concept of the natural number structure, individual objects such as the natural numbers are merely positions in a structure whose only relevant properties are relational ones, that is, those determined by the interrelations with the other numbers in the structure.

This treatment of Dedekind's and Hilbert's axiomatic approaches is clearly related to modern versions of non-eliminative structuralism in the work of Resnik, Shapiro, and Parsons (among others). This concerns, in particular, the structural conception of objects as positions in a mathematical pattern or structure.[39] Also relevant here is the fact that the axiomatic method is described by Cassirer as a *top-down* approach in mathematical concept formation. An axiom system specifies a higher-level concept or a set of abstract conditions that any model of the theory has to satisfy. Axiomatically defined concepts are thus, in Cassirer's own terms, "logically prior" and thus independent of the more concrete instances satisfying them. For instance, in the context of Dedekind arithmetic, the specification of a natural number structure is independent of concrete number systems which meet the conditions laid down in the axioms. Similarly, in the case of geometry, Cassirer rightly describes Hilbert's axioms as "hidden definitions" of abstract properties of the Euclidean space that are independent of any concrete or intuitive objects. Thus, Hilbert's axioms are not viewed as descriptive statements about the properties of an intuitively given domain. Rather, they function prescriptively, as definitions of abstract conditions that any space has to satisfy in order to count as Euclidean.[40] Compare again Cassirer on this general point in axiomatic concept formation:[41]

> The determination of the individuality of the elements is not the beginning but the end of the conceptual development; it is the logical goal, which we approach by the progressive connection of universal relations. (Cassirer 1910, 94)

Notice that this account conforms with Cassirer's general remarks on the top-down nature of mathematical concept formation presented in Section 3.1. Given the concepts-first or "relations over relata" view stated here, it seems natural to interpret Cassirer's understanding of modern axiomatic theories (both Dedekind's arithmetic and Hilbert's geometry) as an early version of *ante rem* structuralism.[42]

Let us turn to the second geometrical method discussed in Cassirer's book, that is, the systematic use of transformations in projective geometry and in Klein's Erlangen program. Here too, Cassirer observed a shift in attention from particular geometrical figures to the study of their invariant form. This focus on invariant relations and the notion of the structural equivalence of geometrical objects is again closely related to the modern debates in structuralism. In particular, one way in which the structuralist thesis is usually characterized in modern philosophy of mathematics is based on the notion of structural properties. Roughly put, it is argued that mathematical theories study *only* structural properties of their objects, i.e., properties not concerning their "internal nature" but rather how these objects "relate to each other".[43] Properties in this sense are usually characterized in terms of the invariance under isomorphism, i.e., invariance under structure-preserving mappings. As we saw, a similar focus on invariant properties can also be found in Cassirer, most explicitly in his discussion of Klein's program: geometrical properties specified relative to a given transformation group are the structural ones.

[39]See Shapiro (1997), Resnik (1997), and Parsons (1990). Compare, for instance, Parsons' discussion of the structural characters of objects: "By the 'structuralist view' of mathematical objects, I mean the view that reference to mathematical objects is always in the context of some background structure, and that the objects have no more to them than can be expressed in terms of the basic relations of the structure" (1990, 303).

[40]See, in particular, Schlimm (2013) for a detailed investigation of this prescriptive-descriptive distinction in modern axiomatics.

[41]Heis gives a detailed study of Cassirer's top-down approach in mathematical concept formation and describes this as a direct consequence of his Kantian view of mathematics. See, in particular, Heis (2007).

[42]This form of structuralism is discussed in most detail by Shapiro. See, in particular, Shapiro (1997).

[43]Compare, for instance, Linnebo on such a general characterization of structuralism: "Very roughly, mathematical structuralism is the view that pure mathematics is the investigation of abstract structures, and that all that matters to mathematics is purely structural properties of objects" (Linnebo 2008, 60).

Can Cassirer's discussion of Klein's group-theoretic approach also be read as a top-down structuralism? To address this, it is important to see how the relation between abstract geometrical concepts specified in terms of transformation groups and the concrete geometrical figures instantiating them is understood in his work. Arguably, the situation is not as straightforward here as in the case of modern axiomatics. At some places in his work, in particular regarding his discussion of the abstract types or "essences" of particular figures, this relation seems to be conceived of not in a top-down, but rather in a *bottom-up* manner. Thus, figure types are constructed here in terms of transformations by means of an act of conceptual abstraction from concrete figures. Compare, for instance, Cassirer's discussion of Klein's method in his lecture notes "The Concept of Group and the Theory of Perception" (1944):

> Every particular triangle, every particular circle is to be considered as something in and by itself. Its location in space, the lengths of the sides of the triangle or of the radii etc. belong to its "nature", which latter cannot be defined except with reference to particular local circumstances. Our geometrical concepts ignore these individual differences—or, as we usually say, they abstract from them. But the term "abstraction" itself is not very clear; it needs a sharper and more precise determination. This determination is easily to be found if we look at Klein[']s theory of geometry, based upon his conception of transformation-group. There we find immediately that there are various *degrees* of abstraction that lead us to higher and higher universality. (Cassirer 1944, 191)

Notice Cassirer's use of the term "degrees of abstraction" in this context: geometrical concepts are induced by abstraction from the concrete figures in a given space. This transformations-based method of abstraction is described as a gradual process: given Klein's approach, there is the possibility to gain higher levels of abstraction by considering groups of more general transformations. This characterization of Klein's method does indeed seem related to modern versions of *in re* structuralism. According to this approach, mathematical structures cannot be thought of independently of their instances. Rather, they exist only in so far as they can be instantiated by concrete mathematical systems or objects.[44] Applied to the context of geometry, this is to say that figure types are not to be thought of as "bona fide" objects, but are ontologically dependent on their concrete instances.

In spite of passages such as the above one, there are convincing reasons to believe that Cassirer also understood Klein's group-theoretic approach as supporting a top-down structuralism about geometrical knowledge, similar to the case of modern axiomatics. First, recall from Section 3.1 that the general logic of concept formation in modern mathematics is described as a top-down process in Cassirer (1910). In particular, it is argued there that mathematical concepts are not constructed by means of conceptual abstraction from concrete instances but rather by the specification of abstract conditions that can be satisfied by concrete objects. Given this, it seems plausible that Cassirer's reference to abstraction in his discussion of Klein's method is not understood in a logical sense, but rather in a weaker, psychological sense. Compare again Cassirer on this distinction between a psychological and a logical dependency relation between abstract concepts and intuitive objects:

> It is true that, in the psychological sense, we can only present the meaning of a certain relation to ourselves in connection with some given terms, that serve as its"foundations." But these terms, which we owe to sensuous intuition, have no absolute, but rather a changeable existence. We take them only as hypothetical starting-points; but we look for all closer determination from their successive insertion into various relational complexes. It is by this intellectual process that the provisional content first becomes a fixed logical object. (Cassirer 1923, 94)

In turn, Cassirer leaves no doubt that in the logic of concept formation, abstract concepts are "logically prior" to their concrete instances:

[44] See again Shapiro (1997) for closer discussions of *in re* structuralism.

> Concept and judgment know the individual only as a member, as a point in a systematic manifold; here as in arithmetic, the manifold, as opposed to all particular stipulations [*Setzungen*], appears as the real logical prius . . . The determination of the individuality of the elements is not the beginning but the end of the conceptual development; it is the logical goal, which we approach by the progressive connection of universal relations. (Cassirer 1923, 94)

Besides these general remarks, there is also textual evidence in the book that Cassirer viewed the transformation-based approach in geometry as a top-down method of concept formation. In the context of nineteenth-century projective geometry, this becomes particularly clear in his discussion of Poncelet's work and the central idea of a transfer of relational structure exemplified in his principle of continuity:

> Above all, [Poncelet] is concerned to guard the transference of relations, which he assumes as basic, from any confusion with merely analogical or inductive inference. Induction proceeds from the particular to the universal; it attempts to unite hypothetically into a whole a plurality of individual facts observed as particulars without necessary connection. Here, however, the law of connection is not subsequently disclosed, but forms the original basis by virtue of which the individual case can be determined in its meaning. The conditions of the whole system are predetermined, and all specialization can only be reached by adding a new factor as a limiting determination while maintaining these conditions. From the beginning, we do not consider the metrical and projective relations in the manner in which they are embodied in any particular figure, but take them with a certain breadth and indefiniteness, which gives them room for development. (Cassirer 1923, 81)

Thus, whereas in inductive reasoning one infers from concrete instances to general laws, this direction is reversed in modern geometrical demonstrations. Universal geometrical relations are specified independently of their instantiations.[45] In Cassirer's discussion of Klein's group-theoretic account, a similar picture of the logical priority of the groups of transformations over the particular figures is drawn. Consider, for instance, the following remark on the group-theoretic approach in geometry:

> This process has come to its logical conclusion and systematic completion in the development of modern group theory. Geometrical *figures* are no longer regarded as fundamental, as date of perception or immediate intuition. The "nature" or "essence" of a figure is defined in terms of the *operations* which may be said to *generate* the figure. (Cassirer 1944, 24)

This clearly suggests a top-down understanding of Klein's algebraic method of concept formation in geometry. A group of transformations is conceived here in an abstract way, usually expressed analytically in terms of some linear equations, and is thus independent of a particular manifold. It encodes a particular geometrical structure, for the instance the structure of Euclidean of projective geometry, which can be instantiated in different manifolds. Thus, closely analogous to the discussion of formal axiomatics, Cassirer held that Klein's group-theoretic approach gives an abstract specification of geometrical structures independently of their concrete instantiations or representations.

[45]Cassirer, in fact, draws an explicit connection here between Poncelet's new method of reasoning in projective geometry and the general logic of mathematical concept formation outlined in the first chapter of his book. Compare again Cassirer on this point:

> Between the "universal" and "particular" there subsists the relation which characterizes all true mathematical construction of concepts; the general case does not absolutely neglect the particular determinations, but it reveals the capacity to evolve the particulars in their concrete totality entirely from a principle . . . As Poncelet emphasizes, it is never the mere properties of the particular kind but the properties of the genus, from which the projective treatment of a figure takes its start; the "genus," however, here signifies merely a connection of conditions by which everything individual is ordered, not a separated whole of attributes which uniformly recur in the individuals. The inference proceeds from the properties of the connection to those of the objects connected, from the serial principles to the members of the series. (Cassirer 1910, 82)

5. Conclusion

The focus of this paper was on Cassirer's philosophical analysis of the "structural turn" in nineteenth-century geometry. This turn is frequently characterized as the transition to a new understanding of geometry as a pure science of abstract structures. We argued in Section 2, based on the example of duality in projective geometry, that this new conception was largely a consequence of several methodological innovations in the field, in particular the development of modern axiomatics and the systematic use of transformations in geometrical reasoning.

A closer discussion of *Substanzbegriff und Funktionsbegriff* (1910) has shown that Cassirer was not only a perceptive reader of work of relevant mathematicians such as Poncelet, von Staudt, Klein, Pasch, and Hilbert, among others, his philosophical discussion of these methodological developments also led him to formulate a genuinely structuralist account of geometrical knowledge. Thus, Cassirer rightly deserves the title as one of the early structuralist philosophers whose work shows several interesting points of contact with contemporary philosophy of mathematics. It was argued here that Cassirer's position is best viewed as a version of "methodological structuralism" given that it is less concerned with the metaphysical nature of abstract structures than with mathematical practice. Regarding modern geometry, we saw that Cassirer's discussion focused on two structural methods. The first one is Hilbert's use of formal axiomatics presented in *Grundlagen der Geometrie* (1899). The second one is Klein's group-theoretic approach of his Erlangen program which, in Cassirer's view, presents a generalization of the transformation-based approach in modern projective geometry. As we saw, Cassirer gave a decidedly structuralist interpretation of these two approaches to geometry. In particular, he viewed Hilbert's and Klein's methods—that is, the axiomatic definition of spaces and the study of invariants relative to transformation groups—as alternative ways to specify the structural content of a geometry.

In the paper, we focused on two aspects of Cassirer's geometrical structuralism. It was first argued that his discussion of both geometrical methods supports a kind of *top-down* view according to which the relational structure specified by a geometry is "logically prior" to the intuitive and concrete objects or systems instantiating it. The second point considered here concerns Cassirer's notion of "transfer" of relational structure. As was shown, this concept plays a crucial role in his understanding of modern geometry. In particular, Cassirer described the transformation-based reasoning in projective geometry as a form of transfer of the relevant geometrical properties between configurations. In his discussion of modern axiomatics, a comparable notion of transfer occurs in the form of the semantic interpretation of an axiom system in different geometrical systems.

It would be worthwhile to investigate in further detail not only how Cassirer's account is related to modern structuralism, but also to compare it with other philosophers of geometry working in the early twentieth-century. One of Cassirer's contemporaries whose contributions to formal axiomatics are particularly relevant in this respect is Rudolf Carnap.[46] A closer comparative study of their respective structuralist accounts of pure geometry will be work for another day.

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[46]See, in particular, Carnap (2000) and also Carnap (1922) for Carnap's early work on the philosophy of geometry.

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Georg Schiemer
University of Vienna
georg.schiemer@univie.ac.at

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