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After Non-Euclidean Geometry: Intuition, Truth and the Autonomy of Mathematics

Janet Folina

The mathematical developments of the 19th century seemed to undermine Kant's philosophy. Non-Euclidean geometries challenged Kant's view that there is a spatial intuition rich enough to yield the truth of Euclidean geometry. Similarly, advancements in algebra challenged the view that temporal intuition provides a foundation for both it and arithmetic. Mathematics seemed increasingly detached from experience as well as its form; moreover, with advances in symbolic logic, mathematical inference also seemed independent of intuition. This paper considers various philosophical responses to these changes, focusing on the idea of modifying Kant's conception of intuition in order to accommodate the increasing abstractness of mathematics. It is argued that far from clinging to an outdated paradigm, programs based on new conceptions of intuition should be seen as motivated by important philosophical desiderata, such as the truth, apriority, distinctiveness and autonomy of mathematics.

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". . . as the mathematical results shook themselves free from philosophical controversies, they assumed gradually a more stable form, from which further development, we may reasonably hope, will take the form of growth rather than transformation."
(Russell 1897, sec. 46)

1. Introduction

How do mathematics and its philosophy influence each other? Does philosophy support mathematics by providing it with a foundation, or do such efforts impede progress, as Russell suggests? Conversely, how does mathematics influence philosophy? When mathematics is "shaken free" from its philosophical fetters, how does philosophy respond? When does it, as Russell suggests, "borrow of Science, accepting its final premises as those imposed by a real necessity of fact or logic" (Russell 1897, sec. 46)? And when does it resist change, by trying to restrain and steer mathematics in certain directions?

These questions underpin any attempt to understand the 19th century mathematical revolutions and their philosophical impact. Before this time, through Kant, it made at least some sense to think of mathematics as springing from two main sources: geometry and arithmetic (with algebra understood as abstracted and generalized arithmetic or as a symbolic generalization of geometry). Kant produced a philosophy of mathematics roughly based on this two-pronged foundation, which seems engineered for its epistemic symmetry.[1] Though arithmetic is (mainly) based on succession, which is given by the *a priori* form of time while geometry is (mainly) based on the *a priori* form of space, both are synthetic *a priori* in nature, both depend on *a priori* intuition, and the methodology for both involves the so-called "construction of concepts"—the methodology that is distinctive of mathematics, according to Kant. This is what I mean by "epistemic symmetry": despite some methodological differences between geometry and arithmetic, they provide the same *kind* of knowledge (synthetic *a priori*), which is obtained in the same *kind* of way (through construction of concepts by utilizing *a priori* intuition).

Developments in both geometry and algebra, however, challenge the symmetry view and its foundation. Non-Euclidean geometries undermine Kant's idea that there is a spatial intuition, or at least one that suffices to single out Euclidean geometry as *a priori true*. Around the same time, symbolical algebra after calculus challenges the idea that there is any kind of intuition (temporal or otherwise) sufficient to ground the truths of both ordinary arithmetic and the investigation of various types of algebraic structures. The philosophical impact of the two revolutions is slightly different. The emerging sentiment regarding geometry is generally towards empiricism, with particular geometric systems seen as more like applied (sometimes called "mixed") mathematics. In contrast, algebra seems to move philosophy towards a more formalist conception. Nevertheless, whether empiricism or formalism, the net effect is a broad chal-

[1] At least at first glance; there are some who emphasize the asymmetries too, such as the fact that Kant focuses much more on geometry than arithmetic, he does not connect arithmetic very strongly to time or sensibility, and the fact that we use a geometric line to represent time to ourselves. (See, e.g., Shabel 1998, who argues that for Kant "symbolic" constructions, say in arithmetic or algebra, are not a different *kind* of construction from ostensive (geometric) constructions; rather, "symbolic constructions" *symbolize* ostensive constructions in space, entailing that even symbolic constructions presuppose spatial intuition.) For purposes of this paper I begin with a basic interpretation of Kant, which I believe was held by the figures I will be highlighting. (See Section 4 below.)

lenge to Kant's philosophy.[2] The Kantian view is further strained by the proliferation of mathematical subfields (set theory, graph theory, etc.), and by the apparent intermingling of disciplines as in algebraic geometry. By the end of the 19th century it might seem implausible to view mathematics as in any simple way *about*, *limited to*, or *constrained by* spatio-temporal intuition.

Philosophy of mathematics at this time is in some ways as liberated as the mathematics that has shaken itself free of it, with both philosophers and mathematicians considering new ways to make sense of the changes. Alternative views about mathematical reality, truth, method, etc., arise in programs such as logicism, intuitionism, formalism, and platonism.[3] Along with these new philosophies come new ways to understand the relationship between mathematics and natural science.

This paper focuses on how these mathematical and philosophical developments affected Kantian philosophies. By the late 19th century, is there any remaining sense in Kant's view that mathematics has a truth determining content that is constrained by, and tied to, intuition or human cognition via the form of experience? If so, what sense is this; for example, how can one assert that spatio-temporal *a priori* intuition plays a constitutive role in mathematics while accommodating the consistency and real possibility of non-Euclidean geometries? If not—if it is no longer sensible to defend Kant's theory of intuition—then can anything of Kant's general conception of mathematics (and/or cognition) be salvaged? In short, in what ways can Kant be modified, and defended, rather than abandoned after non-Euclidean geometry?

I will compare the ways several figures *adjusted* rather than *rejected* Kant's philosophy, as they wrestled with both the mathematical changes and the philosophical questions that thereby arose. One such adjustment, associated with Marburg Neo-Kantianism, is discussed in recent literature on Cassirer. For example, Heis argues that Cassirer retains Kant's methodological doctrine of the "construction of concepts" while giving up the associated commitment to *a priori* intuition, space and time. So Cassirer takes the second option above: though the theory of *a priori* intuition at its basis is abandoned, there is something worth preserving in Kant's general approach to the philosophy of mathematics (and perhaps cognition even more generally).

This adjustment involves a new spin on Kant's account of the "construction of concepts". For Kant, space and time are the media for constructing mathematical concepts: we construct the concept of "triangle" in space, of "twelve" in time via successive synthesis. So for Kant, "construction of concepts"—the methodology that is distinctive of mathematics—is an activity that lies more or less between two equals, that of concepts and intuition. In contrast, Cassirer, for example, construes the "construction of concepts" as dependent only on concepts, thus detached from intuition. In addition, apriority is also detached from intuition, becoming a category that is relative, embedded, and dependent on science (see Heis 2011; Friedman 2016). Modeled after Dedekind's logicism, Cassirer aims to accommodate the increasingly abstract nature of the subject matter of mathematics.[4]

Another type of reaction involves *modifying* Kant's conception of *a priori* intuition rather than trying to construct a Neo-Kantian philosophy of mathematics without it. That is, rather than *eliminate* intuition owing to challenges to it, some mathematicians—such as Brouwer, Poincaré and Weyl—attempt to *reconceive*, or *reconfigure*, "intuition". Intuition so reconfigured becomes more abstract, less tied to experience, and closer to concepts or cognition, than spatio-temporality was for Kant. For example,

[2]Not just his philosophy of mathematics. Mathematics provides evidence for his general view of cognition as well as the status of philosophy. Without that evidence, the general picture is undermined.

[3]For example, Frege's view that though they are abstract, mathematical objects are just as real as concrete objects—a view distinct from Plato's theory of forms.

[4]See Heis (2011, sec. 3). How this purportedly works, and whether or not it succeeds, I will leave to Cassirer scholars.

Poincaré construes intuition as more of a flexible, "all purpose" cognitive tool; indeed he suggests that it is even required for (the emerging) symbolic logic.

With "intuition" so reconceived, its job is somewhat transformed. Rather than the field or form of sensibility in which objects corresponding to mathematical concepts are "constructed", mathematical intuition primarily takes on the job of domain-determination. That is, intuition determines whether and what concepts determine their objects, and whether and what domain corresponds to a given mathematical concept. Obviously this is a job that lies more on the side of concepts than Kantian intuition, and resembles the role played by schemata for Kant. But I'm getting ahead of myself.

If 20th century "intuition" moves away from space and time in this way, then we might think of the difference between a view like Cassirer's and that of the intuitionists in the following way. While Cassirer sees mathematics in terms of construction of concepts without intuition, the intuitionists see mathematics as governed by intuition without construction of concepts.

It may be tempting to see this as a distinction without a difference. However, I will argue that there are important philosophical differences between the two methods of accommodating the changes in mathematics, and its relationship to natural science. In particular, retaining intuition enables certain core elements of mathematics to remain conceived as isolated from shifts in the mathematical-scientific landscape. Thus for example, perhaps geometry needs to be seen as more closely connected to physics; but that does not mean arithmetic needs to be similarly reconceived. For those who wished to preserve a more traditional vision of mathematics, adjusting rather than eliminating intuition can be understood as advantageous in at least the following, related ways.

First, intuition blocks the potential lure of *holism* about mathematics and science. Second, it does this by providing a more *absolute* foundation for mathematics than, say, Cassirer's construction of concepts without intuition. Third, this is because intuition provides a foundation for mathematics that is *internal* to mathematics, and thus the aspects of mathematics that remain grounded in intuition are regarded as isolated from scientific shifts. Fourth, this approach better preserves the *autonomy* of (at least a core of) mathematics from natural science. Fifth, this is because mathematical truth is *independent of the methodological role* of mathematics in natural science; that is, mathematical truth requires neither scientific applicability nor empirical confirmation. Sixth, this view can thus clearly preserve the *apriority* of mathematics as well as its *distinctiveness* from natural science. These are not distinctions without a difference; they are philosophical advantages of an intuition-based reconstruction of Kant.

2. Overview of Kant's Philosophy of Mathematics

Famously for Kant, mathematical knowledge is synthetic *a priori*. In fact, mathematics is presented as a paradigm case of synthetic *a priori* knowledge in arguing for the legitimacy of the general category, putting philosophy in its good company. (See for example, Kant 1781/87, Preface to the Second Edition, B8, etc.) Though philosophy and math are both synthetic *a priori*, Kant thinks the two differ in methodology. Philosophy is "merely discursive"; philosophers only analyze concepts. In contrast, mathematicians can also "construct their concepts"; as Kant famously claims, "[p]hilosophical knowledge is the knowledge gained by reason from concepts; mathematical knowledge is the knowledge gained by reason from the construction of concepts" (Kant 1781/87, B741).

Construction of concepts is the method of considering an arbitrary object or instance of a mathematical concept, allowing one to advance to general judgments. Of course, general judgments can be valid without constructing any concepts, as is codified in logic, in rules for the universal quantifier. For Kant, however, constructing concepts enables the same universality as is obtained through universal generalization, while also adding to, or supplementing, the information provided by the (explicit) concepts alone.

When I construct (the concept of) a triangle, for instance, I consider it as an object in space and time. So facts about spatio-temporality—such as the existence of space in which to extend a line and the ability to iterate any construction operations—add to the information provided by the concept of triangle alone. Thus, conclusions about *triangles* can on this view contain more information than conclusions about the mere *concept of triangle*. For Kant this is because constructing the concept of triangle—considering triangles as spatio-temporal objects—enables my judgment to be ampliative, or synthetic. Yet the judgment remains *a priori* since the information added comes only from the spatio-temporal form—the *a priori* form of all experience. That is, the added *content* that is distinctive of mathematical judgments comes from the *a priori form* of experience; and "construction of concepts" is for Kant how this addition occurs in mathematics, and is possible *a priori*.[5]

3. The Challenge of Non-Euclidean Geometries

Non-Euclidean geometries provide a paradigm case of mathematical change that demands a philosophical response. They prod philosophy to ask about the nature of mathematics, mathematical knowledge, and the relationship between mathematics and natural science. They also seem to refute Kant's view that Euclidean geometry is necessarily true, as following from the *a priori* form of space. Some reactions that I will compare in this section are empiricism about (at least certain aspects of) geometry, geometric conventionalism, and the related idea of the relative, or non-absolute, *a priori*. Each of these represents a philosophical shift that attempts to accommodate, rather than challenge or obstruct, the mathematical developments. The existence of real geometric alternatives raises the question whether mathematics really is absolutely *a priori*, and each of these reactions acknowledges the new role of natural science in geometry.

One response to non-Euclidean geometry associated with Russell and Helmholtz[6] is to shift some mathematical truth—geometry—to the empirical, folding some geometric content into physics. This entails rejecting Kant's epistemic symmetry doctrine, singling out geometry as the discipline that needs to be understood in a new way.[7] Given the consistent alternatives, geometric axioms can be regarded as (more like) physical hypotheses—an attitude that seems supported by the fact that one needs the conjunction of physics *and* geometry for relevant empirical tests. This empiricist-holist reaction effectively *re-allocates* certain aspects of mathematics to physics; it shifts the determination of geometric axioms, or systems, away from pure mathematics across the boundary to physics.[8]

Conventionalism is a somewhat similar, though subtler, reaction to non-Euclidean geometry. It views geometric systems

[5]This is of course so rough an explanation it is a caricature. There is a huge literature by Kant scholars on this topic. For just a few examples, see Carson (1999), Friedman (1985), and Shabel (1998, 2006).

[6]For example, Helmholtz (1876/78) strikes me as articulating a fairly strong version of empiricism. However, see Patton (2016) for cautions about this characterization.

[7]Geometric empiricism may seem even more attractive after general relativity.

[8]Philosophy of geometry in the late 19th/early 20th centuries is filled with attempts to make sense of this emerging asymmetry in mathematics; conventionalism, empiricism, holism, etc., are each engaged in very similar efforts. (For an account of the subtle relationships between the views of Einstein, Helmholz and Poincaré, see Friedman 2009.) Furthermore, the idea that geometry was more like applied mathematics pre-dates the actual mathematical revolutions. (One can see such remarks by Gauss, Bolzano and others.) So perhaps Kant's epistemic symmetry thesis was never very plausible to the mathematicians, after all.

as measurement systems; as such they are neither *a priori* nor empirical in any straightforward sense. "Convention" is a new scientific category, promoted by Poincaré,[9] for this intermediary status. As conventions, geometric systems are stipulative; however, unlike some conventions, geometric systems are not arbitrary stipulations. Conventionalism accepts that geometry is closely connected to physics, but it does not see geometry as merely incorporated into physics. Rather, along with other conventional aspects of science, geometry maintains a separation from physics owing to its distinctive methodological role.

Scientific conventions arise for Poincaré when there is more than one option, yet the choice between options is neither straightforwardly empirical nor determined by purely *a priori* criteria. Poincaré points out that empirical testing *presupposes* certain conventions, so it cannot *decide* them. He agrees with the empiricist-holist that we can only test the conjunction of physics and geometry. The difference is that the conventionalist emphasizes the fact that some parts of the whole are (and should be) treated differently from other parts. In particular, some parts can be (and usually are) generally isolated from revision. Moreover, what is so isolated is typically not arbitrary; geometry is shielded from revision because it plays a different role in scientific testing than the more empirical parts of physics. In fact, geometry is part of the testing apparatus, or framework, rather than what is tested. This does not mean that conventions are never revised; rather, revising aspects of a testing apparatus involves a *different kind of process* than revising some other content in light of outcomes based on *using* that apparatus.

According to conventionalism, geometric systems and some other fundamental physical principles thus function as a third category between the empirical and the *a priori*. They are not as absolute as what is *a priori* imposed, since we are at times willing to try other conventions, other frameworks. But they are not directly tested because they are part of what makes empirical testing possible. Thus Poincaré carves out a new category for the boundary between mathematics and science, considering some of what used to be considered as mathematics (particular geometric systems), and some of what used to be considered empirical (particular mechanical principles) as "conventions". Importantly, however, it leaves much of science and mathematics conceived as it was: the highly empirical parts of science remain empirical; the core of mathematics remains *a priori*.[10]

Both empiricism and conventionalism are consistent with a view about intuition like Brouwer's (which will be further discussed in the next section). Brouwer famously declares that he gives up the apriority of space but not that of time, building (pure) mathematics only on the foundation provided by the *a priori* form of time (1913, 80).[11] These reactions—empiricism (about geometry), conventionalism (about principles on the border between geometry and physics) and Brouwerian intuitionism—each more or less isolate geometry as the "problem". They segregate geometry, protecting the rest of mathematics from the need for a new foundation, by allowing it to remain conceived as *a priori* in some former sense.[12]

[9]For his explanation of conventionalism, see Poincaré (1898, 1902); for my take, see Folina (2014).

[10]It is important to note that these responses are focused on the determination of particular geometric systems. That is, pure mathematics still includes the study of various geometries; it just cannot determine the "one" that is "true", any more than mathematics can determine the "one" algebraic structure that is "true".

[11]Of course this alters what counts as pure mathematics, since what is "constructible" in, or via, intuitive time may not be identical with what is constructible in space *and* time. On the other hand Brouwer also reconceived time, which changed the grounding in a different way than simply *eliminating* half of it. In fact, Brouwer helps himself to both the continuity of time as well as its "two-oneness". Again, more on this later.

[12]It is important to emphasize that bracketing off geometry in this and other ways is purely philosophical. What it provides is conceptual freedom—from reconceiving the rest of mathematics; it has no direct impact on mathematical *practice*.

A third important reaction to non-Euclidean geometry, however, re-frames the category of apriority as non-absolute, or "relative".[13] Associated with Cassirer, this is a different kind of reaction that is more general and more drastic. Rather than simply yielding some areas of inquiry such as geometry to the empirical, or creating a new intermediary category such as the conventional, this response changes the entire conceptual framework for thinking about mathematics, science, and their relationship. In some sense it takes the conventionalist reaction and extends it to the whole category of the *a priori*. Thus it is a more drastic reaction, since it affects the conceptual framework for all of mathematics. For Kant and others, *a priori* knowledge consists of necessary truths that are knowable independently of experience. In contrast, this view regards the *a priori* as only relatively fixed, determined only by its role in providing a framework for science. When science changes drastically enough so can its framework; the *a priori* is thus not generally permanent on this view.

This conceptual shift enables geometry to retain the (relative) *a priori* label, despite the multiple options—that is, even though no one option is (absolutely) necessary or fixed. Geometry is not absolutely *a priori*, as is shown by the multiple options; but it is "relatively *a priori*" because whichever geometric system is chosen provides part of the framework presupposed in empirical testing. Geometry is thus still "prior to" scientific testing, or the more straightforwardly empirical; so it retains a position that can be seen as in the Kantian tradition.

Reconceiving the category renders the "apriority" of an area of inquiry a question about its methodological role in science rather than its intrinsic properties, such as the fact that its basic truths are necessary or justifiable independently of experience. While it is important to take methodology seriously, the view that all *a priori* knowledge is merely relative is revolutionary. Though perhaps in the Kantian tradition, it nevertheless opposes Kant's view of the *a priori* as necessary, rather than contingent on natural science. Tying mathematics too closely to science—even as its privileged aspects, its "framework"—fails to preserve its *autonomy*, for it seems to make mathematical truth depend on the role or utility of mathematics in natural science. Blurring the boundary between mathematics and natural science, it also fails to reflect the apparent *purity* of mathematical methodology.[14] In these ways the relative *a priori* drastically alters the conception of mathematics as distinct from natural science: in its content, methodology, and epistemic status. Although there is clearly a close relationship between conventionalism and the relative *a priori*, the virtue of the former (as well as empiricism-holism about geometry) is that while it treats particular geometric systems in a new way, it leaves the rest of mathematics alone.[15]

Whether or not geometry is a special case provides an interesting division between responses to non-Euclidean geometry.

[13] This is also a topic I can only mention, on which there is a large and growing literature. I'll just say that merely *including* the relative *a priori* in the category—so dividing apriority into the relative and absolute—is in effect very similar to the partial re-allocation "moves" of geometric empiricism and (especially) conventionalism. In contrast, it is the replacement idea that really contrasts with these other reactions. For important work on the relative *a priori* and its historical roots, begin with Friedman (2001, 2002). That Friedman advocates *replacing* the absolute with the relative *a priori* (rather than merely dividing the *a priori* into the relative and the traditional, absolute) is explicit in his remark that though he views scientific knowledge as differentiated into three levels—the empirical, the relativized *a priori* and the philosophical meta-frameworks which guide our transitions between frameworks—"[n]one of these three levels are fixed and unrevisable. . . " (2002, 20). This is the view of the relative *a priori* that I address here.

[14] Granted, if we can conceive of and interpret non-Euclidean geometry, then neither our concepts alone nor our concepts aided by the *a priori* form of experience determine geometric truth. But reconceiving the category of *a priori* knowledge because of some changes in it seems somewhat drastic.

[15] That is, conventionalism about geometry need not reverberate throughout all of mathematics; it does not imply the conventionality of arithmetic, for example. Again, this is a point about the *philosophy* of mathematics only, not mathematical *practice*.

Does non-Euclidean geometry mean that the nature and role of *all* of mathematics needs to be reconsidered; does it undermine the entire category of the *a priori*, or the whole of a philosophy of mathematics? Does it refute Kant's philosophy? Or can the damage be isolated, leaving much of mathematics conceived in some general Kantian way? Reconceiving the *a priori* as relative seems an example of the former, while limiting empiricism and conventionalism to geometry enables a more Kantian approach to the rest of mathematics—one that retains a limited or reconceived theory of intuition.

Another interesting division is between philosophical responses that merely *accommodate* mathematical changes and reactions that also aim to *steer* future mathematics. The views about geometry so far discussed can be seen as philosophy *following* mathematics, and accommodating the changes that occurred in it. They are reactions that, as Russell said, "borrow of Science, accepting its final premises as those imposed by a real necessity of fact or logic" (Russell 1897, sec. 46). We might now call such responses "naturalistic", or "second" philosophy.

Other responses see the role of philosophy of mathematics as more "first", proactive, and more traditional. For example, the new intuitionists not only react to the existence of alternative geometries; their programs address the rest of mathematics too. That is, rather than merely accommodating the mathematical changes presented by non-Euclidean geometries (and other mathematical developments), they also aim to *steer* mathematics, to actively direct it, and even to limit it, to reflect their philosophical views.

Thus, the new intuitionists play a double role in my analysis. First, both intuitionism and semi-intuitionism provide important examples of adapting, rather than abandoning, Kant's theory of intuition. The conceptions of intuition in question are slightly different from one another, but they each aim to preserve a basic traditional view: one that yields the independence of mathematical truth from natural science, and one that supports the absolute apriority of mathematics. Second, they all preserve a more proactive, prescriptive, traditional role for the philosophy of mathematics.

4. Intuition and the Absolute *A Priori*

Towards the end of the 19th century, some new philosophies of mathematics arise in response to the mathematical developments discussed above. As we saw, non-Euclidean geometries undermine Kant's vision of the epistemic symmetry of arithmetic and geometry, as well as the theory of *a priori* intuition at its basis. If, then, geometry is less pure and less fundamental than arithmetic, there are two main strategies for a Kantian "rescue" regarding intuition. One can retain the Kantian view that intuition supports geometry while offering a new account of arithmetic, or one can reject Kant's view that intuition determines Euclidean geometry but retain Kantian intuition for arithmetic.

Most defenders of intuition at this time take the second path, dropping geometry to rescue a limited form of Kantianism for the rest of mathematics, in particular number theory. That is, the effect of non-Euclidean geometry on Kant's epistemic symmetry view is to "push geometry down", allowing the rest of mathematics to stay at its former epistemic level, with its former philosophical grounding (however these are conceived). Frege takes the opposite path: he retains Kantian intuition for geometry and instead reconceives arithmetic as logical. So, in his vision, geometry remains at its former level with its former philosophical grounding, and instead Frege aims to "lift arithmetic up" a level from that of geometry to that of logic. Interestingly, he regards non-Euclidean geometry as evidence for geometric intuition: unlike arithmetic, for which there is only one option, the consistent geometric alternatives illustrate, for him, that intuition is required in order to single out *which* alternative is true. "For purposes of conceptual thought we can always assume the contrary of some one or other of the geometrical axioms, without involving ourselves in any self-contradictions . . . " (1884, sec. 14). The alternatives are consistent because geometric axioms are synthetic, not analytic truths; intuition is deployed to select, or determine, the Euclidean system as true. For example, Frege regards thinking about non-Euclidean geometries as using "still the same old intuition of Euclidean space, the only one whose structures we can intuit" (1884, sec. 14). In other words, with Kant, Frege believes we can *conceive* of non-Euclidean geometries, but we can only *intuit* Euclidean geometry.[16] It is in this sense that the view constitutes a Kantian rescue (for geometry).

It is a view that contrasts with intuitionism in its characterization of arithmetic as well as of geometry. Frege's program, logicism, "rescues" Kantianism about geometry in part by "promoting" or elevating arithmetic to the logical. This view of arithmetic truth as logical was particularly offensive to the intuitionists for they regarded logical truth as *empty*. In contrast, they saw arithmetic truth as substantive, a genuine body of knowledge, and thus synthetic. Their defense of Kantian intuition centers on non-geometric areas of mathematics, with a particular focus on arithmetic and what it can ground.

Thus, rather than promoting arithmetic to the logical, the intuitionists' Kantian rescue "demotes" geometry as a first step in preserving Kant's vision of mathematics. As Brouwer articulates it:

> But the most serious blow for the Kantian theory was the discovery of non-euclidean geometry, . . . However weak the position of intuitionism seemed to be after this period of mathematical development, it has recovered by abandoning Kant's apriority of space but adhering the more resolutely to the apriority of time. (Brouwer 1913, 80)

So both logicism and intuitionism proceed from the same rough basis: that of rejecting the epistemic symmetry view associated with Kant. Whether arithmetic is "promoted" to the logical or geometry is "demoted" to the empirical (or the conventional), the effect is in some ways very similar: both sides recognize arithmetic as more fundamental than geometry. Here, however, the views diverge sharply.

Of course, merely demoting geometry would not suffice to preserve Kant's philosophy. First, Kant himself said very little about the role of time in the methodology of arithmetic and algebra. There are a few remarks, for example, the famous passage in the *Prolegomena* about the successive synthesis in time in knowing $7 + 5 = 12$. Kant also mentions the importance of "symbolic construction" for algebra (in contrast with ostensive construction for geometry) in the *Critique*. But these few passages are insufficient to flesh out *how* processes like successive addition or symbolic construction are anything like constructing triangles, lines and circles. In addition, Kant's emphasis on successive synthesis in determining the general concept of number takes place in the *Schematism* rather than in a section featuring intuition. It is thus unclear what is left of the Kantian philosophy of mathematics without spatial intuition and geometry.[17]

Second, and perhaps more importantly, mathematics had already moved into quite abstract territory by the end of the 19th century. Symbolical approaches to algebra freed it from its arithmetic fetters, just as geometry was freed from its Euclidean fetters; and both developments occurred despite philosophical

[16] Another source of asymmetry for Frege is that arithmetic is more general and abstract than geometry, and he notes that it is "closer to the laws of thought". But being close to the laws of thought does not in itself rule out intuition in arithmetic. As we shall see, others around this time endorsed just such an intuition—one that is very abstract and fundamental to systematic thinking—as part of the grounding for arithmetic. Thus, Frege's positive arguments for logicism were essential for his project of refuting arithmetic intuition—though of course his particular version was doomed.

[17] Hamilton notes and attempts to fill in this gap in Kant's own work, rather unsuccessfully in my opinion. See Hamilton (1837) and/or Folina (2012).

concerns about meaning, reference and truth.[18] Any theory that defended Kantian intuition for arithmetic would not only need to detach spatial-geometric intuition from temporal-arithmetic intuition; it would also have to offer a new version of the latter. This is precisely what the new intuitionists do.

In the rest of this section, I sketch and compare the ways several mathematicians adapted Kant's concept of intuition to ground the non-geometric areas of mathematics. Each aimed to preserve the traditional view that pure mathematics is *a priori*, autonomous (independent of natural science), a realm of truth, etc. Though the concepts of intuition vary slightly, they are thus each put to similar uses: to ground a fixed basis for core areas of mathematics, to preserve the apriority of these areas, and to balance the autonomy of mathematics with the importance of its (potential) applicability and its connection to natural science.

4.1. Brouwer

Frege elevates, or "promotes", arithmetic to the logical—thus independent of intuition—while preserving Kantianism about geometry as based on the *a priori* intuition of space. In contrast, Brouwer lowers, or "demotes", geometry—eliminating traditional (synthetic) geometry from pure mathematics—to preserve Kantianism about other areas of mathematics. Thus Brouwer sees the threat of non-Euclidean geometry to Kant's philosophy as isolated, as applying only to the apriority of space. And he blames the emerging asymmetry of geometry and arithmetic on geometry. He thus gives up spatial intuition, but he retains Kantian temporal intuition as foundational for mathematics. So, Brouwer's reaction is more or less the opposite of Frege. Frege preserves Kantianism about geometry, elevating arithmetic to logic; Brouwer preserves Kantianism about non-geometric areas of mathematics, demoting synthetic geometry to the empirical.[19]

As mentioned earlier, Brouwer calls non-Euclidean geometry a "serious blow" to the Kantian theory of mathematics, but that, though weakened, "the position of intuitionism. . . . has recovered by abandoning Kant's apriority of space but adhering the more resolutely to the apriority of time" (1913, 80). Thus Brouwer states explicitly that the position he is calling "intuitionism" can be built on the Kantian theory of the apriority of time alone.

He famously describes how this works as follows.

> This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time, as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely; this gives rise still further to the smallest infinite ordinal number ω. Finally this basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i.e., of the "between," which is not exhaustible by the interposition of new units and which therefore can never be thought of as a collection of units. (Brouwer 1913, 80)

Brouwer further asserts that the apriority of time grounds the study of various geometric systems as well as arith-

[18] An interesting example is when the British algebraists finally "shook" algebra free of its Berkeleyan "controversies" in the mid-1800s. Until then, such philosophical concerns seriously impeded mathematical progress in Britain. "Impossible" numbers do not trouble the new intuitionists because they do not require the individual objects of mathematics to be "intuitive". Intuition at this point becomes process-oriented rather than object-oriented.

[19] Or at least the more empirical; pure mathematics includes only the study of geometrical systems via arithmetic means and does not yield the truth of any one system.

metic and algebra—since geometric systems can be represented arithmetically—and thus all of these areas of mathematics are synthetic *a priori*. So time but not space is *a priori*, and it suffices to ground all of genuine pure mathematics for Brouwer.

Does Brouwer simply retain Kantian time, while eschewing Kantian space? This seems partly right. Kant and Brouwer agree that all experience is temporal, and that time as the *a priori* form of all experience provides a foundation for mathematics. But there are at least two differences. First, Brouwer's conception of time is arguably both more phenomenal (rather than objective/inter-subjective) and more tied to the human intellect (rather than sensibility) than Kant's. Second, whereas Kant is frustratingly silent on the exact role of time in mathematics (focusing mainly on the activity of mental *synthesis*, which takes place in time), Brouwer connects the nature of temporal experience somewhat more explicitly to its role in grounding mathematics.

According to Brouwer, the "fundamental phenomenon of the human intellect" is that it has a temporal form. Not only do our experiences occur in phenomenal time, but our thoughts emerge from this temporal basis too. We experience things as connected via time,[20] but also as separated by time, since experience is sequential. So we experience one thing and then another. Since thoughts are intellectual experiences, the same goes for them.

The first step towards mathematics is to abstract away from the content of these experiences. So, say I experience the taste of pizza and then the taste of wine. The first step towards mathematics is to abstract the sensorial aspects of the pizza taste and wine taste from those experiences. If I abstract enough content, what is left is simply one experience and then another experience; thus what is left is something more formal—the form of: one thing and another; or first and then next, or second. This is what Brouwer calls a "two-oneness". Because we could (and most of us presumably do) have another experience after the second, the relationship between the second and the next forms another two-oneness; and so on.

Brouwer then simply asserts that this process, which "creates" some finite ordinal numbers, can be repeated indefinitely. This can be thought of as the next step towards mathematics—essentially the iteration of the basic successor function. The natural number structure thus emerges as the (abstracted and iterated) temporal structure of experience—when we focus on the *separation*, the differences, between two experiences.

Additionally, Brouwer regards time as supplying the background *connectedness* between the experiences that occur at these different times. So mathematics has access to continuous domains as well as discrete domains via the single intuition of time.[21] Although only discrete mathematical systems can be regarded as sets or collections, this enables more of mathematics to be grounded in the one *a priori* intuition, time. Thus, for Brouwer, time is continuous, always supplying a background "between" any two experiences. And sets are "lifted out" of this background owing to the fact that we can abstract discrete number structures by iterating the basic structure left from the *imprint* of successive experiences on the form of time. In these ways Brouwer has contributed quite a bit of detail in explaining how mathematical objects and domains can be thought of as emerging from, and reliant on, intuitive time. Mathematical objects, domains, systems etc. can—roughly—be thought of

[20] I take it that this is because the entity having those experiences also experiences herself as a single person, thereby unifying and connecting the individual experiences into a single consciousness.

[21] Brouwerian time can be envisioned as a kind of idealized measuring stick, indefinitely long and unfolding, or emerging, at one end. Each person has her own such stick and the marks on the stick are "drawn" from the person's experiences. These are separated from each other by their position on the stick, but connected since they are all on the same stick. If this is the right kind of analogy, one can immediately see that there are worries about the solipsistic nature of mathematics on Brouwer's conception.

as idealized and generalized temporal structures of our experiences.

4.2. Poincaré

The same contrast with Frege applies to Poincaré. Frege elevates arithmetic to the logical—thus not in need of intuition—while preserving some form of Kantianism about geometry. In contrast, like Brouwer, Poincaré downgrades geometry[22] in order to preserve Kantianism about other areas of pure mathematics, including arithmetic. However, in addition to defending conventionalism about geometry, unlike Brouwer, Poincaré further modifies the *a priori* intuitions that he appeals to in defending Kant.

Rather than space and/or time, Poincaré's fundamental mathematical intuition is that of indefinite iteration; he also claims the intuitive continuum as an important source of mathematical information. The latter is closer to a Kantian conception of intuition (space) than the former, though it plays a less central role in his philosophy of mathematics. Poincaré claims that the intuitive continuum is necessary to organize "brute sensation" into the coherent experiences that we have (1913, 44). The intuitive continuum is thus more explicitly tied to sense experience for Poincaré. In contrast, the intuition of indefinite iteration seems more like a form of cognition, or thinking, rather than the form of experience, or sensibility.[23]

The intuition of indefinite iteration is thus particularly interesting, for it illustrates an important shift in thinking about "intuition" from Kant's spatio-temporality. Poincaré appeals to the intuition of iteration in his early arguments against logicism (e.g., 1894), though he does not explicitly defend this intuition until later writings. His eventual defense (e.g., in his 1905–06 papers) is fairly sophisticated; and he retains intuition as a way to support a semi-Kantian philosophy of mathematics throughout his philosophical work.[24] Iteration also links Poincaré's mathematical intuition to those of both Brouwer and Weyl: Brouwer, because he relies on the fact that we can *indefinitely iterate* the mathematical process extracted from intuition, and for whom time is the *a priori* form of the *intellect*; Weyl, because he follows Poincaré in regarding iteration as a fundamental mathematical intuition.

A main focus related to this intuition is the principle of mathematical induction. Poincaré says induction is mathematical reasoning "par excellence", and "only the affirmation of the power of the mind which knows itself capable of conceiving the indefinite repetition of the same act when once this act is possible . . . " (1902, chap. 1, part 6). Since intuition here is of a mental power, it is clear that this conception is quite different from Kant's forms of experience, space and time. For Poincaré, intuition provides a kind of self-insight into the mind's power to apprehend infinite domains via indefinite iteration of their defining operations. That is, we can apprehend a domain as infinite when we "see" that the construction acts that define or produce it can be repeated without end. And the ability to conceive of the domain in this construction-iterating way is also what grounds our knowledge that induction results in true conclusions about these domains.

Poincaré's remarks connecting intuition to induction relate also to his general views about what makes a domain of objects definite, or determinate. He contrasts two approaches to infinity: that of the "Cantorian" and that of the "pragmatist" (assigning himself to the latter group). He also refers to the issue of "extension":

> The pragmatists adopt the point of view of extension, and the Cantorians the point of view of comprehension. (When a finite

[22] To the conventional rather than the empirical, but this is still a demotion.

[23] A question to be dealt with at another point is if so, whether it is really an "intuition" in Kant's sense. This question, however, applies to all post-Kantian accounts of intuition.

[24] For an explication and defense of these arguments see Folina (2006).

collection is concerned, this distinction can be of interest only to the theorists of formal logic; but this distinction seems to us much more profound when infinite collections are concerned.) If we adopt the point of view of extension, a collection is formed by the successive addition of new members; we can construct new objects by combining old objects, then with these new objects construct newer ones, and if the collection is infinite, it is because there is no reason for stopping. From the point of view of comprehension, on the other hand, we begin with the collection in which there are pre-existing objects . . . (Poincaré 1913, 67–68)

And continuing on the next page:

> Another source of divergence arises from the manner of conceiving the definition . . . Let us note in passing that there are definitions which are incomplete in the sense that they do not define a particular thing but rather an entire genus. They are legitimate and they are even the ones most frequently used. But according to the pragmatists it is necessary to understand therein the set of the particular objects which satisfy the definition and which could finally be defined in a finite number of words. (For the Cantorians this restriction is artificial and devoid of meaning.) (Poincaré 1913, 69)

Poincaré here addresses the issue of the proper relationship between mathematical concepts and sets. Under what conditions does a definite set exist? When does a concept determine a set?

Though he is arguing against realism here, he is not endorsing strict constructivism. In the second quote, for example, he admits that some acceptable definitions do not provide explicit construction-instructions for the objects concerned. In fact, he agrees that one can define an entire "genus" at once. Nevertheless, as a semi-constructivist, he thinks even general definitions of this sort should provide insight into the nature of the individual objects so determined. The first quote tells us a little bit about the conditions that provide this insight.

For finite sets, as Poincaré admits, the difference between the realist and the anti-realist is not important. But our insight into the objects that form an infinite set relies on what he calls the point of view of "extension" or composition. Collections are formed by successive constructions, and infinite collections are just those for which there is "no reason for stopping". Thus, our insight into infinite sets comes, if not from explicit constructions, at least from understanding how we *would* define/construct/access the individual objects if we wanted. And such an understanding is provided by intuition—the same intuition that underpins mathematical induction. So for Poincaré, intuition is what grounds our understanding of infinite mathematical sets.

As I mentioned, Poincaré also appeals to a second intuition—of the "intuitive continuum". This is not an operational intuition; furthermore, it creates even more distance between his and a strict constructivist view. Yet its epistemological role is similarly to ground the mathematical commitment to definite domains—such as the real numbers—on the basis of an idea (the continuum). That is, just as the intuition of iteration grounds our commitment to a definite simply infinite set; so the intuitive continuum grounds our commitment to a definite domain of real numbers.[25] In this way, intuition generally shifts for Poincaré towards something that grounds the practice of associating certain concepts with determinate mathematical domains.

4.3. Weyl

In *The Continuum*, Weyl produces a predicative system of analysis based on arithmetically definable properties only (and on restrictions to the quantifiers). In response to the set theoretic

[25] And indeed other n-dimensional continua, such as topological spaces. It must be emphasized that such domains are not *sets* for Poincaré. This is because they are not formed by a collecting activity; nor do they provide objects to which further set theoretic operations can be applied. So, for example, he agrees that the real numbers do not have the same cardinality as the natural numbers; but he denies that they have a higher cardinality, by denying that they form a collection with a cardinality in the first place. The same reserve would apply to other continua, such as spaces.

paradoxes he aimed to build a theory of real numbers in a strictly non-circular constructivist way. Predicative analysis loses some results from classical analysis but at the time he "saw no other possibility" as he famously remarks:

> It is not the purpose of this work to cover the "firm rock" on which the house of analysis is founded with a fake wooden structure of formalism—a structure which can fool the reader and, ultimately, the author into believing that it is the true foundation. Rather, I shall show that this house is to a large degree built on sand. I believe I can replace this shifting foundation with pillars of enduring strength. They will not, however, support everything which today is generally considered to be securely grounded. I give up the rest, since I see no other possibility. (Weyl 1918, 1)[26]

His so-called "genetic" point of view limits mathematics to what can be built explicitly (at least in principle), from some basic operations/intensions and—for Weyl—some basic set like the natural numbers. Though Weyl articulates this differently, with Poincaré intuition is central to his account of what makes a set of objects definite.

Weyl helps himself to the basic set, $\mathbb{N}$, which already depends on intuition in his view. "The intuition of iteration assures us that *the concept 'natural number' is extensionally determinate*" (110). New mathematical objects are then produced via "the mathematical process" by explicit definitions on $\mathbb{N}$.

Weyl ties the nature (and thus level) of an object to its definition, and each type of definition on a set creates a new "sphere of existence". So for Weyl, although the set of natural numbers and the set of rationals $m/1$ are very similar, their objects are not identical (61). This may seem strange to us, but for Weyl the *way* an object is defined determines what object it *is*. So the natural number m and the rational number $m/1$ are not identical objects. Thus for him there is a very close relationship between a concept or definition, and its extension, the set so defined.

[26] All citations in this section are to Weyl (1918).

Nevertheless, it is important to note that set identity remains extensional for Weyl:

> How two *sets* (as opposed to *properties*) are defined . . . does not determine their identity. Rather, an objective fact . . . is decisive; namely, whether each element of the one set is also an element of the other, and conversely. (Weyl 1918, 20)

So sets, and the objects of mathematics in general, are extensional as in classical mathematics. And the mathematical process is the formation of new sets, or extensions, from properties defined on old sets (28).

What governs which properties form new sets? Those sanctioned by intuition. For Weyl, "[n]o one can describe an infinite set other than by indicating properties which are characteristic of the elements of the set" (23). Furthermore, "the sense of a concept is logically prior to its extension" (110). So the requirement of concepts for sets is not a mere human weakness, or epistemological requirement; it is a *logical* requirement. Finally, not all senses of concepts correspond to "extensionally determinate" sets (109).

Weyl appeals to intuition for his basic category and also to justify his restriction to explicit definitions on this category.

> The intuition of iteration assures us that *the concept "natural number" is extensionally determinate*. (Certainly, every version of arithmetic must extract *this* basic fact from intuition.) However, the universal concept "object" is not extensionally determinate—nor is the concept "property", nor even just "property of natural numbers." (Weyl 1918, 110)

Both "object" and ''property of natural numbers" are too open-ended to yield definite extensions.

Weyl and Poincaré thus have a similar vision of the epistemic role of intuition (though Weyl is more explicit)—that of grounding how and which concepts determine their domains. They also both focus on the intuition of indefinite iteration as the primary mathematical intuition. Iteration provides a basis for mathematics in the form of insight into the natural number structure. In addition, iteration is foundational for any insights gained through symbolic logic, "since logical inference consists of iterating certain elementary logical inferences . . . " (19; see also 48). Thus, both Poincaré and Weyl reconceive intuition as a more general epistemic tool—a tool that both mathematics and logic depend on, a tool for thinking rather than for providing the sensorial background for "constructing" concepts. This seems quite a big shift from Kant's spatio-temporality.

4.4. Analysis

I have described the reactions of several mathematicians to mathematical developments of the 19th century, especially non-Euclidean geometry. Each aimed to preserve some aspects of Kant's philosophy of mathematics they thought were correct. Each renounced the apparent symmetry of arithmetic and geometry to do so, showing that they conceived Kant's appeals to space and time to be fairly independent of each other.[27]

Frege preserves intuitive space, arguing that intuition is needed to single out Euclidean geometry from other, non-Euclidean, options. However, he eliminates intuitive time for mathematics, attempting to show that arithmetic (and later, analysis) is analytic, a development of logic, rather than synthetic, and a development of intuition.

Brouwer's position can be seen as the opposite of Frege. He gives up intuitive space, arguing that non-Euclidean geometry shows that Kant was wrong about it, since there is more than one viable option.[28] But he preserves intuitive time, which for him retains a fairly Kantian connotation. He argues that time yields various mathematical structures (both discrete and continuous) when we abstract from its experiential content and iterate the formal relationships, or structures, revealed by this abstraction process.

Poincaré and Weyl both move quite a bit further from Kantian spatio-temporality than both Frege and Brouwer. For them it is neither space nor time, but indefinite iteration that is highlighted as the central mathematical intuition. Of course iteration takes place in time (at least for us), but it is iteration rather than time that is identified as the main *a priori* intuition for mathematics. Neither stresses time as the source of iteration, which is thus rather further from Kantian intuition than mathematical intuition is for Brouwer. Though like Brouwer, they also both allude to the intuitive continuum, unlike Brouwer this is mentioned as a second intuition in addition to iteration, rather than a property of intuitive space or time.

Here Poincaré and Weyl part company. For Poincaré the intuitive continuum has a spatial connotation: we have "the intuitive notion of the continuum of any number of dimensions whatever because we possess the capacity to construct a physical and mathematical continuum". This is *a priori* because the intuition is strictly speaking "merely the awareness that we possess this faculty" for constructing continua (1913, 44). This thus links the intuitive continuum to the intuition of iteration, which is also merely the *awareness* of our ability to indefinitely repeat certain operations.

For Weyl, in contrast, the continuum he wished to capture mathematically has a temporal connotation. The intuitive continuum is given in the "constant form of my experiences of consciousness by virtue of which they appear to me to flow by successively" (1918, 88). Though he also applies his ideas to continuous spaces, the source of the intuitive continuum is phenomenal time. In addition, Weyl argues at this point (in a further contrast with Poincaré) that the mathematical process

[27] See note 1 above.

[28] Of course, Brouwer and others were writing later than Frege (1884); they had the benefit of more work supporting the real possibility of non-Euclidean geometries.

cannot completely represent the phenomenal, intuitive continuum. Regarding this, he cites Bergson for the "deep division" between mathematics and our experience of the continuity of time (90).

Despite the differences in connotation (spatial versus temporal), for both Poincaré and Weyl, the continuum is a separate intuition, or resource, distinct from the intuition of indefinite iteration. Yet there is a final divide between them on this matter. Poincaré cites as evidence of a foundation in intuition, our ability to construct physical and mathematical continua; one might well argue that this was rather *ad hoc*, designed to fill in and support the mathematics and science on which he worked. Weyl's intuitive continuum, in contrast, eludes scientific-mathematical description—a problem that drove him from one foundation to another, from predicativism to Brouwer's intuitionism then to formalism, or axiomatics.

We see, then, the wide variety of attempts to defend Kant's philosophy of mathematics. Uniting the intuitionists, though, is their common vision and goals. They each strive to uphold Kant's view that *a priori* intuition is necessary for mathematics, by focusing on non-geometric areas of mathematics, and "demoting" geometry to the empirical or conventional. And they all oppose alternative foundations for the non-geometric areas, such as logicism, formalism and platonism. They see geometry as a special case, one that does not imply anything new about the nature of the rest of mathematics. And for the rest of mathematics they aim to preserve a traditional philosophical approach: one that not only accommodates existing mathematical changes, but also guides and directs future mathematics.

5. Conclusion

I have tried to explain some of the pressures on philosophy of mathematics after Kant, in light of mathematical developments during the 19th century. I've particularly focused on those who are inclined towards a Kantian account of mathematical knowledge, and for whom the mathematical changes necessitate some fairly significant philosophical adjustment. Perhaps, as I think, philosophical issues are what often motivate mathematics to shake itself free from an old conception and its fetters;[29] but philosophy, in turn, must also react to and reflect the changes that thereby come about, as Russell asserted.

The philosophical accounts I have highlighted share the idea of *adjusting*, rather than *eliminating*, *a priori* intuition as a foundation for mathematics—despite its alterations. Though each view has its own flavor, they share some important common features. They also represent more proactive philosophies that aim to guide and restrain mathematics, rather than to merely accommodate it. What remains is to defend these efforts—at least so we understand why someone might adopt this approach. Why might a foundation based on an adjusted *a priori* intuition be preferred over other reactions such as the relative *a priori*, formalism or holism-empiricism?

As I've interpreted them, our central figures reconceive intuition to be less tied to sensibility and more tied to cognition. Of the three here considered—Poincaré, Brouwer, and Weyl—it is Poincaré and Weyl who most alter the concept of intuition that they defend. Yet, even Brouwer's "two-oneness" seems a more cognitive, structural conception of mathematical intuition than Kantian space or time (despite Brouwer's appeal to intuitive time in explaining two-oneness). The question remains, why "water down" Kantian intuition in this way; with intuition so weakened, or altered, do these views really count as defending Kant? Are they plausible, or at least appealing?

The new conceptions of intuition are more abstract, or operational, which admittedly moves intuition away from a strict

[29] That is, the impetus for mathematical change is often philosophical in nature, even if it is mathematicians who are doing the philosophizing.

Kantian approach, where intuition is so tied to sense experience as to be the form of *sensibility*. However, the benefit of this move is that the more abstract conception arguably grounds a wider domain of mathematics. That is, mathematics had already changed; as stated above, simply dropping geometry out of the domain of the synthetic *a priori* would not suffice to ground its increasingly formal and abstract subject matter. The only options were to reject or adjust intuition.

Though the proposed concepts of intuition are somewhat new, the epistemic role of intuition remains basically Kantian for Brouwer, Poincaré and Weyl. None of them are "realist" so they share with Kant the idea that mathematical knowledge is not a reflection of an independently existing reality. Instead, mathematics is governed and constrained by what we *bring to* experience in order to cognize it; and intuition is one of the things we so bring. Though modified more towards cognition than sensibility, the new conceptions preserve these general Kantian roles for intuition. They also enable a conception of mathematics that is Kantian in the following more specific ways.

First, the intuition grounding mathematics remains *a priori*, and it is purportedly common to all humans. Second, intuition is thus independent of shifts in both natural science and conceptual frameworks. Third, intuition delivers an account of mathematics that is substantive but necessary, i.e., synthetic *a priori*. Fourth, and finally, intuition governs the relationship between concepts and their objects or domains.[30] Thus, as for Kant, intuition remains part of an explanation of our (human) epistemic access to the objects falling under a given mathematical concept. It also remains part of an explanation of the stability and definiteness of mathematical domains *without realism*. The benefits of adapting Kantian intuition over other philosophies of mathematics that rejected it—such as axiomatics, empiricism and/or the relative *a priori*—are therefore substantial. Intuition avoids the main pitfalls of realism, while preserving the traditional conception of mathematics as a domain of truth that is *a priori*, absolute, and substantive, with an internal methodology that renders it autonomous from natural science. The attractions seem obvious.

[30] For example, iteration grounds domains like the natural numbers for Poincaré and Weyl; time grounds both continuous and discrete domains for Brouwer; and the absence of a separate intuition of sets prohibits domains such as the set of all objects or all sets.

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Janet Folina
Macalester College
folina@macalester.edu

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