Journal for the History of Analytical Philosophy

Volume 6, Number 3

jhaponline.org

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Francesca Biagioli

Hermann von Helmholtz's geometrical papers (1868–1878) have been typically deemed to provide an implicitly group-theoretical analysis of space, as articulated later by Felix Klein, Sophus Lie, and Henri Poincaré. However, there is less agreement as to what properties exactly in such a view would pertain to space, as opposed to abstract mathematical structures, on the one hand, and empirical contents, on the other. According to Moritz Schlick, the puzzle can be resolved only by clearly distinguishing the empirical qualities of spatial perception from those describable in terms of axiomatic geometry. This paper offers a partial defense of the group-theoretical reading of Helmholtz along the lines of Ernst Cassirer in the fourth volume of *The Problem of Knowledge* of 1940. In order to avoid the problem raised by Schlick, Cassirer relied on a Kantian view of space not so much as an object of geometry, but as a precondition for the possibility of measurement. Although the concept of group does not provide a description of space, the modern way to articulate the concept of space in terms of transformation groups reveals something about the structure and the transformation of spatial concepts in mathematical and natural sciences.

Special Issue: Method, Science, and Mathematics: Neo-Kantianism and Analytic Philosophy
Edited by Scott Edgar and Lydia Patton

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1. Introduction

Hermann von Helmholtz played a key role in the neo-Kantian project of a renewal of Kant's transcendental philosophy not least because of his objections to Kant in the light of later scientific developments. In the eyes of his contemporaries and, more recently, according to scholars such as Friedman (1997), Ryckman (2005), DiSalle (2006), Hyder (2009), and Patton (2014), Helmholtz formulated both compelling objections to Kant and rejoinders to such objections within the framework of a Kantian epistemology. More generally, his epistemological work is a *locus classicus* for clarifying the issue of which aspects of Kant's philosophy are up to the best current scientific theories and which ones should be refurbished or even rejected on account of theory change. However, this issue becomes more controversial, when it comes to spelling out such aspects.[1] One of the reasons for this, I believe, is that Helmholtz's arguments received different interpretations in the light of mathematical procedures that were developed only later or were not available to him at the time he wrote. Therefore, I have argued elsewhere that a more comprehensive study of the reception of Helmholtz in neo-Kantianism is no less essential to a correct assessment of the aspects of his epistemology that do admit a Kantian interpretation (Biagioli 2016).

This paper deals with the more particular case of the reception of Helmholtz in the debate on the articulation of the concept of space in terms of transformation groups. The question whether Helmholtz foreshadowed such an approach relates to the so-called "Riemann-Helmholtz-Lie problem of space," that is, the problem of determining the necessary and sufficient conditions for a Riemannian metric of constant curvature. As this label suggests, the problem first posed by Helmholtz (1868) found a solution only in the third volume of Sophus Lie's *Theorie der Transformationsgruppen* (1893)[2]. Accordingly, it became commonplace to attribute to Helmholtz implicitly group-theoretical considerations also with regard to his empiricist philosophy of geometry.[3] Helmholtz maintained that the axioms of geometry have an empirical origin in our observations of the behavior of solid bodies rather than being *a priori* knowledge in Kant's sense (i.e., necessary and universally valid). Helmholtz's objection to a Kantian method of approaching measurement based on the necessary precondition of Euclidean geometry was that non-Euclidean displacements are imaginable under different empirical conditions. Nevertheless, Helmholtz considered the possibility of generalizing the Kantian notion of the form of spatial intuition to a manifold of constant curvature, which would include Euclidean and non-Euclidean geometries as special cases. In other words, Helmholtz's objection to Kant seems to amount to the fact that the general properties of space can be identified as the invariants of a larger group of transformations than the Euclidean group.

Although much of what Helmholtz says can be given a consistent reading along these lines, several problems arise in the above reading. Which are exactly the general properties of space? How are they distinguished from the more specific ones?

[1] On Hermann Cohen's and Alois Riehl's opposed readings of Helmholtz see Biagioli (2014). More details about the controversial aspects in the literature on Helmholtz's characterization of space will be given in Section 3.1.

[2] Translated by Joël Merker as *Theory of Transformation Groups: General Properties of Continuous Transformation Groups. A Contemporary Approach and Translation* (2015).

[3] See, e.g., DiSalle (2006, 77–78): "Poincaré's group-theoretical account of space (Poincaré 1902, 76–91) is only a psychologically more detailed, and mathematically more precise, articulation of Helmholtz's brief analysis."

Given infinite logical possibilities, what are the criteria for selecting a hypothesis or a class of hypotheses when it comes to the actual structure of space?

I will rely on Moritz Schlick's reading of Helmholtz to argue that these problems did not find an adequate solution in the more recent literature. I will argue for a partial defense of the group-theoretical reading in terms of Cassirer's historical reconstruction of the problem of space, according to which the very idea of using transformation groups to articulate the notion of space served the purpose of exploring new hypotheses.

The first part of the paper discusses some of the evidence for the group-theoretical reading of Helmholtz in his epistemological writings. Subsequently, the main argument for using group theory to spell out Helmholtz's considerations (in particular the thought experiments about the world in a convex mirror) is traced back to Felix Klein (1898). The second part of the paper deals with Schlick's critical remarks against this reading, which had imposed itself after Henri Poincaré formulated a similar thought experiment about the non-Euclidean world in *La Science et L'Hypothèse* (1902). The concluding section discusses how Cassirer elaborated on the philosophical implications of Klein's and Helmholtz's methodologies in the fourth volume of *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neueren Zeit* (1940)[4].

2. Helmholtz and the Group-Theoretical View of Geometry

2.1. A place for group-theoretical considerations in Helmholtz's geometrical papers: 1868–1878

Helmholtz formulated what he believed to be a set of necessary and sufficient conditions for a Riemannian metric of constant curvature in 1868. Later group-theoretical considerations in the work of Lie relate, in particular, to Helmholtz's requirement that any point of a system in motion can be moved continuously to the place of any other (i.e., the free mobility of rigid bodies). As pointed out by Lie, Helmholtz seems to presuppose a false inference from the latter condition to a similar requirement of free mobility at the infinitesimal level. Lie showed that, nevertheless, a proof of Helmholtz's conjecture can be obtained by restricting free mobility and the other conditions to the relations of infinitesimally near points (Lie 1893, 460–64). Alternatively, Lie showed that Helmholtz's conditions—as originally formulated by him—can be used to characterize Euclidean and non-Euclidean motions, insofar as the operations with rigid bodies form a group (of collineations).[5] However, such a characterization is univocal only for finite regions of space (1893, 470–71).

Lie's group-theoretical considerations enabled him to gain clarity over what can and cannot be proved under Helmholtz's conditions. In addition, Lie believed that the use of group-theoretical expressions was at least "compatible" with what Helmholtz said in a time where the language of group theory was not yet available (Lie 1893, 469).

Further evidence for attributing to Helmholtz group-theoretical procedures is found in his epistemological lectures for a wider audience, "Über den Ursprung und die Bedeutung der geometrischen Axiome" (1870) and "Die Tatsachen in der Wahrnehmung" (1878).[6] Although these lectures do not contain

[4]Appeared posthumously in English translation as *The Problem of Knowledge: Philosophy, Science, and History since Hegel* (1950).

[5]Informally, operations form a group iff: (i) the product of any two operations of the group also belongs to the group; (ii) for every operation of the group, there exists in the group an inverse operation; (iii) there is in the group an identity substitution. See Yaglom (1988, 12–13) for a formalization of the original definition, which was first given by Évariste Galois with regard to sets of permutations. As pointed out by Wussing (2007), the abstract concept of group originated from the generalization of Galois's and Camille Jordan's notion in the works of Klein and Lie, among others.

[6]Translated as "The Origin and Significance of Geometrical Axioms" and

new mathematical results, they were the main references for the argument about the possibility of imagining non-Euclidean spaces. This is also due to the fact that, in 1869, Eugenio Beltrami had pointed out to Helmholtz that his previous characterization of rigid motions included non-Euclidean motions. Another reason is that the premises of Helmholtz's argument are taken not from purely mathematical researches, but from his theory of vision (Helmholtz 1867). In his epistemological writings, Helmholtz emphasized that the starting point of his geometrical investigation was somehow induced from the observation that solids or parts of our bodies can be brought to congruent coincidence without changing in shape and size, as a sort of generalized "fact" (Helmholtz 1921, 15, 41, 135–36). The argument of 1870 relies on Helmholtz's empiricist approach for the view that spatial intuitions are inferred from experiences rather than possessed *a priori*, as Kant claimed. Helmholtz argued that Kant's claim was contradicted by the possibility of inferring different intuitions under the hypothesis of different mechanical laws.

Helmholtz described in a thought experiment what the measurements in a convex mirror would look like under the hypothesis that rigid bodies can be superimposed in a congruent manner in the vicinity of the center and tend to contract in the measure that they move towards the edges of the mirror. To an external observer, the objects in the mirror would appear to move as in a pseudospherical space.[7] For every measurement in our world, there would be a corresponding measurement in the mirror. This means that a hypothetical inhabitant of the mirror, whose bodily and visual experiences would be subject to the same conditions, would be able to judge correctly about distances by visual estimation. If someone who has only had experience of a flat space would suddenly find herself in such a world, she would perceive very distant objects as nearer and converging lines would appear to diverge as she approaches. However, it is imaginable that she would become accustomed to these phenomena and learn to adjust her judgments after some time. Helmholtz's conclusion is that spatial intuition or immediate knowledge about space is in fact the result of a more complex cognitive process activated by the regularity of the phenomena. Therefore, contrary to Kant's pure intuitions, spatial perception according to Helmholtz can adapt even to intuitions we never had (Helmholtz 1921, 5).

Helmholtz's argument concerning geometrical axioms is that these can be derived from empirical facts and perhaps even refuted by experience. An example of this is imaginable along the lines of the above thought experiment, insofar as the laws of Euclidean geometry (especially the axiom of parallels) would be approximately valid only at the center of the mirror. The same laws would be contradicted ever more apparently at the periphery. According to Helmholtz, this shows that the particular choice of Euclidean or non-Euclidean axioms in representing space depends on the mechanical behavior of our most fixed bodies. Only free mobility is a precondition for the possibility of measurement itself, as "all spatial measurement, and therefore in general all magnitude concepts applied to space, presuppose the possibility of the motion of spatial structures whose form and magnitude one may take to be unchanging despite the motion" (Helmholtz 1921, 24).

In 1870, Helmholtz used this argument to reject Kant's view of (Euclidean) geometry as a paradigm of *a priori* knowledge (i.e., apodictic and independent of experience). He clarified the latter point in 1878 by distinguishing the idea of a general form of intuition from the specific assumptions expressed by the axioms of geometry. In the light of this distinction, it becomes clear that Helmholtz's objection to Kant concerns the status of the latter assumptions in particular. He wrote:

"The Facts in Perception." Both papers are found in the English edition of Helmholtz's *Epistemological Writings* (1977).

[7] Helmholtz's thought experiment is modelled on Beltrami's (1868) interpretation of non-Euclidean geometry on a surface called a "pseudosphere."

> I believe the resolution of the concept of intuition into the elementary processes of thought is the most essential advance in the recent period. This resolution is still absent in Kant, which is something that then also conditions his conception of the axioms of geometry as transcendental propositions. Here it was especially the physiological investigations on sense perception which led us to the ultimate elementary processes of cognition. These processes had to remain still unformulable in words, and unknown and inaccessible to philosophy, as long as the latter investigated only cognitions finding their expression in language. (Helmholtz 1921, 143)

This argument relates to the general, discursive concepts of traditional logic, which Kant contrasted with the single and immediate intuition of space in the Transcendental Aesthetic (Kant 1787, 40). A related question arises whether the same cognitive processes would find a more adequate representation in mathematical terms. Helmholtz himself proposed such a representation by identifying the structure of space as a three-dimensional manifold that satisfies the free mobility of rigid bodies.

Although the group-theoretical view of geometry was unavailable to Helmholtz, group theory presented itself as a natural candidate for a more precise articulation of his view. Firstly, it offered a model-based account of non-Euclidean measurements. Secondly, the above distinction between general and special assumptions about space seemed to find a more precise expression in terms of inclusion of one group of transformations into a larger group. Thirdly, there is evidence that Helmholtz deemed it more plausible that the idea of a generalized form of intuition would find an adequate expression in mathematical terms. He wrote in Appendix 3 of the printed version of the same lecture:

> Kant's doctrine of the a priori given forms of intuition is a very fortunate and clear expression of the state of affairs; but these forms must be devoid of content and free to an extent sufficient for absorbing any content whatsoever that can enter the relevant form of perception. But the axioms of geometry limit the form of intuition of space in such a way that it can no longer absorb every thinkable content, if geometry is at all supposed to be applicable to the actual world. If we drop them, the doctrine of the transcendentality of the form of intuition of space is without any taint. Here Kant was not critical enough in his critique; but this is admittedly a matter of theses coming from mathematics, and this bit of critical work had to be dealt with by the mathematicians. (Helmholtz 1921, 162–63)

In order to explore these ideas further, the following section discusses Klein's argument for a group-theoretical interpretation of Helmholtz's thought experiment. The idea of using group theory to clarify Helmholtz's distinction between general and specific properties of space is discussed in the following section. Finally, it will be argued that the emergence of the concept of group in this debate opens the door to a possible development of the Kantian theme in Helmholtz's considerations.

2.2. Klein's reading of Helmholtz

As mentioned earlier, Lie was the first to point out that Helmholtz's investigation of the foundations of geometry contained implicitly some group-theoretical considerations. However, Lie showed that Helmholtz's own considerations required substantial revisions, when reformulated in group-theoretical terms.

This section addresses the related question whether group theory offers a plausible interpretation of Helmholtz's epistemological argument about the geometrical presuppositions of measurement. According to philosophers such as Bertrand Russell, there was a clear distinction between the epistemological focus of Riemann's and Helmholtz's earlier investigations into the foundations of geometry and the more technical results of Klein's and Lie's works on projective geometry and group theory (Russell 1897, chap. 1). Russell maintained that it was only in a third phase of this debate that mathematicians such as Henri Poincaré, Moritz Pasch, and Giuseppe Veronese addressed the problem of the origin of geometrical concepts from a new epistemological perspective. However, independently of Poincaré's better-known discussion, a group-theoretical reading of Helmholtz's epistemological argument was given by Klein in his review of the third volume of Lie's *Theorie der Transformationsgruppen*. Klein delivered this lecture to the Physico-mathematische Gesellschaft of the University of Kazan, when Lie was awarded the first Lobachevsky prize, in 1897.

Part of the reasons for the delayed reception of Klein's ideas is the delayed reception of his mathematical work on non-Euclidean geometry itself.[8] Klein presented the first projective model of non-Euclidean geometry based on a projective metric in 1871. According to this model, geometries are classified into elliptic, hyperbolic, and parabolic. Klein laid down the fundamental ideas for a group-theoretical view of geometry in his "Vergleichende Betrachtungen über neuere geometrische Forschungen" (1872)[9], which is best known as the "Erlanger Programm," after Klein's appointment as a professor at the University of Erlangen in the same year. As the projective model of non-Euclidean geometry showed, geometrical properties can be characterized in different ways as the invariants relating to a particular group of transformations. For example, it is a well-known fact that displacements and rotations in Euclidean geometry do not alter such properties as distances, the measure of angles, parallelism, and the distinction between lines and curves. However, only the latter property is generally preserved by projections. Klein identified the group of transformations that is common to Euclidean and non-Euclidean geometries as collineations. He proved the equivalence of his earlier classification of geometries with the three cases of manifolds of constant curvature according to Beltrami's theory (i.e., more, less than and equal to 0, respectively) by identifying the corresponding transformation groups. It followed a proof that metrical projective geometry is independent of the theory of parallels, insofar as the axiom of parallels is valid only in the third case (i.e., in parabolic geometry).

Although the ideas of the Erlanger Programm have been considered to be very influential in retrospect, Klein himself did not draw much attention to it until only after the development of essential requirements for the implementation of such a project by other mathematicians, in particular Lie. So, it was not by chance that Klein especially emphasized the significance of the group-theoretical view for the theory of measurement in his review of Lie's work. Klein began by presupposing that the points in space can be represented by the 3-tuples of a continuous numerical manifold or Dedekind's axiom of continuity. In general, Klein conceived of axioms as "the postulates by which we read exact assertions into inexact intuition" (Klein 1890, 572). Klein maintained that each geometry deserves an axiomatic definition according to the group of transformations that acts upon its elements. Such a definition, in Klein's view, depends on "conceptual properties" independently of the particular and arbitrary choice of coordinates (Klein 1898, 588). The consideration of a transformation group entails a definition of the particular properties of figures as relative invariants. According to Klein, this is the "approach of metrical geometry, when the fact of the free mobility of rigid bodies (or the 'statements of congruence') is taken as primitive" (1898, 588). This indicates that Klein attributed to Helmholtz a group-theoretical approach to the foundation of metrical geometry.

However, in the second part of the paper, Klein showed how the same conclusions follow from the "geometrical foundations" of group theory independently of the presupposition of the numerical representation (Klein 1898, 593). In Klein's view, the formulation of postulates is justified by the fact that there is a lower limit to empirical measurement. Klein's approach to the problem of space emerges from the following consideration about the upper limit:

[8] In what follows, I rely above all on Hawkins (1984) and Rowe (1992).

[9] Translated by Mellen W. Haskell as "A Comparative Review of Recent Researches in Geometry" (1892–1893).

> Correspondingly, when it comes to taking into consideration the topologically different forms of space for the determination of the geometry of actual space, we are faced not so much with an arbitrary but with an inner consequence. Our empirical measurement has also an upper limit, which is given by the dimensions of the objects accessible to us or to our observation. What do we know about spatial relations in the infinitely large? To begin with, absolutely nothing. Therefore, we rely on the postulates that we formulate. I consider all of the different topological forms of space equally compatible with experience. The fact that we put first some of these forms of space in our theoretical considerations (i.e., the original types, that is, the properly parabolic, hyperbolic, and elliptic geometries) and finally select parabolic geometry (i.e., the usual Euclidean geometry), depends solely on the principle of economy. (Klein 1898, 595)

The problem under consideration consists of determining the class of all surfaces in elliptic, hyperbolic, and parabolic space that are locally isometric to the Euclidean plane.[10] Klein restricted his consideration to the manifolds of constant curvature and singled out Euclidean geometry for reasons of simplicity, in a way that closely reminds one of Poincaré (1898). As pointed out by DiSalle (2006), this line of argument differs from Helmholtz's in one important consequence: for Helmholtz, the ultimate grounds for selecting the geometry of space are empirical rather than conventional. Nevertheless, Klein deemed the group-theoretical view "conceptual" in Helmholtz's sense, namely as implicit in but not univocally determined by the series of impressions associated with motions.[11] For example, Klein introduced the distinction between metrical and projective geometry "not as arbitrary or indicated by the nature of the mathematical methods, but as corresponding to the actual formation of our space intuition, in which mechanical experiences (concerning the movement of rigid bodies) combine with experiences of visual space (concerning the different kind of projection of intuited objects)" (Klein 1898, 593). In this way, Klein's geometrical argument enabled him to avoid a classical objection against Helmholtz's approach, which found one of its clearest formulations in Poincaré:

> For Helmholtz and Lie the matter of the group existed previously to the form, and in geometry the matter is a numeric manifold (*Zahlenmannigfaltigkeit*) of three dimensions. The number of dimensions is therefore posited prior to the group. For me, on the contrary, the form exists before the matter . . . We escape in this way also an objection which has often been made to Helmholtz and Lie. But your group, say these critics, presupposes space; to construct it you are obliged to assume a continuum of three dimensions. You proceed as if you already knew analytical geometry. (Poincaré 1898, 40)

Hyder sheds light on Helmholtz's reliance on analytic geometry by observing that, as a physicist, Helmholtz emphasized that numeric values were needed to pick out a system of rigid (self-identical) motions (Hyder 2009, 171). As pointed out by Hyder, this presupposition enabled Helmholtz to account for the possibility of determining congruence as a regulative demand in Kant's sense. However, it remains true that circularity would arise, if the numerical representation of space was to be taken as part of a fundamental analysis of space in terms of transformation groups.

Whereas Poincaré believed that this undermined Helmholtz's approach, Klein avoided circularity by showing that the foundation of metrical geometry based on the statements of congruence and the numeric representation of space can be rephrased from a geometrical point of view via the projective model of non-Euclidean geometry. According to Klein, this ultimately

[10]Klein's solution to this problem is found in Klein (1890). This is known as a distinct problem of space, which is now called "Clifford-Klein" or "the problem of the form of space" (Torretti 1978, 151).

[11]It is worth adding that Klein reconsidered his earlier argument after Einstein's general relativity by admitting that the form of space in the infinitely large may depend on the distribution of matter (Klein 1928, 270; see also Torretti 1978, 152).

depends on the general scope of the mathematical reasoning at work in the foundation of measurement. With regard to Helmholtz's reasoning, Klein traced back a geometrical point of view to Helmholtz's physiology of vision as follows:

> Helmholtz was presumably far from the typical projective way of thinking (in the sense of von Staudt). It must be added that, in the years of Helmholtz's mathematical work, projective geometry was usually considered to be a special field; the insight into its foundational meaning for every geometrical speculation was not widespread at all. Or maybe Helmholtz, as a natural scientist, was fundamentally reluctant towards the abstraction that lies at the foundation of projective geometry. In the introduction to his Göttingen notice from 1868 he distances himself from a foundation of geometry that would put forward the properties of visual space, because even the blind can acquire correct representations of space. Interestingly, this is in contrast though with the fact that Helmholtz himself is continuously led to deal with projective questions by his extensive optical investigations. He deals with these questions by auxiliary means of his own invention, but also sometimes by means of general reasoning. (Klein 1898, 598)

Not only does Helmholtz foreshadow the projective and group-theoretical view of geometry according to Klein, but the passage above suggests that there is some continuity between Helmholtz's psychological considerations on spatial intuitions and the generalization to mathematical reasoning. Along these lines, Klein maintained that his model of non-Euclidean geometry provides an adequate interpretation of Helmholtz's thought experiments about free mobility in a hyperbolical space. The fundamental ideas of Helmholtz's argument find a precise expression by saying that the measurements in our world and the corresponding measurements in the mirror belong to the larger group of collineations (1898, 599).

This example sheds further light on Klein's view of geometrical knowledge as based on axioms, where axioms impose conceptual constraints on imprecise intuitions. Klein relies on the general level of mathematical reasoning to justify, in addition, the introduction of ideal elements—which play a fundamental role in projective geometry—and the axiom of continuity, which enables the observer to postulate a bijective correspondence between points and numbers in measurement (Klein 1898, 594).[12]

Klein's reading of Helmholtz certainly accounts for the relevance of Helmholtz's psychological investigations to his geometrical reasoning. However, it is not clear to what extent Helmholtz himself met Klein's requirement of idealization. Nor did Klein discuss the status of what Helmholtz, referring to Kant, called the form of spatial intuition. If it were to be characterized in terms of imprecise empirical intuitions, the form of intuition would lose its general character. If identified with the full-blown mathematical analysis of a projective metric, such a form would lose its immediacy and would coincide fundamentally with conceptual thinking. The latter option is suggested by Helmholtz's remark about the resolution of intuition into intellectual processes and inspired several strategies for an intellectualization of intuition in neo-Kantianism. The following section contrasts Schlick's fundamental objection to this tradition with Cassirer's strategy.

3. Schlick and Cassirer

The group-theoretical articulation of space was called into question after Einstein's general relativity for two main reasons. Firstly, in the above classification of the geometries that figure as possible candidates for the representation of physical space there is no place for the hypothesis of a variable curvature. However, the latter hypothesis received an unexpected application in Einstein's space-time theory. Secondly, and more fundamen-

[12]This axiom does not feature in Helmholtz's considerations. However, in Helmholtz (1887) he explains that approximation suffices for the purposes of physics even without using irrational numbers.

tally, the traditional approach to the problem of space failed to account for the possibility that the geometry of physical space depends on empirical factors. Klein's and Poincaré's conjecture was that a comparison of geometries alone would suffice to determine which hypothesis would be the most convenient. On the contrary, the curvature of space-time in Einstein's gravitation theory depends on the distribution of matter. If a general form of space in the Kantian sense is to be distinguished from the specific properties of space in relativistic physics, a further generalization to the invariants of Einstein's space-time theory is required.[13]

A comparison between the classical and the relativistic problem of space is beyond the purposes of this paper. This section focuses on the related question whether, nevertheless, the articulation of space in terms of transformation groups provides a plausible interpretation of Helmholtz's ideas.

3.1. Specific or general properties of space?

Schlick distanced himself from the received interpretation of Helmholtz in his comments on the centenary edition of Helmholtz's *Schriften zur Erkenntnistheorie* (1921),[14] which was published by Schlick himself in collaboration with the physicist Paul Hertz. In what follows, I will refer to Schlick's comments on "Die Tatsachen in der Wahrnehmung," namely, the central work for the articulation of Helmholtz's view of the form of intuition.[15] On the one hand, Schlick and Hertz referred to Lie for a rigorous solution of the problem of space via explicitly group-theoretical considerations. On the other hand, Schlick denied that the same kind of considerations account for Helmholtz's distinction between the general form of intuition and the properties that are being specified as geometrical axioms. After drawing this distinction, Helmholtz provided a few examples of axioms of ordinary (i.e., Euclidean) metrical geometry: between two points only one straight line is possible; through any three points a plane can be placed; through any point only one line parallel to a given line is possible (Helmholtz 1921, 128). However, he did not clarify with further examples what properties exactly would count as more general. According to Helmholtz, it should be possible to derive the "most essential features" of spatial intuition from the order of what exist "one beside another." As a further step, we learn to compare magnitudes "by observing congruence of the touching hand with parts or points of the surfaces of bodies, or congruence of the retina with parts and points of the retinal image" (1921, 127). So, the question arises whether such knowledge can be given an axiomatic formulation in the light of later mathematical developments. Schlick admitted that modern geometers tended to answer this question affirmatively, although not everyone agreed on the particular axiomatization. Lie and Klein identified the more general structure of space that results from Helmholtz's distinction as a projective metric, which would include Euclidean and non-Euclidean geometries as special cases. Russell (1897) identified the common properties of Euclidean and non-Euclidean geometries as continuity, homogeneity, and having a finite number of dimensions in projective geometry (without a metric). Poincaré (1898) ruled out both interpretations by identifying the only "qualitative" geometry as the *analysis situs*.[16]

[13] Famously, Hermann Weyl worked on such a generalization from 1918 to 1923. See Ryckman (2005) for a detailed discussion of Weyl's work and its relation to the Kantian tradition.

[14] Translated by Malcom F. Lowe as *Epistemological Writings* (1977).

[15] Schlick commented on Helmholtz's most philosophical papers (i.e., Helmholtz 1870, 1878) while leaving to Paul Hertz the comments on Helmholtz's mathematical papers (Helmholtz 1868, 1887). It might be objected that such a division obscures the connection between the philosophical and the mathematical aspects of Helmholtz's work. In the following, I suggest that this partly depends on Schlick's own attempt to clarify the different aspects of Helmholtz's notion of space.

[16] The "analysis situs" or "science of position" in which Poincaré himself made essential contributions is known today as "topology."

Schlick advocated a wholly different interpretation, for two reasons. Firstly, in order to uphold Helmholtz's distinction, "the 'general form' will have to be understood as the indescribable psychological component of spatiality which is imbued in sense perception" (Schlick in Helmholtz 1921, 172–73). Secondly, Schlick called into question Helmholtz's distinction between the form and the matter of spatial intuition. What Helmholtz called "form" belongs to the content of intuition in the Kantian sense. For example, Helmholtz compared spatial intuition to the color system. However, only space and time are forms of intuition according to Kant. In Schlick's view, a more consistent naturalization would reduce spatial intuition to sensuous contents. The result of such a reduction is what Schlick identified as a subjective spatial intuition or acquaintance with the spatiality of sense perception. Insofar as the physical concept of space can be characterized mathematically, Schlick identified the latter with a formal construction (Schlick in Helmholtz 1921, 167).[17]

In other words, in order to uphold Helmholtz's distinction, Schlick rejected another important aspect of his conception of space, that is the view that the formation of spatial concepts—as in Kant's theory of space—offers an account for the applicability of geometry in physics. It was in virtue of this aspect that Helmholtz called Kant's doctrine of the *a priori* given forms of intuition "a very clear expression of the state of affairs" in the above quote. In the changed context of Schlick's interpretation, on the one hand, axioms were introduced as arbitrary assumptions, whose consistency with one other had to be proved according to David Hilbert's axiomatic method. The same method enabled the definition of geometric objects as the abstract terms that stand in the relations established by the axioms. On the other hand, the physico-mathematical representation of space-time that is presupposed by Einstein's theory of measurement was in contradiction with the approximately Euclidean space of our everyday experiences: modern physics has gone over to the assumption of non-Euclidean metrics and ascribed to space a variable measure of curvature (Schlick in Helmholtz 1921, 185). Schlick therefore deemed a favorable interpretation of Helmholtz the view that our (empirical) intuitions would adapt to the non-Euclidean metrics under different physical laws. However, under this assumption, nothing remains of Kant's *a priori* intuition, which in Kant's original sense of "a priori" would be universally valid and necessary (see Schlick in Helmholtz 1921, 181 n 60). In order to account for presuppositions of measurement such as Helmholtz's definition of "rigidity," Schlick advocated Poincaré's geometrical conventionalism, according to which it has to be stipulated that some physical bodies satisfy free mobility.

As Friedman pointed out, Schlick's reading of Helmholtz presupposes scientific and mathematical concepts that were unavailable to Helmholtz himself. More importantly, Schlick tended to ascribe to Helmholtz his own philosophical assumptions about causal realism, which are sometimes inconsistent with Helmholtz's view of empirical knowledge. According to Friedman: "In the physiological optics Helmholtz comes very close to the view that lawlike relations among our sensations—arrived at by inductive inferences in accordance with the principle of causality or the lawlikeness of nature—are constitutive of their relationship to an external world" (Friedman 1997, 33). Friedman goes on to argue that the final piece of the puzzle was Helmholtz's mathematical contribution to the Helmholtz-Lie problem of space. This enabled him to distinguish between the general concept of space, which is a precondition for the possibility of measurement, on the one hand, and the different metrical geometries that depend on experience, on the other. Ryckman summarized the same argument by saying that: "Helmholtz

[17] See Neuber (2012) on the development of this argument in Schlick's thought. The above distinction between phenomenal and physical space goes back to Schlick (1916) and remains a tenet throughout his later views.

argued against the Kantian philosophy of geometry while retaining an inherently Kantian theory of space" (Ryckman 2005, 73–74). It follows that geometrical conventionalism is not the only consistent reading of "rigidity" in Helmholtz's sense. The notion of a rigid body is "*constitutive* of the concept of congruence on which geometrical measurement rests" (2005, 71).

However, it remains true that Schlick formulated a compelling objection, when it comes to clarifying the supposed properties of what Helmholtz calls the general form of intuition: once such properties have been given a precise mathematical formulation, it is not clear what distinguishes them from the more specific properties, which Helmholtz identifies as geometrical axioms. Such a difficulty is still present in the literature, where several interpretations have been proposed. Some interpreters put at the center the abstract notion of a differentiable manifold, which they deem an intellectualized representation of our natural knowledge about space (Torretti 1978, 166). It follows from this reading that the general properties of space can be stated by the axioms of analytic geometry, although these were not part of the traditional axioms of metrical geometry, which for Helmholtz are empirical (Torretti 1978, 166–67; Lenoir 2006, 201). Others point out that "Helmholtz's intuition of space is richer than Schlick's, because it operates with the free mobility of rigid bodies and therefore includes the idea of constant curvature of space" (Pulte 2006, 198). This characterization, however, would be too narrow to account for the variably curved space-time of general relativity. To put it in group-theoretical terms, given the possibility of generalizing spatial concepts to ever more abstract invariants, the problem arises of clarifying which invariants should be "free of content" enough for the purposes of measurement.

3.2. Articulating the concept of space as an ongoing process

One possible strategy for dealing with the problem posed by Helmholtz is to identify a privileged class of hypotheses concerning the structure of actual space among all the logical possibilities. Given the fact that several geometries would be equally compatible with experience, our choice will depend ultimately on rational grounds (e.g., the principle of the economy of thought in the above quote from Klein, or simplicity, according to Poincaré[18]). Cassirer defended the same strategy in *Substanzbegriff und Funktionsbegriff* (1910). However, he substantially revised his argument in his later work, *Zur Einstein'schen Relativitätstheorie* (1921).[19] Here, Cassirer admitted that the above strategy proved to be insufficient for the solution to the relativistic problem of space. So now he identified the "a priori" of space as the more general function of spatiality that finds its expression in the line element of a Riemannian space (Cassirer 1923, 433). Such *a priori* knowledge allows for the variably curved space-time of Einstein's gravitation theory, insofar as no specific metrical structure has to be attached to space itself.

While Cassirer's argument takes into account the revolutionary aspect of Einstein's theory, it has been objected that not much remains of what Kant called "a priori." Schlick wrote in the concluding argument against a mathematical formulation of

[18]Poincaré's position is best known as geometrical conventionalism, with regard to his thesis that one geometry cannot be more "true" than another, but only more "convenient" (Poincaré 1898, 42). Poincaré's conventions, however, are not arbitrary. In the same paper, for example, he goes on to argue that "reason has its preferences," although these are not as "imperative" as Kant's aprioricity: "It has its preferences for the simplest because, all other things being equal, the simplest is the most convenient" (1898, 42). According to Poincaré, Euclidean geometry is simpler, because only the Euclidean group has the translations as a normal subgroup.

[19]Both works are found in the English edition of Cassirer's *Substance and Function and Einstein's Theory of Relativity* (1923).

the distinction between general and specific properties of space:

> Some neo-Kantians (as P. Natorp, E. Cassirer) have tried to conceive of the a priori nature of spatial intuition in the genuine sense of Kant (thus not in Helmholtz's psychological interpretation), but such that it does not comprise the stipulation of some or other specific Euclidean or non-Euclidean geometry. They seem, however, to be defeated by their effective failure to say what are the a priori laws of spatial intuition which, in their opinion, then still remain. (Schlick in Helmholtz 1921, 173)

More generally, Cassirer's earlier philosophical project of a neo-Kantian philosophy of mathematics that would account for the applicability of some parts of mathematics seemed to be called into question. As observed by Heis (2011), Cassirer's mature philosophy of mathematics seems to focus on the problem of a unitary account of "pure mathematics" instead. The group-theoretical articulation of space plays nonetheless a central role both in Cassirer's early neo-Kantianism and in the later developments of his philosophy of mathematics (see Ihmig 1997). Important sections of the *Philosophy of Symbolic Forms* are devoted to the same topic. Some of Cassirer's latest papers and lectures from the American period are devoted to the concept of group (in Cassirer 2010).

Without attempting to provide a comprehensive discussion of this complicated issue here, I will limit myself to drawing attention to another work that is worth considering for Cassirer's final position on the problem of space, that is, the fourth volume of *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neueren Zeit: Von Hegels Tod bis zur Gegenwart*. The original version of this work dates back to 1940; however, it first appeared in English translation in 1950.

Considering the formation of geometrical concepts in modern geometry, Cassirer called into question the idea of the necessary applicability of geometry in the received view. The loss of certainty followed from the transformation of mathematics from the science of specific objects (i.e., numbers and magnitudes) to the study of structures. Paraphrasing a passage from Klein's *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, Cassirer deems the concept of group "characteristic of a wholly intellectual mathematics that has been purged of all intuition; of a theory of pure forms with which are associated not quantities or their symbols, numbers, but intellectual concepts, products of thought, to which actual objects or their relations *may*, but *need* not, correspond" (Cassirer 1950, 30). What is counterintuitive about these kinds of concepts is that geometrical properties can be determined univocally only relating to transformation groups. Absolute distances and the measure of angles, for examples, are invariant under translations, which form a normal subgroup of the Euclidean or "principal" group, after Klein's terminology. However, Klein's projective model of non-Euclidean geometry of 1871 shows that the same properties are altered by collineations. Therefore, Klein (1872) claimed that the task of geometry in general is not to study the properties of particular figures, but, given a manifold and a transformation group on it, to investigate the properties that are invariant under this group. It follows that every geometry is equivalent when it comes to its truth. Cassirer observed that, nevertheless, geometrical concepts can be ordered from the more specific to the most general according to their relative invariants. Affine transformations form a larger group than translations, insofar as they map lines to lines and preserve the relations of parallelism. Collineations preserve only such properties as, of points: to lie on the same line, of curves: to be a conic. Finally, the transformation of the *analysis situs* can turn even straight lines into curves (Cassirer 1950, 33–34).

In Schlick's view, the same fact about the progressive abstraction of modern geometry is in open contradiction to the supposed necessity and generality of Kant's *a priori* intuitions: not only is intellectual mathematics wholly detached from spatial intuitions, but it is also insufficient to deal with the epistemological problem of space. According to Schlick's conventionalism, simplicity offers a selection criterion only when the whole system of science is considered, which accounts for the use of non-Euclidean geometry in general relativity.

Although Cassirer fundamentally agreed with the latter consequence of Schlick's argument, he relied on group-theoretical considerations to reintroduce epistemic constraints on the classification of geometrical concepts. The reason for this is twofold. On the one hand, group theory offers a unified perspective on geometry as part of intellectual mathematics. On the other hand, the structure of geometrical concepts tells us something about the concept of space and how we should proceed in the empirical investigation of spatial relations:

> In proportion as geometrical concepts gain in elegance and precision, the "world of space" is transformed for us, and other and deeper strata come to light. "The gradual separation of affine and projective geometry from metric," says Klein, "may be compared with the procedure of the chemist, who isolates increasingly valuable constituents from a compound by using constantly stronger analytical reagents; our reagents are first affine and then projective transformations." The analogy lies in this, that in progressing from metric to affine and projective geometry we lay bare, deeper and deeper layers of spatial forms because we reach those basic spatial elements that prove invariant not only with respect to the relatively limited transformations of the "principal group," but to those going on and on without end. (Cassirer 1950, 34)

What can be anticipated *a priori* is not a more or less general form of space as given once and for all, but, firstly, the very fact that the investigation of spatial concepts reveals several layers of forms, corresponding to the required degree of precision of measurement. The space of sense perception, for example, approximates the geometrical properties of a Euclidean space. However, the generalized form of spatial intuition studied by Helmholtz and the space-time of special relativity correspond to the larger group of collineations. Secondly, according to Cassirer, the history of science shows a tendency to achieve higher standards of precision, since increased precision in the formation and specification of mathematical concepts, and thus increased idealization, proceeds without limit. Riemannian manifolds of variable curvature, for example, appeared to be purely analytical speculations before general relativity. In Cassirer's eyes, the application of Riemannian geometry in relativistic physics, however unprecedented, confirmed a general historical tendency (Cassirer 1923, 440–42).

Cassirer agreed with Helmholtz's empiricist philosophy of geometry that the concept of the form of intuition can and has to be generalized in the light of scientific developments. Nevertheless, in reconsidering the necessity and general validity of *a priori* knowledge with regard to the formation of geometrical concepts, Cassirer opened the door to a transcendental argument.

As Patton noticed, Helmholtz's epistemology in general poses the problem of justifying the construction of systems of signs that do not map onto the known properties of the objects considered independently: what justifies the scientist in her belief that the theory nonetheless describes the phenomena as represented in the sign system? Patton summarizes Helmholtz's answer by saying that "we must trust that the regularities of nature that that system describes map on to regularities in nature" (Patton 2009, 284–85). Following Cohen, Cassirer addressed the same problem by taking the *a priori* knowledge that is implicit in the history of science as a condition of experience in the Kantian sense.

Cassirer makes this point particularly clear in the following consideration about the group-theoretical reading of Helmholtz's derivation of spatial concepts:

> Helmholtz' exposition was based in particular on the concept of group, though he had not grasped this idea so definitely nor ap-

> plied it so explicitly as did Lie and Klein afterward. It was shown that in a three-dimensional space of constant curvature displacements are possible that depend upon six parameters; thus there is a sextuply infinite variety of movements. But now the point of difficulty in the problem, from an epistemological standpoint, was shifted. In order to obtain a trustworthy insight respecting the meaning and origin of the various systems of axioms it is no longer enough to analyze the idea of space; it is essential to keep in mind the group concept itself. And in that regard we are in a most favorable position, for the logical character of group theory is hardly in doubt, being immediately obvious from its applications and its evolution. (Cassirer 1950, 42)

Cassirer refers to Poincaré for a clarification of the idealizing process that leads from our experiences with solid bodies to the mathematical concept of group. Regarding the question whether the general properties of space thus derived can receive an adequate formulation in terms of axioms as well, Cassirer relies on the following definition of axiom, which is borrowed from Klein's *Elementarmathematik vom höheren Standpunkte aus* (1925, vol. 2: 202): "The axioms of geometry are, as I believe, not arbitrary but rational propositions that in general are occasioned by the perception of space and are regulated as to their individual content on the grounds of their purposiveness" (in Cassirer 1950, 43).[20]

4. Concluding Remarks

The earlier discussions suggest that Helmholtz's empiricist approach led quite naturally to the idea of using the concept of group as a primitive notion to articulate the concept of space. However, several problems emerged from the later attempts to provide Helmholtz's reasoning with a more precise mathematical formulation. The first is that the approach of metrical geometry adopted by Helmholtz cannot provide a complete analysis of the concept of space or there would be a problem of circularity with such an analysis, if it would tacitly presuppose knowledge of analytic geometry (in particular of the properties that depend on the numeric representation of space). Poincaré's geometrical conventionalism offers a possible solution to this problem by introducing all metrical concepts on a top-down approach, as stipulations based on an explicit axiomatic formulation. However, as pointed out by Schlick, it will be impossible to provide a consistent description of the most essential properties attributed to spatial intuition by Helmholtz, and generally our intuitive notion of space will have nothing to do with the construction of the physico-mathematical concept of space.

I argued that a different way to overcome the charge of circularity can be traced back to the earlier reception of Helmholtz's idea by Lie and especially Klein, who had developed himself a twofold approach to mathematical structures: from the geometrical viewpoint of projective geometry, on the one hand, and under the assumption of 3-tuples of real numbers to represent space, on the other. In Klein's view, the outcomes of both approaches are translatable into one another, because the formation of mathematical concepts rests ultimately on rational grounds (i.e., the introduction of the conceptual postulates that regulate the relations within a particular domain). In Klein's methodology, this approach enabled the clarification of the relations between different branches of mathematics and the discovery of new applications. Cassirer spelled out some of the philosophical implications of Klein's structuralist methodology by interpreting the concept of group as an *a priori* concept in the sense of Marburg neo-Kantianism, namely, a presupposition for the formation of spatial concepts that reveals itself historically.

Two further problems have been emphasized by Schlick's objections against the group-theoretical characterizations of Helm-

[20] I have slightly modified the English translation of *Zweckmäßigkeitsgründe* in order to render the epistemological significance that Klein and Cassirer attach to this term.

holtz's mathematical reasoning given in retrospect by Lie, Klein, Poincaré and still taken for granted in most of the literature. First, would such a characterization be consistent with Helmholtz's distinction between spatial intuitions and geometrical axioms? Second, even if a distinction between more and less general characteristics can be made in terms groups and subgroups of transformations, it remains unclear what properties exactly should be included in the general ones in order to account for the preconditions for the possibility of measurement. How do we know that the generalized form of intuition is the right one and will not be replaced by a more general or even completely different form?

I believe that the interpretative issue whether the characterization of Helmholtz's foundation of measurement via the projective model of non-Euclidean geometry would be at least consistent with what Helmholtz says can be answered affirmatively based on the textual evidence considered earlier.

I have looked at Cassirer for a possible way to overcome the second, more complex issue whether generalizing the form of intuition is a viable strategy for dealing with the foundations of measurement. A thorough discussion of Schlick's concerns about such a strategy would require us to say much more about how Cassirer accounts for generalization in science also in connection with his philosophical interpretation of general relativity. I have relied on his account of group-theoretical geometry from 1940 here, because, in this connection, he draws particular attention to the heuristic aspect of mathematical researches such as Riemann's and Klein's. The same aspect finds a powerful expression in the modern way to articulate space by laying down a group of transformations, along with a variety of possible combinations, instead of immutable truths. So, the fact that it might be problematic to provide a list of characteristics of the form of space that are supposed to be valid once and for all, as pointed out by Schlick, does not prove problematic to Cassirer's account. While Cassirer admits that the generalization can proceed without an end, he imposes an epistemic constraint on the generalizing process, insofar as idealization is guided by the idea of exploring new hypotheses. This is the tendency that Cassirer drew back to Helmholtz's generalization of the form of intuition, and that, according to him, found confirmation in the revolutions of twentieth-century physics.

Acknowledgements

This research has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 715222). I am grateful to Georg Schiemer and my colleagues in Vienna for their support and advice on my research. I wish to thank Lydia Patton, Scott Edgar, Jeremy Heis and an anonymous referee for their very helpful comments on a previous draft of this paper. I also wish to remark that the current paper is my own work, and no one else is responsible for any mistakes in it.

Francesca Biagioli
University of Vienna
francesca.biagioli@univie.ac.at

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