Journal for the History of Analytical Philosophy

Volume 6, Number 3

jhaponline.org

The Epistemological Question of the Applicability of Mathematics

Paola Cantù

The question of the applicability of mathematics is an epistemological issue that was explicitly raised by Kant, and which has played different roles in the works of neo-Kantian philosophers, before becoming an essential issue in early analytic philosophy. This paper will first distinguish three main issues that are related to the application of mathematics: (1) indispensability arguments that are aimed at justifying mathematics itself; (2) philosophical justifications of the successful application of mathematics to scientific theories; and (3) discussions on the application of real numbers to the measurement of physical magnitudes. A refinement of this tripartition is suggested and supported by a historical investigation of the differences between Kant's position on the problem, several neo-Kantian perspectives (Helmholtz and Cassirer in particular, but also Otto Hölder), early analytic philosophy (Frege), and late 19th century mathematicians (Grassmann, Dedekind, Hankel, and Bettazzi). Finally, the debate on the cogency of an application constraint in the definition of real numbers is discussed in relation to a contemporary debate in neo-logicism (Hale, Wright and some criticism by Batitksy), in order to suggest a comparison not only with Frege's original positions, but also with the ideas of several neo-Kantian scholars, including Hölder, Cassirer, and Helmholtz.

Special Issue: Method, Science, and Mathematics:
Neo-Kantianism and Analytic Philosophy
Edited by Scott Edgar and Lydia Patton

The Epistemological Question of the Applicability of Mathematics

Paola Cantù

1. Introduction

A recent trend in the history and philosophy of mathematics has shown that Kant's philosophy has influenced not only neo-Kantian philosophers, but also early analytic philosophers. For example, it has been convincingly shown that Kant's agenda influenced not only Cassirer's reaction to formalism and platonism (Heis 2011; Mormann 2008; Heis 2015; Yap 2017), but also Hilbert and Frege. In this paper, I will focus on a specific question, the explanation of the (successful) application of mathematics to empirical sciences, in order to evaluate the heritage of Kantian and neo-Kantian perspectives on early and contemporary analytic philosophy.[1] I will claim that Kant's formulation and solution of the problem of the application of mathematics, together with other elements of his understanding of mathematical knowledge, influenced the development of several "neo-Kantian" philosophers and mathematicians (Helmholtz, Hölder, Cassirer), who rightly maintained the idea that the question is primarily epistemological rather than metaphysical or ontological. Furthermore I will show that the particular way in which Kant formulated the question of the applicability of mathematics and in which neo-Kantians discussed it, influenced not only early analytic philosophy (Frege), but also some recent developments such as neo-logicism and its relation to structuralism.

In the first section I mention different formulations of the problem of the application of mathematics: does it concern the application of different systems of numbers or just of real numbers to magnitudes? Should magnitudes be considered as abstract or concrete magnitudes? Is the application of mathematics always successful?

This conceptual analysis of the problem is supported in the second section by a historical investigation of the differences between Kant's position in relation to this problem, several neo-Kantian perspectives (Helmholtz and Cassirer in particular, but also Otto Hölder), early analytic philosophy (Frege) and late 19th century mathematicians (Dedekind, Hankel, and Bettazzi).

In the final section of the paper, I will suggest a comparison between different positions concerning the specific question of the definition of real numbers by abstraction in contemporary neo-logicism (Hale, Wright, and Batitsky), and the mentioned positions of Frege, Hölder, Cassirer, and Helmholtz. This last part focuses on the applicability of real numbers at the crossroads of geometry, measurement theory, and theory of magnitudes. The discussion of the epistemological problem of the application of mathematics to the world will thus be limited to the case of the notion of real numbers as developed in analysis and theory of magnitudes respectively: are real numbers in their analytic definition applicable to physical magnitudes? This restriction of the problem is particularly interesting, because it focuses on issues that were central to Kant's doctrine but also related to the development of new axiomatic theories, and to the reactions of neo-Kantian philosophers.

I will claim that the heritage of Kant's objective to identify some general principles (conditions) for the application of math-

[1] This issue will be tackled from a limited perspective, because the point here is not to answer the philosophical question of whether an analytic structure underlies our representations of the continuum and of extension, but rather, and more modestly, to trace certain conceptual crossovers between different philosophical traditions. Yet, we will come back to the question in the final remarks, as we will consider whether the difference between an additive and an order approach to the definition of magnitudes plays some significant role in the neo-logicist debate.

ematics to the world is still alive in the contemporary debate in philosophy of mathematics, especially in contemporary discussions of the application constraint in the debate between logicism and structuralism. I will distinguish several neo-Kantian strategies to cope with the problem: (1) Cassirer's insistence on concept formation; (2) Frege's insistence on the application constraint in the definition of real numbers; and (3) Helmholtz's remarks that not all physical magnitudes are measurable and thus that mathematics is not always (successfully) applied to the world.

2. A Conceptual Analysis of the (Successful) Application of Mathematics

Contemporary philosophy of mathematics is very much concerned with the application of mathematics to empirical sciences. Yet, there are many different ways to tackle the problem, and different ways to solve it.

Firstly, it is not clear whether the application of mathematics should be considered a metaphysical, an ontological, or an epistemological challenge. Colyvan, for example, remarks that it is an epistemic problem, even if it is often presented—in association with the indispensability argument—as an ontological or metaphysical claim.[2] Secondly, it is not clear whether it should ground mathematical knowledge, or explain its fruitfulness. Finally, it is not always clear whether the whole of mathematics is taken into account in this debate, or rather only some systems of numbers.

To disentangle several issues that are often mixed up in the philosophical discussion, I should like to first distinguish three main questions related to the application of mathematics, and then refine this analysis, based on some historical evidence recalled in Section 3:

(A) justification of mathematical knowledge and/or entities by means of an indispensability argument;
(B) philosophical explanations of the successful application of mathematics to the understanding of the physical world; and
(C) foundational discussions of the notion of real number (Frege's application constraint), and of mathematical and physical measurement.

2.1. The indispensability argument

The indispensability argument, which is so often discussed in the recent literature on philosophy of mathematics, is aimed at justifying mathematics, and concerns mathematical theories that are applied to scientific theories, and in particular to physical theories. There are various formulations of the argument. In the ontological version, the argument claims that the objects of such mathematical theories exist, or that the theories are true; in the epistemological version, it claims that we are justified in believing that the objects of such mathematical theories exist, or in believing that the theories are true (Colyvan 2001, 6–16; Panza and Sereni 2013, 196–203).

(A1) The *indispensability argument* aims to infer a metaphysical truth concerning the existence of abstract mathematical entities from the successful application of mathematics (or of a part of mathematics) to scientific theories (Quine, Putnam).
(A2) An epistemic version of the argument has recently been suggested by Colyvan (2001).

[2] According to Colyvan it is a challenge that should be tackled both by realists and anti-realists, because it concerns "the different methodologies of mathematics and empirical sciences", and the fact that "mathematics seems to proceed via *a priori* means and yet finds applications in *a posteriori* science" (Colyvan 2012, 120). "The core problem, it seems to me, is that of explaining the appropriateness of mathematical concepts—concepts developed by a priori methods—for the description of the physical world. Of particular interest here are cases where the mathematics seems to be playing a crucial role in making predictions" (Colyvan 2012, 116).

(A3) A related but different argument involves the justification of the use of abstract elements in mathematics, even if they do not find application in any empirical science, because they are taken to be inseparable from the real part of mathematics that can be successfully applied to scientific theories, and because certain ideal elements might become indispensable in future applications (Cassirer).

These kinds of argument use the application of mathematics to justify mathematics, and in the latter case even to justify the very part of mathematics that is not and cannot be successfully applied to science.

2.2. Philosophical explanations of the successful application of mathematics

Rather than justifying the certainty or the truth of mathematics, other authors aim to understand why mathematics can be used as a tool to investigate reality. Attention is focused on *philosophical explanations of the successful application of mathematics*, which appear as a form of unreasonable effectiveness (Steiner 1995). Different kinds of philosophical explanations have been given:

(B1) an explanation based on the idea that the universe is written in mathematical language (Galilei 1623, 171);
(B2) a transcendental explanation of the mathematical nature of the world, ultimately relying on the constitution of the subject, which interprets reality by means of the *a priori* forms of intuition: space and time (Kant), or by means of specific processes of concept-formation (Cassirer, Hölder);
(B3) a structuralist explanation: the application of mathematics to the world is based on similarities (homomorphisms) between mathematical structures and systems of magnitudes (Bettazzi, Hölder, representational theory of measurement);
(B4) an empirical explanation: mathematics can be applied successfully only when it effectively produces an axiomatic system that can be put in some structural correlation to conceptual models of physical reality (Helmholtz);
(B5) a logicist explanation: mathematics can be applied to the description of the world, given that it contains only logical concepts (Frege).

These explanations are of quite different kinds: transcendental vs. transcendent, structuralist vs. logicist, foundational vs. empirical. These explanations presuppose a philosophical understanding of mathematical knowledge, and are focused on an epistemological issue: the fruitfulness of mathematics.

2.3. Foundational analysis of the notions of real number and measurement

Another issue is related to a specific case of the application of mathematics to scientific theories: measurement by real numbers. Different positions can be distinguished:

(C1) Some authors question the idea that real numbers are actually used in the measurement of physical magnitudes (Helmholtz).
(C2) Other authors believe that measurement should be considered as an essential feature of the notion of real number, and is a feature that should especially occur in the definition of real numbers themselves (Frege's application constraint).
(C3) Some scholars distinguish between physical measurability and mathematical conditions of measurement, thereby claiming that the applicability of real number systems to mathematical systems of magnitudes depends on the possibility of defining both independently, and successively determining a homomorphism between their respective structures (Bettazzi and Suppes' representational theory of measurement).

These issues, like the indispensability argument, are centred on a foundational problem, i.e., the mathematical definition of real number, and the mathematical and physical notions of measurement, but they are strictly related to mathematical and scientific practice.

My claim in the next section will be that several distinctions—with an exception made for some ontological (or even metaphysical) versions of the indispensability argument—are instances of epistemological questions that either arose in Kantian and neo-Kantian philosophy, or were developed in order to account for some Kantian transcendental *desiderata*.

3. A Historical Survey

3.1. Kant and the applicability of mathematics

Before analyzing neo-Kantian responses to the question of the application of mathematics, I will briefly recall Kant's conception and consider the development of new theories of numbers and magnitudes that have suggested a careful reconsideration of his approach.

Kant believed in the successful application of mathematics to the world and tackled both the epistemological version of question (A) and question (B) from the point of view of transcendental idealism (B2), which allows us to explain both mathematical certainty as well as its applicability to the world without making recourse to the nature of the things it is applied to. Kant's solution is essentially philosophical, and does not really take into account the details of mathematical practice.

Kant was very much concerned about the applicability of mathematics to objects of experience, and believed he had found a solution to the problem in transcendental philosophy itself (Friedman 1990, 218). Contrary to the traditional view defended by Galileo that the world is written in mathematical language (B1), Kant believed that the constitution of the subject makes it necessary to describe the world mathematically.

> This transcendental principle of the mathematics of appearances yields a great expansion of our a priori cognition. For it is this alone that makes pure mathematics in its complete precision applicable to objects of experience, which without this principle would not be so obvious, and has indeed caused much contradiction. (Kant 1998, 289/B206–07)[3]

With respect to the question of the application of real numbers (C), it should be noticed that the notion of real number changed between Kant's time (when the Newtonian definition as ratio of magnitudes was dominant) and the end of the 19th century (when they were defined independently from magnitudes, as extensions of rational numbers). Kant considered mathematics as the science of measuring, but did not consider irrational numbers as proper numbers, because their properties are determined by geometrical magnitudes.

It is well known (Friedman 1990) that Kant maintained the traditional distinction between geometry and arithmetic, as he claimed that irrational numbers are themselves not numbers, but only rules for approximation of numbers (*Ak*, 11, 210.13–14),[4] whose properties are not included in the concept conceived by the intellect alone, but are determined by the consideration of geometrical magnitudes (*Ak*, 14, 57–58). At the same time, Kant

[3]See also the following passage quoted in Heis (2011, 784 n 71), a passage that will be useful to help us understand Cassirer's discussion of the same theme:

> Thus although in synthetic judgments we cognize a priori so much about space in general or about the shapes that the productive imagination draws in it that we really do not need any experience for this, still this cognition would be nothing at all, but an occupation with a mere figment of the brain, if space were not to be regarded as the condition of the appearances which constitute the matter of outer experience; hence those pure synthetic judgments are related, although only mediately, to possible experience, or rather to its possibility itself, and on that alone is the objective validity of their synthesis grounded. (Kant 1998, 282–83/A157/B196)

[4]The abbreviation *Ak* is used for Kant (1900–).

also claimed that "mathematics is a science of measuring the magnitude of things, or how many times something is posited in a thing" (*Ak*, 29: 49; cited in Sutherland 2006).

Kant sometimes used the term magnitude (*Grösse*) to indicate an extensive or geometrical magnitude, and sometimes to indicate the abstract notion of quantity, but he clearly distinguished between the two meanings, when he introduced the notions of *quantitas* and *quantum*.[5] The traditional separation between arithmetic and geometry, and the distinction between *quantitas* and *quantum* are discussed by authors who formulate an explicit application constraint (Frege, Bettazzi).

3.2. New mathematical developments: Grassmann, Riemann, Hankel and Dedekind

If several results in algebra had already questioned the relation between geometry and algebra in Kant's time, there is no doubt that this question came definitively to the forefront afterwards, and came to be discussed at length in the 19th century. Two main references are Bernhard Riemann's theory of manifolds (1854), which opened up a different understanding of continuity and of measurement in the infinitely small, and Hermann Grassmann's extension theory (1844) and arithmetic (1861).[6] Grassmann developed three new results that question Kant's approach to mathematics: (1) a separation of abstract extension theory from concrete geometry, which challenges the relation between mathematics and three-dimensional space; (2) the use of inductive properties in the determination of the properties of natural numbers; and (3) a general investigation of the properties of mathematical operations that will later be read by Hankel (1867) and Stolz (1885–86) as a formalistic or merely symbolic presentation of the arithmetical operations.

Further changes in geometry contributed to calling into question the application of numbers. A major change was of course the development of non-Euclidean geometries, in which I include not only hyperbolic and elliptic geometry, but also hyperspatial geometry and non-Archimedean geometry, which all contributed to questions about (1) the specific relation between geometry and empirical space as well as (2) Kant's use of the notion of pure intuition of space, and thus his solution to the applicability problem of mathematics. Another fundamental domain was that of projective geometry, especially because of the introduction of ideal elements (Heis 2011, 763).

Besides, the notion of application came to the forefront when the notion of function started to be rigorously investigated both

[5] See Kant (1998, 632/B745):

> But mathematics does not merely construct magnitudes (*quanta*), as in geometry, but also mere magnitude (*quantitatem*), as in algebra, where it entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of magnitude. In this case it chooses a certain notation for all construction of magnitudes in general (numbers), as well as addition, subtraction, extraction of roots, etc. and, after it has also designated the general concept of quantities in accordance with their different relations, it then exhibits all the procedures through which magnitude is generated and altered in accordance with certain rules in intuition; where one magnitude is to be divided by another, it places their symbols together in accordance with the form of notation for division, and thereby achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction (of the objects themselves), which discursive cognition could never achieve by means of mere concepts.

[6] As a detailed investigation of the question of the applicability of mathematics in the works of Grassmann and Riemann would go beyond the scope of this paper, I will only recall here that Grassmann made recourse to the notion of application to explain both the relation between extension theory and geometry, and how we can use geometrical examples to grasp the content of an abstract theory, while Riemann investigated the relation between the application of the theory of manifolds to space and the measurement of continuous magnitudes. See for example the following relevant literature, even if not focused exclusively on the question of applicability: Petsche, Lewis, Liesen, and Russ (2011), Flament (2005), Lewis (2004), Cantù (2003), Schubring (1996), Otte (1989), Banks (2013), Tappenden (2006), Ferreirós (2006), Gray (2007), and Torretti (1978).

in connection with algebraic symbolism (see e.g., Lagrange 1797, quoted in Panza 2015) and in connection with the analytical notion of magnitude (Du Bois-Reymond 1882).

Two other crucial references are certainly Dedekind's definitions of natural number (1888) and real number (1872). The latter conflicts with Kant's conception in at least three ways: (1) it separates the notion of real number from intuition; (2) it does not introduce real numbers as ratios of quantities; and (3) it does not offer a solution to the applicability problem. More generally, Dedekind made the tension between the analytic and the synthetic definitions of real number explicit, so that after his definition new foundational questions concerning the need to introduce an application constraint in the definition of real numbers arose (C).

The axiomatic investigation of magnitudes, and of measurable magnitudes in particular (Stolz 1885–86; Veronese 1891; Hölder 1901; Bettazzi 1890) constituted not only a reaction to Dedekind's definition of real numbers, but also an alternative way to cope with the problem of the applicability of mathematics. The various solutions are quite different from one another, but they either accept Dedekind continuity as an essential aspect of magnitudes, or argue why this should not necessarily be the case. These theories introduce a distinction between an abstract mathematical notion of extensive magnitude, and a physical notion of extensive magnitude. The conditions for mathematical measurability do not coincide with the conditions for physical measurability.

This distinction is mixed up with another distinction between two conceptions of magnitude and of measurement: an additive approach based on the priority of the notion of addition, and an order approach based on the priority of the notion of order in the axiomatic formulation.[7] I think that this distinction, which is not so significant from a strictly mathematical point of view (given that all authors in the end consider the interrelations between algebraic properties, order properties, and eventually topological properties), makes sense from a philosophical perspective, and is of some utility in the analysis of Helmholtz's perspective.

According to the additive definition of magnitude, the primitives are an equality or inequality relation and an operation of addition: an order relation can sometimes be defined by means of the primitives, but the definition of magnitude is independent from order. Magnitude is anything that can be compared, and order is not necessary to determine whether something is equal to or different from something else, even if it is needed to determine whether one thing is bigger or smaller than the others. The additive approach has its origins in the Greek theory of proportions, and was applied to physical measurement by Helmholtz (1887; see Darrigol 2003; Biagioli 2016): to measure a class of physical magnitudes one should identify a physical operation that can be executed on such quantities, and then investigate its algebraic properties (commutativity, associativity, etc.), so as to choose the appropriate system of numbers to measure them. Besides, the additive approach is compatible with the idea that real numbers might be defined as ratios of quantities, an idea that is maintained by Bettazzi. So, the additive approach can be linked on the one hand to the notion of physical magnitude, and on the other hand to the idea that real numbers can somehow be constructed or defined from an abstract mathematical notion of magnitude. The answer to the applicability problem depends on the relation between these two notions of magnitudes. The additive approach applies only to extensive magnitudes. Other magnitudes need some kind of indirect measurement.

[7]This distinction is not identical with Darrigol's distinction between "a narrow concept of measurement in which the additivity and divisibility of quantities is required in a concrete sense, and a more liberal concept in which the ordering of quantities is the only requirement" (Darrigol 2003, 518–19). The latter distinction conflates, according to my interpretation, the distinction between the order and the additive approach with the distinction between physical and mathematical measurability.

According to the order-based definition of magnitude, the primitives are an order relation and an operation (of successor, division, passage to the limit, . . .). Given that the presence of an order relation (even a trivial one) is sufficient to introduce a measurement, and that magnitudes are defined as ordered structures, any magnitude is *ipso facto* a measurable magnitude. The notion of a non-measurable magnitude does not even make sense. The order approach often accepts Dedekind's continuity from the outset (Hölder 1901), thus using numerical systems that play a fundamental role in algebra to measure physical quantities. This is paradoxical, if one conceives measurement as the application of numbers to reality, because rational numbers are rich enough for the physical approximations that are used in the measurement of physical quantities (Helmholtz 1887). But the paradox vanishes if one conceives magnitudes as abstract mathematical concepts, and in particular either as something that codifies the properties of real numbers as a complete order field, or as some kind of functions. The order approach, depending of course on the properties required for the ordering, might apply to intensive quantities too.

3.3. Frege and Bettazzi on the application constraint

It is well known that one of the epistemological questions that most enthused Frege with respect to the analysis of the principles of arithmetic was its applicability, which "alone elevates arithmetic from a game to the rank of a science. So applicability necessarily belongs to it" (Frege 1903, §91). Frege shared with Kant the idea that the applicability of numbers is an important aspect of the legitimacy of their definition: this is also one reason for his dissatisfaction with Dedekind's definition of real numbers:

> The path that is to be pursued here thus lies between the old way of founding the theory of irrational numbers, the one H. Hankel used to prefer, and the paths followed more recently. We retain the former's conception of real number as a relation of quantities . . . , but dissociate it from geometrical or any other specific kinds of quantities and thereby approach more recent efforts. At the same time, on the other hand, we avoid the drawback showing up in the latter approaches, namely that any relation to measurement is either completely ignored or patched on solely from the outside without any internal connection grounded in the nature of the number itself . . . our hope is thus neither to lose our grip on the applicability of arithmetic in specific areas of knowledge nor to contaminate it with the objects, concepts and relations taken from those areas and so to threaten its peculiar nature and independence. The display of such possibilities of application is something one should have the right to expect from arithmetic notwithstanding that that application is not itself its subject matter. (Frege 1903, §159; English translation in Hale 2002, 306)[8]

The Kantian flavor of Frege's remarks is related to his idea that the applicability of real numbers to magnitudes (i.e., the fact that they can be used to measure magnitudes) should be included in their definition, without thereby mentioning the things to which mathematics is applied (C3). In other words, the definition of real number should contain some general condition of applicability, without presupposing an effective application to a given domain of entities. But the necessity of introducing an application constraint for real numbers is a way to react to the new axiomatizations of real numbers and of magnitudes mentioned above, while at the same time maintaining the usual definition of real number as a relation of magnitudes. Frege's approach is foundational: the answer to the question of the applicability of mathematics should be given inside mathematics, in the definition of its primitives, which are logical concepts. Such an answer can be given, because in the logicist perspective mathematics is logic, and can thus be applied to any possible domain of objects (B5).

[8]Hale claims that "while Frege does not mention either Cantor or Dedekind explicitly here, it is certain that he had them in mind when he spoke of 'the paths followed more recently'."

It should be noted that Frege was not the only one who discussed the necessity of introducing an application constraint in the definition of real numbers. Rodolfo Bettazzi, as well as other members of the Peano school, made similar foundational remarks, as he compared the definition of number as a ratio of quantities and the definition of number based on order (Bettazzi 1887). Bettazzi claimed that there are two points of view according to which one can introduce and develop the idea of number: one might consider number as representing magnitudes in their relation to a magnitude of the same species, or as a purely analytical entity, regardless of the application it might receive in the measurement of magnitudes.

Using the first method, the number originates in the consideration of magnitudes and in the advantage of having an entity that represents them. Number, according to this point of view, is therefore an entity that reminds us of the way a magnitude can be obtained from the unity of its category. With this method, number appears as representing magnitudes and can be immediately used to investigate them. The definitions of concepts and operations relative to numbers must therefore stem from the corresponding definitions of magnitudes (Bettazzi 1887).

It should be noted that when Bettazzi mentions magnitudes, he does not intend either geometric or physical magnitudes, but rather an abstract notion of magnitude, i.e., the element of a class of homogeneous magnitudes characterized by a relation of equality or inequality, and by an additive operation.

Even more interesting is Bettazzi's comparison between the mentioned synthetical (traditional) way to introduce real numbers as ratios of magnitudes, and the analytical (Dedekindian) way. Bettazzi claims that according to the analytic point of view, the properties of numbers depend on the formal properties of certain abstract operations, because numbers are first introduced as the elements of the given operations and can be generalized only if the properties of those operations are preserved and certain impossibilities eliminated. For example, natural numbers are generalized into integers so as to make subtraction possible, integer numbers are generalized into rational numbers so as to make division possible, rational numbers are extended by the introduction of certain real numbers so as to allow the operation of extracting the root of any positive number, and so on. A main difficulty is, according to Bettazzi, that one does not know exactly where one should stop in this procedure of generalization or when one would have enough numbers to measure magnitudes (Bettazzi 1887).

3.4. Helmholtz, Hölder and Bettazzi

Evaluating Helmholtz's position with regard to the application of mathematics is a difficult task, firstly because Helmholtz's definition of number changed over time, secondly because his fundamental article *Zählen und Messen* (1887) does not concern real numbers, and finally because he is mainly concerned with measurability rather than applicability.[9] But his empiricist approach is very interesting because it shows a back-and-forth movement between axiomatization and application, and because he separates the mathematical conditions and the physical requirements for the measurability of a system of magnitudes. It is for this very reason that Helmholtz has been considered an essential figure in the transition between the classical conception of measurement (measurement is the comparison between a magnitude and a homogeneous unit; numbers are defined by means of measurement), and the representational conception of measurement (measurement is a correlation between a system of magnitudes and a system of numbers, which need to be defined independently from one another—see Michell 1993; Darrigol 2003; Diez 1997; Suppes and Zinnes 1963).

[9] In the following, I will rely heavily on Darrigol's 2003 article *Number and Measure*, but I will maintain the translation "magnitude" rather than "quantity" for the term *Grösse*.

In an essay on the foundations of the natural sciences written in the early 1840s, Helmholtz, following Kant, defined numbers as ratios of magnitudes:

> An object, considered with respect to quantity [*Quantität*], is called magnitude [*Grösse*]; hence we can regard as magnitude every object that can be thought of as decomposed into equal parts. To measure means to determine the amount [*Menge*] of such parts; a determined amount is called number; a single part is called unit of measurement [*Maasseinheit*]. (Königsberger 1903, 128).

In a further passage of the same essay Helmholtz adds:

> The science of the connection [*Verbindung*] of magnitudes according to quantity is arithmetic. It can be purely developed according to the laws of common logic from the concepts presented here. It leads to the well-known number forms [*Zahlenformen*] of positive, negative, integer, fractionary (including irrational, i.e. fractionary with infinitely great denominator), real and imaginary numbers. (Königsberger 1903, 128–29)

Arithmetic seems here to be the logical treatment of operations between magnitudes, and real numbers are also mentioned. By contrast, in the 1887 article, where ordinal numbers are introduced independently of magnitudes, and arithmetic derived through Grassmann's mathematical induction and the operation of addition and multiplication, Helmholtz merely mentions irrational ratios, which can be approximated by rationals in a sufficient way in the cases discussed by physicists (Darrigol 2003, 553).

Physical measurable magnitudes are then defined independently from numbers. To determine whether a physical magnitude is measurable, one has to verify whether a specific method of comparison can be introduced between certain physical magnitudes, and whether a concrete additive operation satisfying certain properties can be introduced between them. The properties that the relation and the operation should satisfy constitute an implicit characterization of the mathematical conditions for the applicability of numbers to magnitudes. They will be made explicit by Hölder (1901), but are discussed in a way that is more similar to Helmholtz's approach in Bettazzi's *Theory of Magnitudes* (1890), where order is not required for the definition of magnitude, but is required to characterize the set of measurable magnitudes. Both Bettazzi and Hölder formulated a more general requirement for the applicability of a system of numbers to systems of magnitudes: a representation theorem, showing that there is a homomorphic mapping between the two systems that preserves the relevant relations and operations (Bettazzi 1890; Ehrlich 2006; Hölder 1901). In a representational theory of measurement, one defines numbers and magnitudes independently, and then has to justify the assignment of numbers to magnitudes, and also specify the degree to which this assignment is unique (Suppes 1951).

Helmholtz's representational approach to measurement is based on addition and not on order, as required by Hölder and by most contemporary approaches to the so-called representational theory of measurement. Helmholtz introduced an additive approach to measurement. This explains why he believed that not all physical magnitudes might be measurable. A trivial order can be determined between almost any kind of physical magnitudes, whereas this is not the case for an additive operation.

To summarize, Helmholtz's explanation of the successful application of mathematics is based on the importance accorded to applications and has an empirical flavor (B4): mathematics can be applied successfully only when it effectively produces an axiomatic system that can be put in some structural correlation to conceptual models of physical reality. For this reason, and also because real numbers are not strictly necessary to measure physical magnitudes, Helmholtz's conception is quite opposed to the idea of an application constraint in the definition of real number (C1).

The fact that Helmholtz developed a representational view of measurement, that he did not investigate the application of the system of real numbers, and that he formulated no application constraint, might suggest that the representational view already offers a solution to the applicability problem (C3). Yet the question remains: how can one have access to physical systems of magnitudes independently of their mathematical representation? The difference between Hölder's and Bettazzi's perspectives (B3) and Helmholtz's conception (B4) consists in the fact that the former have a mathematical rather than a physical or empirical understanding of measurement, and thus define measurable magnitudes as having some mathematical properties. (Hölder assumes that measurable magnitudes should be Dedekind-continuous, while Bettazzi requires some less strict condition on order.)

Helmholtz is well aware that there are many systems of physical magnitudes that do not satisfy either the mathematical or the physical requirements for measurement. Some kind of indirect measurement is often possible, but Helmholtz does not take it for granted that mathematics always has a successful application.[10] On the contrary, as Darrigol remarks, experience (application to physics) is required to decide between multiple options in geometry, i.e., "the application decides the axioms",[11] whereas in arithmetic, the axiomatic definition of number is independent of experience, and the latter is required to determine which physical magnitudes are measurable, i.e., "the axioms control the applications" (Darrigol 2003, 555–56).

3.5. Cassirer

Cassirer's views on the application of mathematics have been discussed in the literature (see e.g., Mormann 2008; Heis 2010, 2011; Yap 2017) with respect to the methodology of mathematical and physical sciences (sameness thesis), to the applicability of ideal elements in physics, and to mathematical concept formation. In the following section I will analyze the results of these inquiries in light of the tripartite conceptual analysis of mathematical application mentioned in §2.2.

Cassirer interpreted Kant's recourse to intuition in geometry as a means to describe the specific kind of synthesis that is at stake in mathematics, and seemed to wipe out Kant's opposition between real numbers and magnitudes (Cassirer 1910, 1922). He agreed with Couturat that the objectivity of mathematics consists in the fact that it can refer to all possible objects, but he claimed that the formal character of mathematics cannot be explained from the logicist perspective, but only from the perspective of a critical theory of knowledge, which "begins where logicism ends" (Cassirer 1907, 44–45). This is an explicit criticism of a logicist explanation of the successful application of mathematics to science (B5). And a critical theory of knowledge should explain why mathematical concepts "have their function and their proper application solely within empirical science itself" (1907, 42–43).

Applicability is a philosophical question that can be tackled when the mathematical analysis of concepts is finished. It is thus certainly not a foundational question as in (C2), and it cannot be used to decide which definition of real number should be preferred (against Frege and Bettazzi). The separation of the application problem from mathematics is a way to guarantee mathematical freedom. The application of mathematics is ex-

[10] See Helmholtz (1887, 385–86; English translation in Helmholtz 1977, 98):

> Apart from the magnitudes discussed so far, which are directly recognizable as such because they can be conjoined by addition, there still remains a series of other relationships, also expressible using denominate or non-denominate numbers, for which an additive conjunction with ones alike in kind is not yet known. . . . Specific gravity, thermal conductivity, electrical conductivity, thermal capacity and so on are similar magnitudes.

[11] On the reaction of Cassirer to Helmholtz's naturalization of the *a priori*, see Patton (2009). For a more general discussion of objectivity in Cassirer and Helmholtz with respect to psychology and physiology see Edgar (2015a,b).

plained by philosophy, because the latter is directed neither to mathematics nor to physics, but to the connection between them.

The formal character of mathematics—its possibility of expressing a relation that can be applied to all things—relies on a peculiar act of abstraction, which "is not directed upon the separating out of the quality of a thing" (negative abstraction), but brings "to consciousness the meaning of a certain relation independently of all particular cases of application, purely in itself" (positive abstraction; see Cassirer 1923b, 39; Yap 2017, 16–17).

> The function of 'number' is, in its meaning, independent of the factual diversity of the objects which are enumerated; this diversity must therefore be disregarded when we are concerned merely to develop the determinate character of this function. Here abstraction has, in fact, the character of a liberation; it means logical concentration of the relational connection as such with rejection of all psychological circumstances, that may force themselves into the subjective stream of presentations, but which form no actual constitutive aspect of this connection. (Cassirer 1923a, 39; cited in Yap 2017, 16–17)

Cassirer's answer to the question of the successful application of mathematics is still transcendental, although it does not rely on the pure forms of intuition, but rather on a process of concept-formation that characterizes mathematical generality (B2).[12]

The mathematical notion of real number does not include any reference to the objects it can be applied to:

> The whole 'being' of numbers rests, along these lines, upon the relations which they display within themselves, and not upon any relations to an outer objective reality [*gegenständliche Wirklichkeit*]. They need no foreign 'basis' [*Substrat*], but mutually sustain and support each other insofar as the position of each in the system is clearly determined by the others. (Cassirer 1907, 38).[13]

Like Frege and Bettazzi, Cassirer believes that no consideration of intuitive relationships between magnitudes should be necessary to understand the concept of real number, but unlike them, he seems to be quite satisfied with Dedekind's definition, and believes that the arithmetical realm is sufficient:

> We thus see that, to get to the concept of irrational number, we do not need to consider the intuitive geometric relationships of magnitudes, but can reach this goal entirely within the arithmetic realm. A number, considered purely as part of a certain ordered system, consists of nothing more than a 'position'. (Cassirer 1907, 49).

We have seen so far that in his early writings Cassirer maintains a Kantian solution to the question of the application of mathematics. Heis remarks that in later writings the question of the applicability of mathematics is related to the distinction between real and ideal elements in mathematics and in geometry. According to Heis, Cassirer developed a twofold strategy to explain the "essential applicability of mathematics in the exact sciences" (Heis 2011, 788). On the one hand Cassirer tries to show that even ideal elements find an application in physics, on the other hand he considers the unity between real and ideal elements as indissoluble, so that the application of the real part of mathematics implies the acceptance of the ideal part of mathematics. The first argument looks like a standard indispensability argument: we should accept ideal elements because they are applied in physics (A2),[14] whereas the latter is a variant of an indispensability argument (A3): even if only real elements find an application in physics, they are so strictly intertwined with ideal elements in mathematics that they should all be accepted in order to maintain our mathematical theories.

[12] As an anonymous reviewer pointed out to me, this is not original to Cassirer, but present already in Natorp's writings. See e.g., Mormann (2018).

[13] Passages like this are used in the literature to show that Cassirer developed a form of non-eliminative structuralism (Reck 2013).

[14] Or, more precisely, they could become indispensable in future applications, as was the case for Riemannian geometry, which was first considered as a mere abstract possibility, and then played an extraordinary role "in the grounding and construction of Einstein's theory of gravitation" (Cassirer 1923b, 432, 440). I thank an anonymous reviewer for this remark.

Heis suggests that this new kind of argument becomes necessary when Cassirer abandons the sameness thesis, which amounts to the following claim: "Mathematical knowledge and physical knowledge are of the same kind. Both are characterized by the introduction of 'ideal elements' which in both areas play essentially the same role" (Mormann 2008, 152). The Kantian argument based on processes of concept-formation does not hold anymore, when the parallelism between mathematical and physical knowledge changes into a relative independence of the methodologies of different scientific disciplines in *Philosophie der symbolischen Formen* (1923a).

To summarize, in Cassirer's early writings, the applicability of mathematics is not explained by the mathematical definition of real number, but by positive abstraction, i.e., a theory of concept formation that is made explicit by critical philosophy, and holds both for mathematics and physics (Sameness Thesis). There is no formulation of an application constraint, because Cassirer accepts Dedekind's definition, which does not explain the application of real numbers to the measurement of magnitudes. Finally, the question of the justification of mathematics becomes more important in later writings, when Cassirer develops the idea that each special science has its own methodology.

4. A Variety of Neo-Kantian Lineages in the Effort to Define Real Numbers by Abstraction

In this section I briefly sketch out two ways to define real numbers by abstraction: Hale's definition by abstraction from ratios of quantities, which satisfies an application constraint, and Shapiro's Dedekindian definition by abstraction from concepts of rationals, which does not satisfy the constraint. I then analyze Batitsky and Wright's criticism of Hale's ontological perspective in light of the historical survey presented in section §3. Firstly, I claim that Batitsky's and Hale's discussion about the neo-logicist definition of real numbers as ratios of elements of complete quantitative domains is based on a distinction between a mathematical and a physical notion of magnitude that recalls the distinction between the order and the additive approach in the axiomatizations by Hölder, Bettazzi and Helmholtz. Secondly, I claim that there are two different neo-Kantian issues that characterize the neo-logicist project by Hale, and the structuralist project by Shapiro, respectively: the search for a general principle that might explain the applicability of real numbers without reference to specific objects of application (a transcendental *desideratum*), and the belief that the application constraint need not be satisfied whenever the process of concept formation is not conditioned by empirical applications (Cassirer). Even if Hale and Wright do not explicitly use neo-Kantian language, their arguments for and against the application constraint in the definition of real numbers certainly have a neo-Kantian flavor. In particular, I believe that the distinction between an ontological and an epistemological point of view on the application constraint can be better understood against the background of the neo-Kantian perspectives mentioned in §3.

4.1. Two variants of abstractionism for real numbers: Hale and Shapiro

The applicability constraint has often been referred to as Frege's constraint by neo-logicists.[15] It is an essential part of Bob Hale's project to extend the neo-Fregean axiomatization of arithmetic to real numbers. The main neo-logicist idea is that of introducing real numbers by principles of abstraction, i.e., principles that seek "to give necessary and sufficient conditions for the identity of objects mentioned on its left-hand side in terms of the holding of a suitable equivalence relation between entities of some other

[15] Bob Hale (2002, 305) introduced the name "Frege's constraint", following a suggestion by Crispin Wright.

sort" (Hale 2000, 100–01). While Frege introduced real numbers as relations of magnitudes, Hale goes back to the "traditional" definition as ratios of magnitudes, as he introduces real numbers according to the following abstraction principle: given a and b, elements of a complete quantitative domain, and c and d, elements of another complete quantitative domain (the compared magnitudes need not be homogenous), the ratio $\frac{a}{b}$ is equal to the ratio $\frac{c}{d}$ if and only if equimultiples of their numerators stand in the same order relations to equimultiples of their denominators (Hale 2000, 107).

There are different ways to define real numbers via abstraction principle that do not satisfy the application constraint, e.g., Shapiro's Dedekindian way, where "successive abstractions take us from one-to-one correspondence on concepts to cardinals, from cardinals to pairs of cardinals, from pairs of finite cardinals to integers, from pairs of integers to rationals, and finally from concepts of rationals to (what are then identified as) reals" (Wright 2000, 318–19). While Wright is quite satisfied with this way, believing that the application constraint should be required only in the definition of natural numbers, Hale shares with Frege a dissatisfaction with Dedekindian definitions. Hale thus tries to satisfy the application constraint, and avoids starting from a given set of numbers (e.g., the natural numbers), but defines real numbers by abstraction on any complete quantitative domain (Hale 2000, 104).

4.2. Batitsky's criticism of Hale's ontological assumptions: Helmholtz's view

Philosophically, the relevant question is why one should defend the application constraint in contemporary philosophy of mathematics. This is the question raised by Batitsky as he evaluates Hale's introduction of real numbers and the contemporary representational approach in the theory of measurement. According to Batitksy, Hale's epistemological commitment to the idea that quantitative domains should have quite a rich structure in order to allow the definition of real numbers, makes it impossible to apply the numbers so introduced to physical quantities that have a less rich structure and that are nonetheless considered as measurable in the perspective of measurement theory. This reproach was not directed at Frege because, as Hale remarks, in his time most mathematicians believed that a continuous structure should be used to introduce measurement (e.g., Hölder), and because most physical concrete magnitudes that needed to be measured were actually continuous. What Batitsky does not understand is why a contemporary philosopher should still defend Frege's application constraint.

I think that here there are two issues which are conflated together. Batitksy interprets the application constraint as something that concerns numbers in general and not specifically real numbers, whereas philosophically speaking the question that Frege wanted to explain is why real numbers are specifically relevant for measurement. It is true that physically speaking, they are not, but still what Frege actually wanted to define is a theory of ratios of quantities. I think that Bettazzi's discussion of the analytic and synthetic introduction of real numbers is an interesting way to explain the difference in opinion between Hale and Batitksy.

On the other hand, it is interesting to compare Batitsky and Helmholtz, as Batitsky criticizes what he calls Hale's ontological commitment, i.e., the commitment "to the necessary and a priori existence of at least one complete quantitative domain" (Batitsky 2002, 289), whereas Batitksy believes that . . .

> . . . the structure of physical relations and operations for comparing and aggregating quantities, unlike the structure of mathematical relations and operations on numbers, is not a matter of an a priori stipulation, but is subject to experimentally and theoretically motivated refinements and revisions. . . . None of these refinements and revisions in our conception of quantities, however, led us to alter our axioms characterizing the structure of the real numbers (which

> still remains that of a complete ordered field). This, of course, is how it should be. For while we may agree with Hale that quantities are abstract entities (or, more cautiously, that quantities do not belong to the same ontological category as concrete objects to which quantities are attributed), we should not forget that they are abstractions of physical illations and operations on concrete objects in the world. And this means that the applicability of reals (or other kinds of numbers) as measures of a given quantity, far from being "essential to their very nature", is and always has been contingent upon the extent to which the physical world allows comparison and aggregation of objects having that quantity. (Batitsky 2002, 297)

The dialogue between Hale and Batitsky is based on a distinction between a mathematical notion of magnitude (such as the one developed by Bettazzi or Hölder), i.e., a set of axiomatic conditions that define the mathematical conditions for the applicability of measurement, and a physical notion of magnitude (such as the one recurring in Helmholtz's writings), i.e., a set of concrete physical conditions (a method for comparison, a concrete operation) for measurement. Like Helmholtz, Batitksy claims that in the case of measurement, the axioms control the application, and not vice versa.

Interestingly, the distinction between an additive and an order-based definition of magnitude could be compared with the distinction between Hale's and Shapiro's introduction of reals by abstraction, but also more generally with different neo-logicist perspectives. Although this is not my objective here, in any case I will suggest some lines along which I would like to develop the topic in a future paper. Even if the focus is not necessarily on the operation of addition, there are neo-logicist strategies that avoid assuming an unrestricted comprehension version and instead start from some specific operations (for example certain abstraction operators). The passage from arithmetic to real analysis is accomplished by a piecemeal addition of the relevant abstraction principles, whose choice is certainly dependent on the mathematical context. Other approaches on the contrary accept unrestricted abstraction principles and then add existence assertions for properties, thereby avoiding having to define abstraction operators relative to certain contexts and trying to follow a uniform procedure (Linsky and Zalta 2006; Ebert and Rossberg 2016).[16] More generally, one should verify whether the additive and order approach reflects a difference between a mathematical or logical point of departure of the investigation, as if there were approaches that start from mathematics and try to determine its analytic structure, whereas other approaches start from logic and then try to include mathematics in its scope. From an epistemological perspective, the two procedures are radically different, because in the first procedure, one reasons case by case and does not assume the existence of the totality of individuals, whereas in the second case the domain is determined from the beginning.

4.3. Two neo-Kantian alternative versions of abstractionism: Hale and Wright on the application constraint

Hale answers Batitksy's criticism by distinguishing three readings of Frege's constraint. The first, according to him, is "anodyne", and therefore does not constitute a genuine philosophical solution to the problem of the applicability of real numbers to the world. Hale is here developing a neo-Kantian version of abstractionism, which he believes is necessary in order to give a philosophically intrinsic explanation of the applicability of real numbers in their own definition:

> There is, first, an entirely anodyne reading on which the Constraint demands no more than, and would be met in full by, the

[16]What makes it even more difficult to compare different perspectives is the fact that few authors (Wright, Hale, and Shapiro) have gone beyond the realm of arithmetic, discussing the analytic structure of real analysis: see e.g., Wright (1997, 2000), Shapiro (2000, 2003), Hale (2000, 2002, 2004), and Hale and Wright (2001).

> availability of representation theorems of the kind proved in standard measurement theory. Second, there is the exorbitant reading which requires, if the reals are to be defined by abstraction, that the abstraction should be over an equivalence relation or relations holding among suitably chosen actual lengths, masses, etc. Third, there is a reading on which the Constraint requires more than the first but appreciably less than the second: that an adequate definition of the reals must, as Dummett puts it, "display the general principle underlying the use of the real numbers to characterise the magnitude of quantities of these and other kinds [i.e., masses, lengths, velocities, etc.]". (Hale 2002, 312)

By contrast, the dialogue between Hale and Wright suggests a connection with Cassirer's approach. Wright distinguishes between a metaphysical and an epistemological interpretation of the constraint, and believes that structuralism answers the epistemological issue (even if not in an immediate way, because some further thought is needed to grasp the structural similarities that explain applications), but discounts the metaphysical issue, whereas Hale's interpretation is bound to answer both issues. According to Wright, a structuralist need not respect Frege's constraint (Wright 2000, 326):

> Frege's Constraint is justified, it seems to me, when—and I am tempted to say, only when—we are concerned to reconstruct a branch of mathematics at least some—if only a very basic core—of whose distinctive concepts can be communicated just by explaining their empirical applications. However, the fact is that both our concepts of the identity of particular real numbers and, more importantly, the entire overarching conception of continuity, as classically conceived—the density and completeness of the range of possible values within a parameter determined by measurement—are simply not manifest in empirical applications at all. Rather, so one would think, the flow of concept-formation goes in the other direction: the classical mathematics of continuity is made to inform a non-empirical reconceptualization of the parameters of potential variation in the empirical domains to which it is applied. (Wright 2000, 328–29)

Paraphrasing Wright, one could say that, as in Cassirer, mathematics is a matter of concept-formation, and only some parts of mathematics actually find application in the empirical world. Whenever this is not the case, as e.g., in the theory of continuity, for example, it is perfectly sound to ignore the application constraint.

To summarize, Batitsky's line of reasoning shares several aspects of Helmholtz's argumentation; Wright's remarks on the acceptability of the Dedekindian way, and on the effective application of certain abstract parts of mathematics, remind us of Cassirer's approach. But even apart from these similarities, the contemporary debate cannot be fully understood without reference to the Kantian and neo-Kantian discussion of the question of the application of mathematics.

On the one hand, the Kantian and neo-Kantian insistence on the epistemological aspect of the application constraint contributes strongly to the understanding of contemporary discussions on structuralism, neo-logicism, and abstractionism. On the other hand, the search for some general principles that might explain the possible application of mathematical concepts to scientific theories is an epistemological heritage of the Kantian and neo-Kantian tradition. Finally, the historical survey of different neo-Kantian strategies (Cassirer, Helmholtz, Hölder, and Frege) might help to clarify the general question of the application of mathematics, distinguishing different problems (the epistemological justification of mathematical knowledge, the epistemological question of why mathematical theories are fruitful in the description of scientific phenomena, and foundational concerns about the best definition of real number), and multiple solutions to them.

Paola Cantù
Aix Marseille Univ, CNRS, CEPERC, France
paola.cantu@univ-amu.fr

References

Banks, Erik C., 2013. "Extension and Measurement: A Constructivist Program from Leibniz to Grassmann." *Studies in History and Philosophy of Science Part A* 44: 20–31.

Batitsky, Vadim, 2002. "Some Measurement-Theoretic Concerns about Hale's 'Reals by Abstraction'." *Philosophia Mathematica* 10: 286–303.

Bettazzi, Rodolfo, 1887. "Sul concetto di numero." *Periodico di matematica per l'insegnamento secondario* 2: 97–113, 129–43.

———, 1890. *Teoria delle grandezze*. Pisa: Spoerri.

Biagioli, Francesca, 2016. *Space, Number, and Geometry from Helmholtz to Cassirer*. Cham: Springer.

Cantù, Paola, 2003. *La matematica da scienza delle grandezze a teoria delle forme. L'Ausdehnungslehre di H. Grassmann*. PhD thesis, Università di Genova.

Cassirer, Ernst, 1907. "Kant und die moderne Mathematik: Mit Bezug auf Bertrand Russells und Louis Couturats Werke über die Prinzipien der Mathematik." *Kant-Studien* 12: 1–49.

———, 1910. *Substanzbegriff und Funktionsbegriff*. Berlin: Cassirer Verlag.

———, 1922. *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neuren Zeit*, vol. 2. Berlin: Cassirer Verlag.

———, 1923a. *Philosophie der symbolischen Formen*, vol. 1. Berlin: Cassirer.

———, 1923b. *Substance and Function, and Einstein's Theory of Relativity*. Chicago: Open Court.

Colyvan, Mark, 2001. *The Indispensability of Mathematics*. Oxford: Oxford University Press.

———, 2012. *An Introduction to the Philosophy of Mathematics*. Cambridge: Cambridge University Press.

Darrigol, Olivier, 2003. "Number and Measure: Hermann von Helmholtz at the Crossroads of Mathematics, Physics, and Psychology." *Studies in History and Philosophy of Science* 34: 515–73.

Dedekind, Richard, 1872. *Stetigkeit und irrationale Zahlen.* Reprinted in Dedekind (1932), pp. 315–34.

———, 1888. *Was sind und was sollen die Zahlen?* Reprinted in Dedekind (1932), pp. 335–91.

———, 1932. *Gesammelte mathematische Werke*, vol. 3, edited by R. Fricke, E. Noether and Ø. Ore. Brunswick: Vieweg & Sohn.

Diez, José, 1997. "Hundred Years of Numbers: An Historical Introduction to Measurement Theory 1887–1990. Part I: The Formation Period. Two Lines of Research: Axiomatics and Real Morphisms, Scales and Invariance." *Studies in History and Philosophy of Science* 28: 167–81.

Du Bois-Reymond, Paul, 1882. *Die allgemeine Functionenlehre. Metaphysik und Theorie der mathematischen Grundbegriffe: Grösse, Grenze, Argument und Function*. Tübingen: Laupp.

Ebert, Philip A., and Marcus Rossberg, eds., 2016. *Abstractionism: Essays in Philosophy of Mathematics*. New York: Oxford University Press.

Edgar, Scott, 2015a. "Intersubjectivity and Physical Laws in Post-Kantian Theory of Knowledge: Natorp and Cassirer." In Friedman and Luft (2015), pp. 141–62.

———, 2015b. "The Physiology of the Sense Organs and Early Neo-Kantian Conceptions of Objectivity: Helmholtz, Lange, Liebmann." In *Objectivity in Science: New Perspectives from Science and Technology Studies*, edited by F. Padovani, A. Richardson, and J. Y. Tsou, pp. 101–22. Dordrecht: Springer.

Ehrlich, Philip, 2006. "The Rise of non-Archimedean Mathematics and the Roots of a Misconception. I: The Emergence of non-Archimedean Systems of Magnitudes." *Archive for History of Exact Sciences* 60: 1–121.

Ferreirós, José, 2006. "Riemann's *Habilitationsvortrag* at the Crossroads of Mathematics, Physics, and Philosophy." In Ferreirós and Gray (2006), pp. 67–96.

Ferreirós, José and Jeremy Gray, eds., 2006. *The Architecture of Modern Mathematics. Essays in History and Philosophy*. Oxford: Oxford University Press.

Flament, Dominique, 2005. "H. G. Grassmann et l'introduction d'une nouvelle discipline mathématique: l'Ausdehnungslehre." *Philosophia Scientiæ* 5: 81–141.

Frege, Gottlob, 1903. *Grundgesetze der Arithmetik*, vol. 2. Jena: Pohle.

Friedman, J. Tyler and Sebastian Luft, eds., 2015. *The Philosophy of Ernst Cassirer: A Novel Assessment*. Berlin: De Gruyter.

Friedman, Michael, 1990. "Kant on Concepts and Intuitions in the Mathematical Sciences." *Synthese* 84: 213–57.

Galilei, Galileo, 1623. *Il Saggiatore nel quale con bilancia esquisita e giusta si ponderano le cose contenute nella libra astronomica e filosofica di Lotario Sarsi Sigensano scritto in forma di lettera all'illustrissimo e reverendissimo Monsignore D. Virginio Cesarini*. Rome: Giacomo Mascardi. Translated into English as "The Assayer," in *The Controversy on the Comets of 1618*, by S. Drake and C. D. O'Malley. Philadephia: University of Pennsylvania Press, 1960.

Grassmann, Hermann G., 1844. *Die Wissenschaft der extensiven Grösse oder die Ausdehnungslehre, eine neue mathematische Disciplin dargestellt und durch Anwendungen erläutert*, first part, *Die lineale Ausdehnungslehre*. Leipzig: Wigand. Reprinted in Grassmann (1894–1911), vol, I.1.

———, 1861. *Lehrbuch der Arithmetik*. Berlin: Enslin. Partially reprinted in Grassmann (1894–1911), vol. 2.1, pp. 295–349.

———, 1894–1911. *Gesammelte mathematische und physikalische Werke*, 3 vols, edited by F. Engel. Leipzig: Teubner.

Gray, Jeremy, 2007. *Worlds Out of Nothing: A Course in the History of Geometry in the 19th Century*. Dordrecht: Springer.

Hale, Bob, 2000. "Reals by Abstraction." *Philosophia Mathematica* 8: 100–23.

———, 2002. "Real Numbers, Quantities and Measurement." *Philosophia Mathematica* 10: 304–23.

———, 2004. "Real Numbers and Set Theory–Extending the Neo-Fregean Programme Beyond Arithmetic." *Synthese* 147: 21–41.

Hale, Bob and Crispin Wright, 2001. *The Reason's Proper Study: Essays Towards a Neo-Fregean Philosophy of Mathematics*. Oxford: Clarendon Press.

Hankel, Hermann, 1867. *Vorlesungen über die complexen Zahlen und ihre Functionen*. Leipzig: Voss.

Heis, Jeremy, 2010. "'Critical Philosophy Begins at the Very Point Where Logistic Leaves Off': Cassirer's Response to Frege and Russell." *Perspectives on Science* 18: 383–408.

———, 2011. "Ernst Cassirer's Neo-Kantian Philosophy of Geometry." *British Journal for the History of Philosophy* 19: 759–94.

———, 2015. "Arithmetic and Number in the Philosophy of Symbolic Forms." In Friedman and Luft (2015), pp. 123–40.

Helmholtz, Hermann von, 1887. "Zählen und Messen, erkenntnistheoretisch betrachtet." In *Philosophische Aufsätze, Eduard Zeller zu seinem fünfzigjährigen Doctorjubiläum gewidmet*, pp. 11–52. Leipzig: Fues.

———, 1977. *Epistemological Writings*, translated by M. Lowe, edited by R. Cohen and Y. Elkana. Dordrecht: Reidel.

Hölder, Otto, 1901. "Die Axiome der Quantität und die Lehre vom Mass." *Berichte der mathematisch-physischen Classe der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig* 53: 1–64. Translated into English as "The Axioms of Quantity and the Theory of Measurement," by J. Michell and C. Ernst. *Journal of Mathematical Psychology* 40 (1996): 235–52, 345–56.

Kant, Immanuel, 1900–. *Kants gesammelte Schriften*. Berlin: Reimer/De Gruyter.

———, 1998. *Critique of Pure Reason*, edited and translated by P. Guyer and A. Wood. Cambridge: Cambridge University Press.

Königsberger, Leo, 1903. *Hermann von Helmholtz*, vol. 2. Brunswick: Vieweg.

Lagrange, Joseph-Louis, 1797. *Théorie des fonctions analytiques*. Paris: Imprimerie de la République.

Lewis, Albert C., 2004. "The Unity of Logic, Pedagogy and Foundations in Grassmann's Mathematical Work." *History and Philosophy of Logic* 25: 15–36.

Linsky, Bernard and Edward N. Zalta, 2006. "What is Neologicism?" *The Bulletin of Symbolic Logic* 12: 60–99.

Michell, Joel, 1993. "The Origins of the Representational Theory of Measurement: Helmholtz, Hoelder and Russell." *Studies in History and Philosophy of Science* 24: 185–206.

Mormann, Thomas, 2008. "Idealization in Cassirer's Philosophy of Mathematics." *Philosophia Mathematica* 16: 151–81.

———, 2018. "Zur mathematischen Wissenschaftsphilosophie des Marburger Neukantianismus." In *Philosophie und Wissenschaft bei Hermann Cohen*, edited by C. Damböck. Dordrecht: Springer.

Otte, Michael, 1989. "The Ideas of Hermann Grassmann in the Context of the Mathematical and Philosophical Tradition since Leibniz." *Historia Mathematica* 16: 1–35.

Panza, Marco, 2015. "From Lagrange to Frege: Functions and Expressions." In *Functions and Generality of Logic: Reflections on Dedekind's and Frege's Logicisms*, pp. 59–95. Cham: Springer.

Panza, Marco and Andrea Sereni, 2013. *Plato's Problem: An Introduction to Mathematical Platonism*. Basingstoke: Palgrave Macmillan.

Patton, Lydia, 2009. "Signs, Toy Models, and the A Priori: From Helmholtz to Wittgenstein." *Studies in History and Philosophy of Science Part A* 40: 281–89.

Petsche, Hans-Joachim, Albert C. Lewis, Jörg Liesen, and Steve Russ, eds., 2011. *From Past to Future: Graßmann's Work in Context. Graßmann Bicentennial Conference, September 2009*. Berlin and Heidelberg: Springer-Verlag.

Reck, Erich H., 2013. "Frege or Dedekind? Towards a Reevaluation of Their Legacies." In *The Historical Turn in Analytic Philosophy*, edited by E. Reck, pp. 139–70. Basingstoke: Palgrave Macmillan.

Riemann, Bernhard, 1868. "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen (1854)." *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen* 13: 133–42. Reprinted in Riemann (1876), pp. 254–69. Translated into English as Riemann (2016).

———, 1876. *Bernhard Riemann's Gesammelte mathematische Werke und Wissenschaftlicher Nachlass*, edited by R. Dedekind and H. Weber. Leipzig: Tuebner.

———, 2016. *On the Hypotheses Which Lie at the Bases of Geometry*, translated by W. K. Clifford, edited by J. Jost. Cham: Springer.

Schubring, Gert, ed., 1996. *Hermann Günther Grassmann (1809–1877): Visionary Mathematician, Scientist and Neohumanist Scholar. Papers from a Sesquicentennial Conference*. Dordrecht: Kluwer.

Shapiro, Stewart, 2000. "Frege Meets Dedekind: A Neologicist Treatment of Real Analysis." *Notre Dame Journal of Formal Logic* 41: 335–64.

———, 2003. "Prolegomenon to Any Future Neo-Logicist Set Theory: Abstraction and Indefinite Extensibility." *The British Journal for the Philosophy of Science* 54: 59–91.

Steiner, Mark, 1995. "The Applicabilities of Mathematics." *Philosophia Mathematica* 3: 129–56.

Stolz, Otto, 1885–1886. *Vorlesungen über allgemeine Arithmetik*, 2 vols. Leipzig: Tuebner.

Suppes, Patrick, 1951. "A Set of Independent Axioms for Extensive Quantities." *Portugaliae Mathematica* 10: 163–72.

Suppes, Patrick and Joseph L. Zinnes, 1963. "Basic Measurement Theory." In *Handbook of Mathematical Psychology*, edited by R. D. Luce, R. R. Bush, and E. Galanter, pp. 1–76. New York: Wiley & Sons.

Sutherland, Daniel, 2006. "Kant on Arithmetic, Algebra, and the Theory of Proportions." *Journal of the History of Philosophy* 44: 533–58.

Tappenden, Jamie, 2006. "The Riemannian Background to Frege's Philosophy." In Ferreirós and Gray (2006), pp. 97–132.

Torretti, Roberto, 1978. *Philosophy of Geometry From Riemann to Poincaré*. Dordrecht: Reidel.

Veronese, Giuseppe, 1891. *I fondamenti della geometria*. Padoa: Tipografia del Seminario.

Wright, Crispin, 1997. "On the Philosophical Significance of Frege's Theorem." In *Language, Thought and Logic: Essays in Honour of Michael Dummett*, edited by R. Heck, pp. 201–44. Oxford: Clarendon Press. Reprinted in Hale and Wright (2001), pp. 272–306.

———, 2000. "Neo-Fregean Foundations for Real Analysis: Some Reflections On Frege's Constraint." *Notre Dame Journal of Formal Logic* 41: 317–34.

Yap, Audrey, 2017. "Dedekind and Cassirer on Mathematical Concept Formation." *Philosophia Mathematica* 25: 369–89.