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Collections in Early Bolzano

Stefania Centrone and Mark Siebel

There are quite a few studies on *late* Bolzano's notion of a collection (*Inbegriff*). We try to broaden the perspective by scrutinising the forerunner of collections in Bolzano's *early* writings, namely the entities referred to by expressions with the technical term "et". Special emphasis is laid on the question whether these entities are set-theoretical or mereological plenties. Moreover, similarities and differences to Bolzano's mature conception are pointed out. We argue that early Bolzano's particular blend of set-theoretical and mereological features is best interpreted as a set-theoretically enhanced mereology.

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Stefania Centrone and Mark Siebel

1. Introduction

In chapter XII of the second book of Leibniz' *Nouveaux Essais* (Leibniz 1704), Theophilus and Philalethes discuss complex ideas. Let us eavesdrop on their conversation:¹

Phil. With respect to substances there are two sorts of ideas. One of them is of single substances, such as the idea of a man or a sheep; the other one is of several substances put together, such as the idea of an army of men or a flock of sheep. These collective ideas also form a single idea.

Theo. This unity of the idea of an aggregate [*unité de l'idée des agrégés*] is a very genuine one; but fundamentally we have to admit that this unity of collective ideas is merely a respect or relation, whose foundation lies in what each of the individual substances taken by itself has. Thus the only perfect unity these *entities originating through aggregation* have is a mental one, and consequently their very *being* is in a way mental or phenomenal, like that of the rainbow in the sky.

Philalethes points out that a collective idea (such as the idea of an army of men) is an idea in the same sense as the idea of a single substance (such as the idea of one of the army's members). Theophilus agrees but adds that this does not hold for the things falling under these ideas. Whereas the several substances put together in thought exist independently of whether they are thought together or not, their aggregate does not. The latter is rather existentially dependent on a mind aggregating the given

¹Occasionally, we modified existing translations in order to come closer to the original source.

parts, and hence it is a unified whole only insofar as there exists a single idea of it.

In a diary-note on the "Differences between Leibniz' and my Opinions", the Bohemian mathematician and philosopher Bernard Bolzano criticises the psychological character attributed to aggregates by Leibniz (*BGA* 2 B 18/2, 38ff.).² He quotes one of Leibniz' remarks in a letter to the Hanoverian princess Sophie: "L'assemblage des êtres, n'est pas un être" ("A collection of beings is not a being").³ Bolzano comments: "Warum nicht?" ("Why not?"), and he continues in French: "un être, mais non pas une substance" ("a being but not a substance"). "Sein" ("to be") is for late Bolzano tantamount to being causally potent; the corresponding entities are also labelled "wirklich" ("real") by him. He thus seems to be saying that aggregates of causally potent objects are part of the causal order even if they are not substances, the latter meaning roughly that they do not exist in themselves.

Late Bolzano's term for aggregates is "Inbegriff", and we will use the translation "collection" henceforth. In his remark on Leibniz, Bolzano attributes a feature to collections suiting wholes in the sense of contemporary mereology but not sets in the sense of modern set theory. After all, whereas a whole with real objects as its parts is itself real, a set with such objects as its elements has no causal powers because of its mathematical abstractness. There are further places in Bolzano's writings providing support for the mereological conception of collections. For example, his argument for the existence of substances in the *Athanasia* pre-

²*BGA* is the abbreviation for the *Bernard-Bolzano-Gesamtausgabe* (Bolzano 1969-).

³*BGA* (2 B 18/2, 41). In Leibniz (1875-90, VII, 557), we read, however: "l'assemblage de plusieurs Estres n'est pas un Estre". In his introduction to *BGA* (2 B 18/2), Berg assumes that Bolzano is quoting from Feuerbach (1837) and that the quarrel with Leibniz, in his diary-notes, probably originates from the publication of Feuerbach's work. Indeed, we read in Feuerbach: "l'assemblage des Estres, n'est pas un Estre" (339). The book, however, does not appear in the catalogue of Bolzano's original library (Berg and Morscher 2002).

supposes that the collection of all adherences, i.e., particularised properties occurring at a certain time and needing a bearer, is real (*AT* 22). Similarly, one of his arguments for the existence of God in the *Lehrbuch der Religionswissenschaft (Textbook of the Science of Religion)* requires that collections of real objects whose existence is contingent on something are themselves real (*RW* 178).⁴

There are quite a few studies on *late* Bolzano's notion of a collection. We try to broaden the perspective by scrutinising the forerunner of collections in Bolzano's *early* writings.⁵ The most fruitful source is a manuscript on the foundations of mathematics entitled *Allgemeine Mathesis (General Mathematics, AM)*, being intended to result in a second issue of the *Beyträge zu einer begründeteren Darstellung der Mathematik (Contributions to a Better-Founded Presentation of Mathematics, BM)*. The manuscript was written between the publication of the *Beyträge* in 1810 and Bolzano's decision in 1812 to work on a "new logic" which resulted in the *Wissenschaftslehre (Theory of Science, WL)*.⁶ Although Bolzano makes use of the word "Inbegriff" in his early writings, it occurs only rarely and not as a central term of his ontology (*BM* 39, 66; *AM* 54). His initial conception of collections is thus not to be found under the heading "Inbegriffe". As will be presently seen, it rather hides behind what he says about the little word "et" and the kind of composition denoted by it.

Being a study in formal ontology, the *Allgemeine Mathesis* treats the general laws pertaining to *all* things, i.e., not only to "real

things" but also to "things of thought in their merely ideal being" (*AM* 17; see also *BM* 6, 11–12, 16–17). As already mentioned, "real" ("wirklich") in application to objects means *causally potent* in late Bolzano. This must not be identified with *perceptible* because "the individual substances of which... sensible objects are composed" are not perceptible in Bolzano's sense but real (*WL* III, §315, 247).⁷ Aside from real objects, mature Bolzano accepts a large class of non-real or abstract things, such as space and time, merely possible things (e.g., a golden mountain), mathematical objects (e.g., numbers and geometrical figures) and logico-semantic objects (e.g., sentences and ideas in themselves) (*Morscher* 1974, 122–23). As to the latter, he differentiates the word "dog", the subjective idea of a dog occurring in the mind of a particular person at a particular time and the common content of word and act of thinking, viz., what he terms the "objective idea" or "idea in itself" of a dog.

In early Bolzano, things are less clear. At the time of the *Beyträge* and the *Allgemeine Mathesis*, he already makes a distinction between words and the concepts denoted by them.⁸ But he does not strictly keep apart concepts as concrete mental entities and concepts as the abstract contents of mental and linguistic entities. Bolzano's decided distinction between subjective and objective ideas, as well as his distinction between judgements and sentences in themselves, is still in its infancy. There are places where he distinguishes between "truths" and "conceived, recognised truths, true cognitions" and emphasises that "not every truth has to be (by its definition) a cognition, thought or

⁴Similar evidence for the mereological interpretation is to be found in *PU* (§25, 72–73); *WL* (I, §79, 364); and *WL* (III, §353, 407–08; see also I, §68, 307–08) in combination with *WL* (I, §75, 337–38). But note that the thesis that collections of real things are themselves real is in conflict with two of Bolzano's central ontological principles (*Schnieder* 2002, 218–19; *Textor* 1996, 336, 346; contrast *Rusnock* 2012, 828–34).

⁵Brief reflections on Bolzano's early account of collections are contained in *Krickel* (1995, sec. C.II) and *Sebestik* (2010).

⁶Berg mentions this decision in his introduction to the first part of the *Wissenschaftslehre* in the *Gesamtausgabe* (*BGA* 1 11/1, 9).

⁷See also *AT* (85); *WL* (I, §79, 362, §70, 318). In contrast, a "real idea" ("reale Vorstellung") is an idea under which an object can fall, regardless of whether this object is real in the above-mentioned sense. That is, real ideas need not represent real things (*WL* I, §79).

⁸See *BM* (52–53); *AM* (144, 147–48, 160); and many passages from the manuscript "Etwas aus der Logik" (circa 1812; *BGA* 2 A 5, 141–46, 160–61, 163). In §29 of the latter, Bolzano also distinguishes sentences and the judgements expressed by them (*BGA* 2 A 5, 147–48).

imagined by someone" (AM 178). In the same spirit, he declares in the *Beyträge* (BM 39–40): "In the realm of truth, i.e. in the collection of all true judgements, a certain *objective connection* prevails which is independent of our accidental *subjective recognition* of it. As a consequence of this, some of these judgements are the grounds for others and the latter are the consequences of the former." But Bolzano is still nowhere near permanently bearing in mind this distinction.

Some passages from the *Beyträge* suggest that "real" is to be understood in the sense of "perceptible". Real things, we read, are "sensible" and "affect my perceptive faculty" (BM 3, 142). This would mean that, whereas all of early Bolzano's real things are real in late Bolzano's sense, the converse is not true because some causally potent objects are not perceptible. However, other passages in the *Beyträge* refer to things "which possess an objective existence independent of our consciousness" (BM 11–12), and this is a description covering also mathematical and logico-semantic objects. Ideal existence, on the other hand, is the "possibility of being thought"; an ideal thing is a thing "which can in general be an object of our capacity for representation" (BM 3, 12). Since the golden mountain does not exist but is conceivable, early Bolzano would class it as ideal. It is uncertain, however, what he would say about the objective idea of a golden mountain. On the narrow reading of "real", it is not real because it is not perceptible; on the broad reading, it is real because it exists independently from our consciousness.

Bolzano identifies just two laws applying to all things, whether real or ideal (AM 17; see also 32–33, 54). First, there is the "law of being thinkable together" ("Gesetz der Zusammendenkbarkeit"): "each thing can be combined in thought with every other thing". Secondly, there is the "law of relation" ("Gesetz des Verhältnisses"): "each thing stands in a certain relation to every other thing". The kind of composition relevant to the first law is elucidated by contrasting it with a second kind of composition:

1. There is, to begin with, a composition of two concepts into a third indicated by the word *and*; for example, *inkpot* and *sandpot* together result in the concept of a *writing tool*. . . . This composition is possible for any two concepts. . . . 2. A quite different composition is the one . . . taking place between the concepts *house* and *wooden* when one thereof forms the third concept *wooden house*. . . . This kind of composition is no longer applicable to all things. . . . A wooden house is an object that is a *house* and *wooden* at the same time. By contrast, a *writing tool* is neither an object which is an *inkpot* nor an object which is a *sandpot*, but a sum of *inkpot* and *sandpot*. (AM 34–35)

Further examples of the first type of composition are the fusion of the concepts "knife" and "fork" into "cutlery" (AM 33), "the flower" and "the pot" into "the flower in the pot" (BGA 2 B 15, 223) and concepts of arithmetic sums (AM 51–52).

Bolzano points out that both types of composition were already mentioned in the *Beyträge*. The first one was expressed by the connective "et" and the second one by "cum". Although Bolzano prefers to use "quod" for the latter henceforth (AM 34), we remain with "cum". Some pages later, the result of an et-composition receives a telling name:

What originates through [the act of] thinking together is called a whole (*totum*), probably also a *sum*, if need be also a *system*. The things, however, which are thought together are called, in regard to the whole, the *parts* (*partes*), *elements* of it. (AM 38)

Applied to one of Bolzano's examples, "the flower in the pot" refers to a whole/sum/system whose parts/elements are the flower and the pot. It thus seems to denote what Bolzano later terms a collection.

In the following sections, Bolzano's early conception of collections is spelled out in more detail, often by comparing it with his mature view. The main focus is on whether collections are to be treated as set-theoretical or mereological entities. In Section 2, after the differences between these entities are pointed out, four interpretations of "et" are introduced, two of them set-theoretical

and two mereological. In Sections 3–8, the four interpretations are put to the test. Finally, Section 9 provides an overview of the results and a conclusion. In a nutshell, early Bolzano’s theory of collections is best interpreted as a set-theoretically enhanced mereology.

2. Set-theoretical and Mereological Interpretations

One of the central issues in the literature on Bolzano’s advanced theory of collections is whether they could, or even should, be interpreted as *sets* in the sense of *set theory* or as *wholes* in the sense of *mereology*. Roughly, set theory and mereology differ because the former is concerned with the problem of “the many as one”, i.e., with adding together a number of objects to form a new object, and the latter with the problem of “the one as many”, viz., with considering one object as a compound of its parts. This gives rise to some differences in the details (Simons 1987, 10, 13; Simons 1997; Krickel 1995, 18–19; Schnieder 2002, 210).

First, the relation between a whole and its parts—parthood—and the relation between a set and its elements—membership—are not the same. Parthood (more exactly, the relation of being a *proper* part) is characterised by transitivity and irreflexivity: (1) If a is a part of b and b is a part of c , then a is a part of c . (2) Nothing is a part of itself. In contrast, membership is neither transitive nor irreflexive. As to (1), every object a is a member of the corresponding singleton $\{a\}$, in symbols: $a \in \{a\}$; and this singleton, in turn, is a member of the singleton of the singleton of a , in symbols: $\{a\} \in \{\{a\}\}$; but a is not a member of $\{\{a\}\}$. As to (2), the set of all things without a weight is an element of itself because sets do not have a weight.

Secondly, set theory admits of singletons and the null set. As opposed to the latter, the idea of a “null whole”, or a “null individual” being part of every object, is hardly tenable. Moreover, even if every object could be seen as an *improper* part of itself, this principle follows a different tack than the principle that ev-

ery object is an element of its singleton. After all, a whole is by definition and thus always identical with its improper part, whereas a singleton is never identical with its sole element.

Thirdly, sets are extensional. They are identical if they contain the same elements, while the order or arrangement of their elements is irrelevant. Hence, $\{1, 2\} = \{2, 1\}$. Wholes, on the other hand, need not be extensional. For example, a house is a whole of walls, ceilings, windows etc. arranged in a certain way.

Fourthly, sets are usually conceived as abstract in the sense of not entering into causal relationships. In contrast, the ontological status of a whole depends on the status of its parts. Wholes are abstract if their parts are abstract; but they can also be real in late Bolzano’s sense, namely when their parts are real. To use one of Frege’s (1895, 436–37) examples, we can see, touch and wander through the whole of all trees of the Black Forest. But we cannot do this with the corresponding set of trees.

Fifthly, sets are often seen as abstract in the further sense of having no temporal existence, that is, not coming into existence sometime or ceasing to exist at another time. But this is far from obvious because a set seems to exist only if its elements exist. Along these lines, the singleton $\{\text{Leicester City}\}$ did not exist before 1884, when the football club was established. Hence, sets appear to be on a par with wholes in this respect because a whole exists as long as its parts exist. By planting the trees of a wood, we bring the wood into being; and by burning the trees down, we destroy the wood.

Sixthly, the elements of a set need not be held together by common properties or the like. Any number of objects may be summarised in a set, regardless of their ontological status, the particular kind they belong to or their spatiotemporal location. Classical extensional mereology agrees with this insofar as it allows any number of things to build a whole (Simons 1987, chaps. 1–2). But there are also attempts at restricting mereological sums to “integral wholes”, i.e., aggregates forming a unified whole (Simons 1987, chap. 9). Since Bolzano’s collections comply

with the basic law that “each thing can be combined in thought with every other thing” (AM 17; see also 54), it is evident that not all of them are integral wholes. However, since there is disagreement as to the liberty in forming mereological sums, we take the more important question to be whether collections are wholes in the broad sense delimited by the first four characteristics. For the following considerations, we thus assume a broad reading of “whole” according to which any objects whatsoever form such a whole, even, say, Napoleon and the number π .

Bolzano was often accused of confounding set-theoretical and mereological notions. The Kneales (1962, 364) maintained: “Bolzano seems to be in danger of confusing a whole of parts with a set of members.” George (1983, 256) has followed suit: “In Bolzano, set-theoretical and mereological notions tend to run together.” While Berg (1992, 37) argues that there are set-theoretical as well as mereological collections in Bolzano, and Krause (2004, chap. II.7.1) thinks that only collections of *substances* are mereological entities, Krickel (1995), Siebel (1996, chap. 1.6) and Behboud (1997) promote a uniform mereological interpretation. Simons (1997), on the other hand, argues that Bolzano’s theory of collections is neither a set theory nor a mereology, but a general theory covering all kinds of plenties (compare Rusnock 2013, 155). Note, however, that Simons (1997, 105) has in mind a mereology where “a mereological sum is not always to be had, e.g., if some of the objects are concrete and others are abstract, or if they are widely separated in space or time or of widely differing category”. Simons thus operates with the restricted notion of an integral whole.

Let us use “plenties” as a neutral term covering both sets and wholes and “constituent” as a neutral term for elements of sets and parts of wholes. Then the main question of this paper is whether et-composition refers to set-theoretical or mereological plenties. Note that the term “et” is allowed to link both general and singular terms. In the *Beyträge*, the copula in categorical sentences is glossed as expressing “the inclusion of a certain thing, as

individual or kind, under a certain genus” (BM 74). In this spirit, schemata like “A is C” are not only meant to represent “Human beings are mortal” but also “Cajus is a human being” and “Cajus is mortal” (BM 65). Moreover, examples of et-composition with definite articles, such as “the plane surface and the pasteboard on it” and “the flower in the pot”, as well as the fact that Bolzano is interested in what happens when two numbers are thought together, make clear that he has no objections to letting singular terms be connected by “et” (AM 42, 61–62; BGA 2 B 15, 223). To avoid ambiguity, however, we reserve capital letters for general terms and lower-case letters for singular terms henceforth.

Berg (BGA 2 B 14, 13), Centrone (2012b, 6–7; 2012a, 129ff.; 2016, 220–21), Roski (2010, 8–9; 2014, 52) and Sebestik (2010, 23–24) have proposed that we understand “et” by means of set-theoretical union (\cup). They opted for the thesis that the extension of “A et B” is a set, namely the set comprising all those objects being A or B, in symbols: $\{x \mid Ax \vee Bx\}$. The expression “wooden et house” thus denotes the set of all wooden things, whether houses or not, plus all houses, whether wooden or not. Along these lines, it is natural to assume that the singular-term variant “a et b” refers to the pair set $\{a, b\}$, i.e., the set of things being identical with a or b.

The direct mereological counterpart to this interpretation takes “a et b” to name the mereological whole of a and b. For example, “Kim et Tim” does not pick out the abstract set with Kim and Tim as elements but the real whole consisting of Kim and Tim—and hence of their heads, noses and so on. Likewise, a general-term expression “A et B” denotes an object which may be heavily scattered in space and time, namely the totality of every A and every B.

But this is not the end of the story because Bolzano’s comments suggest a second mereological interpretation.⁹ Remember that

⁹The following interpretation was independently found by Blok (2016, 209–12), but he overlooked the difficulties connected with it.

his name for what results from thinking together inkpot and sandpot is “Schreibzeug” (“writing tool”), and that he takes the latter to be “neither an object which is an *inkpot* nor an object which is a *sandpot*, but a sum of inkpot and sandpot” (AM 34–35). In Bolzano’s days, people used quills for writing. Frequently, they owned not only a pot with ink but also a pot with sand in order to dry the ink on the paper. There were even utensils containing an inkpot and a sandpot within one vessel. It might thus very well be that Bolzano had in mind aggregates of an inkpot and a sandpot, including such vessels, when talking about “Schreibzeug”. The same holds for “Tischbesteck”, Bolzano’s name for the result of thinking together knife and fork (AM 33). This term may be translated by “cutlery”, so as to denote all knives and forks. But it is also possible to translate it by “cutlery set”, and then it denotes *ensembles* of a knife and a fork. Understood in this way, “knife et fork” does not refer to the whole of all knives and forks but rather to all wholes of a knife and a fork. The singular-term variant “*a et b*” is assigned the same extension as before, namely the whole consisting of *a* and *b*.

For the sake of completeness, it should be mentioned that the second mereological interpretation gives rise to a second set-theoretical interpretation. According to the latter, “*A et B*” does not refer to the set of all *As* and *Bs* but to all sets of an *A* and a *B*. Analogously, “*a et b*” picks out the same object as under the first set-theoretical reading, namely the pair set containing *a* and *b*.

There are thus two set-theoretical and two mereological interpretations of “et”. To consider just the general-term variants, they take “*A et B*” to refer to

- (S1) the set of all *As* and all *Bs*,
- (S2) all sets of an *A* and a *B*,

- (M1) the whole of all *As* and all *Bs*,
- (M2) all wholes of an *A* and a *B*.

In the next six sections, we chiefly examine whether early Bolzano’s remarks on collections are more in harmony with a set-theoretical or a mereological interpretation. In Section 3, Bolzano’s claim that collections are ideal is discussed. The touchstones in Section 4 are his examples of collections. In Section 5, the issue is whether Bolzano allows for collections having just one constituent or collections containing the same thing twice. Section 6 starts with the question whether a collection’s identity depends on the arrangement of its constituents and continues by comparing early Bolzano’s “multitudes”, “arithmetical sums”, “discrete quantities” and “continuous quantities” with late Bolzano’s “multitudes”, “sums” and “pluralities”. Section 7 addresses Bolzano’s assumption that any two concepts may be combined by “et”. Finally, in Section 8 we take into account two basic inference rules from the *Beyträge* whose conclusions are propositions containing an et-term as subject or predicate. The aim here is to find out which interpretation makes these inferences valid.

To be on the safe side, the *names* Bolzano assigns to collections and their constituents do not provide any evidence for or against one of the given interpretations. From the modern point of view, the German counterparts to terms like “whole”, “sum”, “system” and “part” suggest a mereological reading. But at the beginning of the 19th century, terminology was in a state of flux, and even 75 years later Dedekind (1888, 1ff.) made use of the word “System” for sets. In a footnote to an unpublished manuscript from 1816, Bolzano entrusts a mereological reading to his readers by explaining that “Teil” (“part”) is to be understood in the sense it has “when it is said that soul and body are the two constituent parts [Bestandteile] of man” (BGA 2 A 5, 192). However, a passing remark in just one manuscript does not give permission to draw general conclusions. Moreover, we have seen that Bolzano does not only use the term “Teile” but also the term “Elemente” (“elements”) for the constituents of collections.

3. The Ideality of Collections

A first clue to an adequate interpretation of early Bolzano's collections could be hiding in the fact that he characterises et-composition as "ideal" (AM 36–37, 40). Et-composition "is a merely *subjective* connection of the ideas of the things in our mind, whereby the things themselves (even if they are only things of thought) should not be altered" (AM 35). "That every thing, at least in our imagination, can be joined to every other thing is a feature obviously not pertaining to the *things themselves*, and their being, but only to the ideas of them" (AM 33). Such attributions of ideality may be understood in different ways, the first one being completely innocuous. Just as late Bolzano (EG sec. 3, §6, 101; PU §3, 40–41), so the early one holds that one "can think together even the most disparate things" (AM 42). One of the reasons is that we need not literally put the objects together. The collection of them is thus not real but ideal insofar as they remain at their locations and need not be connected by chains, ropes or any other material links.

However, we also read that a collection "originates through [the act of] thinking together" (AM 38). This would mean that collections are ideal in a second—less innocuous—sense: they are existentially dependent on acts of thinking. Along these lines, the collection of Kim and Tim does not exist until someone gives thought to it, say, by wondering whether Kim and Tim live together. This seems to be what Leibniz claims in the excerpt from the *Nouveaux Essais* cited at the beginning. Moreover, it corresponds to the way in which a close reader of Bolzano, early Husserl, interprets the Leibniz of the *Dissertatio de Arte Combinatoria*. In his first work, *The Philosophy of Arithmetic*, Husserl explains that the whole business of making many into one is the result of a mental act of a specific kind that picks up certain contents and unites them collectively.¹⁰

¹⁰See Husserl (1891, 17); Centrone (2010, 1–12, 81–85); and Leibniz (1666,

The third sense of "ideal" is even more demanding. Bolzano writes that we are concerned with "a merely *subjective* connection of the ideas of the things in our mind" (AM 33). Moreover, he introduces collections by saying that ideas (not things) can be put together to form further ideas or judgements (AM 33); and he writes that "each ideal composition can occur only between two things (ideas[!])" (AM 37). These remarks suggest that collections are ideal in the third sense of being no more, no less than complex subjective ideas with simpler ideas as their constituents (Krickel 1995, 99–104, 123). This would explain why young Bolzano takes collections to be ideal in the second sense of coming into being by acts of thinking together. Collections would be the products of mental acts precisely because they were ideas generated in the mind.

About 75 years later, Dedekind introduced classes in a way quite similar to Bolzano's way:

It very often happens that different things $a, b, c \dots$ are regarded for some reason from a common point of view, are put together in the mind, and it is then said that they form a *system S*. (Dedekind 1888, 1–2)

In his *Grundgesetze der Arithmetik (Basic Laws of Arithmetic)*, Frege comments on this:

A hint of the truth is indeed contained in talk of the common point of view; but regarding or putting together in the mind is no objective characteristic. I ask: in whose mind? If they are put together in one mind, but not in another, do they then form a system? What may be put together in my mind must certainly be in my mind. Do things outside me, then, not form systems? Is the system a subjective construction in the individual mind? Is the constellation Orion therefore a system? (Frege 1893, 2)

Analogous questions may be levelled against young Bolzano,

170): "*Unum* . . . esse intelligitur quicquid uno actu intellectus, s. simul, cogitamus." ("By 'one' we mean whatever we think of in one intellectual act, or at once.")

while his mature self is in line with Frege when explicitly rejecting mentalistic conceptions of collections:

Many people will still not concede to me that among all things, even the most disparate ones, a certain *combination*, a kind of *together* obtains; for they are used to taking the word in a much narrower sense. The maximum they will perhaps concede to me is that between such things a merely *ideal* combination is possible, a combination they will explain in such a way that not *the things themselves* are together, but only their *ideas* can be *thought together* by a thinking being. Hereto I can only reply by recalling yet again that things we can *think together* with the result of being able to say something true about this, as the word is, *ideal whole* they form, must necessarily form a whole to which the quality we talk about actually belongs exactly for the reason that, otherwise, our sentence could not express a truth. (EG sec. 3, §7, 102)

Here Bolzano appears to be criticising the view he himself held some years ago (Krickel 1995, 103–04). Now he thinks that a collection does not come into existence by its constituents being thought together. It rather exists because it has a property not possessed by the constituents in isolation. Collections are thus full-fledged objects existing independently of acts of thinking (EG sec. 3, §6, 100–01; PU §3, 40–41). Consequently, they cannot be complex subjective ideas because the latter do not exist without such acts.

If early Bolzano's collections were complex subjective ideas, then they would clearly not be sets but wholes, albeit wholes of a particular kind, namely mental ones. For in contrast to sets, mental entities enter into causal relationships. Moreover, in contrast to the null set or a singleton, a complex idea must have at least two constituents.

However, while it is fairly obvious that early Bolzano took collections to be existentially dependent on acts of thinking, it is uncertain whether he believed that this holds because they are subjective ideas. After all, there is no inconsistency in claiming

that the collection of, say, the moon and Galileo, although it comes into being only by forming a corresponding idea, is itself not an idea because it consists of a celestial body and a person. In the introduction, it was already emphasised that, in contrast to late Bolzano, his early self does not make a sharp distinction between linguistic expressions, mental acts and their contents. We have the impression that (perhaps under the influence of Kant) he also does not strictly tell apart ideas and what they are ideas of. In the case at issue, keeping up this distinction is particularly difficult for him because he believes that collections come into being by forming ideas of them. In what follows, we thus assume that the early Bolzano, just as the late one, does not conceive of collections as complex ideas. More exactly, while the idea of a collection is trivially a complex idea, the collection itself is an idea at best if its constituents are ideas.

It is tempting to think that collections' coming into being by acts of thinking speaks for a mereological reading of "et" for the very reason that, in this way, they are not abstract in the sense of being atemporal. But remember that a set with a contingent element has a temporally constrained existence, too, because it exists only as long as this element exists. The crucial question is thus whether the assumed *cause* of a collection, namely an act of thinking, militates against the set-theoretical or the mereological interpretation. It appears, however, that plenties of both types, sets and wholes, do not have mental causes. Hence, if ideality in the second sense were a defining characteristic of collections in early Bolzano, they would be sets just as little as wholes. We conclude from this that the property of coming into being by acts of thinking should not be treated as belonging to the *definition* of collections. It is rather an *external* property assigned to them by early Bolzano and, for good reasons, rejected by his later self. Therefore, Bolzano's remarks on the ideality of collections favour neither a mereological nor a set-theoretical reading.

4. Examples of Collections

Let us consider Bolzano's examples of collections. Many of late Bolzano's examples belong to the causal order and hence are hardly sets but wholes. This is true for bodies in general (*AT* 53; *PU* §14, 54), human bodies in particular, their parts, such as eyes and hands, and parts of these parts (*AT* 102), human beings or persons (*AT* 33–34), especially kings, ministers and subjects of a state (*EG* sec. 3, §9, 102–03), a group of travellers (*AT* 34), a group of founders (*WL* I, §83, 398), a society of people (*WL* I, §84, 399; *EG* sec. 3, §11, 104), a state (*EG* sec. 3, §9, 102), a city (*AT* 33), a pile of coins (*WL* I, §84, 399), a watch (*WL* I, §79, 368), a ball (*AT* 304) and a drinking glass (*PU* §4, 41). To some of these entities Bolzano additionally assigns properties entailing that they belong to the causal world: cities can burn, human beings may be benevolent, and a group of travellers may cover a particular distance (*AT* 33–34). Furthermore, he states that bodies produce effects (*AT* 53; *PU* §14, 54), and more generally, that collections may cause a certain outcome (*WL* I, §82, 394) as well as that they sometimes make an impact not to be generated by their parts (*AT* 84). As to the collection of all created beings, he even comes straight to the point by emphasising that it is real (*PU* §25, 72–73).

Similar examples may be found in early Bolzano. In the section of the *Allgemeine Mathesis* entitled "Von dem Begriffe einer Größe" ("On the Concept of a Quantity"), he points out that quantities are a special kind of collection (*AM* 51; more on quantities in Section 6). As examples of quantities he mentions not only a troop of equestrians and a heap of balls, but also the length of a line, velocity, force and degree of heat (*AM* 49–50). Since the latter are measurable, all of these examples seem to be objects entering into causal relationships. The same holds for houses (*AM* 51, 59) and a flower in a pot (*BGA* 2 B 15, 223). They support a mereological interpretation of collections because they lack the abstractness attached to sets.

5. Restrictions on the Constituents of Collections

One of Krickel's (1995, 300) arguments for the thesis that late Bolzano's collections are to be understood mereologically is that Bolzano does not allow for anything like a null set or singletons. According to his official analysis, a collection is "something that has compositeness" (*WL* I, §82, 393–94; see also *EG* sec. 3, §6, 100). He also explains what a collection is by using formulations such as "a whole consisting of particular parts" (*PU* §3, 40; see also *WL* I, §82, 393; *EG* sec. 3, §6, 100). The plural in the latter locution and the word "compositeness" in the former imply that a collection has at least two constituents. There is thus neither an empty collection nor a collection with just one part; and while this result fits wholes, it does not fit sets (Rusnock 2013, 156).

In young Bolzano, the place of the concept "compositeness" is taken by concepts like "connecting", "putting together" and "thinking together". This change in terminology is due to the fact that he assumed collections to come into being by acts of thinking. His later analysis does not have such an implication because a composite object need not be the result of a mental act. However, since young Bolzano's terms denote acts directed at two or more things, it is safe to assume that Bolzano's initial conception of collections, too, does not permit anything along the lines of a null set or a singleton.

Incidentally, early and late Bolzano agree in believing that the respective core concept in their explications of what a collection is, i.e., "compositeness" or "connecting", are simple concepts (*AM* 37; *WL* I, §82, 394; *EG* sec. 3, §6, 100). Moreover, late Bolzano emphasises that the full concept representing collections, the so-called concretum "something that has compositeness", is not simple because it contains not only the abstractum "compositeness" but also contains the all-embracing concept "something" and the copula "has" (*EG* sec. 3, §6, 100).¹¹ In contrast, his early

¹¹For concrete and abstract concepts see *WL* (I, §60).

self makes the mistake of regarding the corresponding concretum, which he takes to be the concept of “an effect . . . whose cause is *composition*”, to be simple (AM 38).

Bolzano’s mature theory of collections includes three no-redundancy restrictions on the parts of a collection (Krickel 1995, 76–77). They determine, with growing strength, that *a* and *b* form a collection only if

(NR1) *a* is not identical with *b* (PU §3, 41),

(NR2) *a* is not a constituent of *b* or conversely

(EG sec. 3, §6, 101),

(NR3) *a* and *b* have no common constituents

(EG sec. 3, §99, 157).

These constraints are further grist to the mereological mill. For whereas they make sense when applied to mereological wholes and their parts, they turn out to be wrong under a set-theoretical reading. As to (NR1), there is no ban on expressions such as “{*a*, *a*” and “{*a*, *b*, *a*” if they are taken to refer to {*a*} and {*a*, *b*}, respectively. As to (NR2) and (NR3), there are sets where one element is an element of another element, e.g., {*a*, {*a*, *b*}}; and there are sets with elements containing a common element, among them {{*a*, *b*}, {*b*, *c*}} (Krickel 1995, 82–83). Hence, these conditions confirm a mereological translation of late Bolzano’s collections.

Unfortunately, there are no explicit considerations regarding no-redundancy to be found in Bolzano’s early writings. The only thing safe to say is that his examples of collections do not violate these constraints. On the other hand, remember that Bolzano takes et-composition to be “possible for any two concepts” (AM 34). Assuming that Bolzano really means *concepts* here, this appears to be in conflict already with the weakest no-redundancy principle. For it allows “Walter Scott et the author of *Waverley*” to denote a collection, whereas there is no such collection according to (NR1) because Walter Scott is identical with the author of *Waverley*. However, “two concepts” could also mean concepts with

two, viz., different, extensions. Remember also that, in its original formulation, the law of being thinkable together states that “each *thing* can be combined in thought with every other *thing*” (AM 17; emphasis added). This could even be interpreted along the lines of the strong principle (NR3): if *a* and *b* have common constituents, the difference between them is not big enough to say that one thing is combined with *another* thing.

At the time of the *Allgemeine Mathesis*, Bolzano concedes only pairwise et-combinations:

It is my opinion, to wit, that each ideal combination is actually of the form: *A et B*; or [*A et B*] *et C*. I wish to say that, in my view, each ideal combination can occur only between *two* things (ideas). But not *A + B + C + D*. (AM 37)

What gave Bolzano this notion? The bracketed addition “ideas” after “things” reminds us of his subjective conception of collections. If collections came into being by thinking objects together, and if our limited mind were able to combine only two objects at a time, then a combination of more than two objects would have to be carried out by combining them step by step. That is, first of all, two of them are put together; then the result of this composition is brought together with a third object; and so on.

Since late Bolzano conceives of collections as objective things and distinguishes objective from subjective ideas, it is no wonder that he eventually rejects his early position:

Someone who considers the constituents of which linguistic expressions for such collections are usually formed (*A* and *B* and *C* etc.) might suspect that the idea is composed of the ideas of the objects *A*, *B*, *C*, *D* . . . and the idea of *combination* expressed by the word ‘and’ repeated as many times as there are objects, less one. . . . Closer consideration, however, reveals this suspicion to be incorrect. For if the idea of a collection of things *A*, *B*, *C*, *D* . . . were really formed in the way the expression ‘*A* and *B* and *C* etc.’ seems to suggest, i.e., if the concept of combination occurred between every two of the ideas *A*, *B*, *C*, *D* . . . , then the connection between the individual parts of every such collection would have to be always

of the kind where a single one of the combined objects, e.g., A , is immediately conjoined only with a single second object B , and the collection of these two immediately conjoined with a single third object C , etc. . . . I believe instead that the concept of connection occurs only once in such ideas, in roughly the way indicated by the expression ‘collection of $A, B, C, D \dots$ ’, no matter how great the number of $A, B, C, D \dots$ may be—thus the entire idea contains nothing more than the concept of a collection and the ideas of the objects $A, B, C, D \dots$ (WL I, §82, 394–95)

Whatsoever happens in the mind of a person who thinks of a collection of n objects, the objective idea grasped by her does not contain “and” (or “et”) $n - 1$ times. In fact, it does not contain this idea at all but is solely composed of the concept of a collection and ideas of the given collection’s elements. Hence, regarding the realm of objective ideas, which is more relevant to logical considerations than the mental (or linguistic) realm, there is no need to decide between the idea expressed by “ A et B et C ” and the one expressed by “ $(A$ et $B)$ et C ” because both are identical with the idea expressed by “the collection of A, B, C ” (EG sec. 3, §95, 155).

6. The Internal Structure of Collections

Late Bolzano uses the term “Menge” for collections in which the arrangement of constituents is irrelevant (WL I, §84, 399–400; EG sec. 3, §§88–89, 151–52; PU §4, 4). The standard translation of “Menge” is “set”. But since it is to be left open whether Bolzano’s *Mengen* are sets in the modern sense, we follow the suggestion of Simons (1997, 95) and translate “Menge” by “multitude”. Anyway, such a collection is extensional: “The parts of which a multitude consists determine it, in fact completely” (EG sec. 3, §89, 152). In this respect, multitudes are on a par with the plenties of modern set theory. The latter do not have any internal structure because their identity depends on nothing but their elements.

At first glance, young Bolzano does not allow for such collections:

A whole is obviously determined if all *singular parts* of it and the *manner of combination* is determined. For example, a house is determined if it is determined how many walls etc. [it consists of]—and how they ought to be combined. (AM 51; see also 59)

If the manner of combination were essential for early Bolzano’s collections, they would lack a central characteristic of sets. However, a closer look reveals that he also accepts collections not determined by the way in which their constituents are combined:

If a whole consists of several equal parts in a particular quantity, then, if *one of these parts and the number* [of them] is given, the whole is determined in a certain way. (AM 59)

This remark could be brought into accordance with the passage cited before by trading on the fact that Bolzano’s formulations express *sufficient* conditions for the determination of collections. Thus, there exist collections whose identity does not depend on the order of their parts. Since these collections are completely determined if their parts are specified, mentioning the order of these parts is *superfluous*. But specifying parts and order is nevertheless *sufficient*, in the logical sense, for determining such collections (just as being a redheaded unmarried male is sufficient for being a bachelor).

A case in point are the plenties called *quantities* (*Größen*). In young Bolzano, a quantity is explicated as a collection which is determined by a unit and one or more numbers (AM 45–46, see also 43).¹² Since quantities are determined by numbers, Bolzano also proposes to call them “*arithmetic wholes*” (AM 51). A unit is “a thing which is not viewed as complex” (AM 41). Note that this does not mean that a unit is *in fact* simple but only that, whether or not it has parts, it is *regarded* as the simplest constituent within a quantity. The paradigm of a quantity is the one

¹²For the sake of simplicity, we do not follow Bolzano in restricting the term “number” to natural numbers (AM 57–58; BGA 2 A 5, 191).

described in the previous quote. It is determined by a unit and a natural number stating how often the unit is contained in the collection. For example, a period of time lasting 5 seconds is a collection containing five times the unit 1 second (AM 46). The same holds, *mutatis mutandis*, for the natural number 5: “The number itself, insofar as it is regarded as a separate thing, may be called a quantity” (AM 50). 5 is a quantity because it is a collection containing a particular unit, namely the number 1, in a stated number, namely five times.

This dovetails perfectly well with what Bolzano says about *addition* (AM 51–52). Addition, in a general sense, is nothing but et-combination because it is the operation of combining given things into a collection. Depending on the manner of combination, there are different kinds of addition leading to different kinds of collections. One of them is “*arithmetic addition*”, and its result is an “*arithmetic sum*” (AM 52). The essential characteristic of arithmetic addition is that it abstracts from the order of the things thought together. For example, when calculating how many seconds are 5 seconds, 3 seconds and 7 seconds taken together, the order is immaterial. An arithmetical sum, as an et-collection whose identity does not depend on the order of the things constituting it, is thus a quantity. This holds in particular for the addition of natural numbers:

If 1 is added in thought to a number, a new number arises. For it arises a particular multitude [Menge]. . . . [T]he number resulting from the sum or thinking together several others is the same, regardless of the order in which one thinks these parts together. For in the concept of a number does not lie the concept of order but only the one of a multitude [Menge]. (AM 61–62)

Note that, here and elsewhere, Bolzano makes use of the term which, in his mature conception, is explicitly dedicated to extensional collections, namely “Menge” (AM 50, 58). Thus, both multitudes in early and late Bolzano share with sets in the modern sense the property that their identity depends on nothing but their constituents. As will presently be seen, Bolzano later adds

that summations of natural numbers also remain unaffected by *aggregation* and *disaggregation* of their constituents (provided that disaggregation leaves the unit 1 intact).

However, even if arithmetic sums and sets share an essential feature, this does not mean that the former are best interpreted as set-theoretical plenties. According to Bolzano, adding two natural numbers results in an orderless et-combination of them. Hence, $2 + 3 = 3 + 2$; and $1 + 4 = 4 + 1$. But given that the result of adding 2 and 3 is identical with the result of adding 1 and 4, these collections can hardly be sets containing the given numbers as elements (Rusnock 2013, 157). Otherwise, $\{2, 3\}$ would be identical with $\{1, 4\}$. In contrast, if we take such et-combinations to be mereological wholes, then $2 + 3$ is not only a plenty containing 2 and 3 but also a plenty containing 1 and 4 because it can be decomposed both into 2 and 3 and into 1 and 4. Hence, Bolzano’s theory of numbers strongly speaks for a mereological reading of collections.

Many years later in his *Einleitung zur Größenlehre (Introduction to the Theory of Quantities)*, Bolzano points out a debatable consequence of his theory:

[I]sn’t it evident that even in the simplest expression of a sum: $1 + 1$, the signs 1 and 1 occurring here do not name the very same thing? If it is to be possible to bring them together and generate the number 2 by this act of bringing together, then the unit conceived of under the second 1 certainly has to be different from the one conceived of under the first. (EG sec. 3, §17, 107)

Conversely, if we assume there to be only one number 1, then natural numbers can neither be wholes nor sets consisting of this number several times. As to sets, since both $\{1, 1\}$ and $\{1, 1, 1\}$ boil down to the singleton $\{1\}$, 2 would be identical with 3 (and any other natural number).¹³ As to wholes, the weakest no-redundancy constraint (NR1) already puts a ban on wholes con-

¹³Multisets allow for multiple occurrences of the same element, but they are not at issue here.

taining the same thing several times. Curiously enough, Bolzano does not consider this to be an objection to his theory of numbers. By contrast, Frege (1884, §§36–44) discusses the problem of similar but distinct units at length.

In the passage from the *Größenlehre* cited above, Bolzano applies the word “sum” (*Summe*). In this context, “sum” is a technical term denoting a special kind of multitude (*Mengen*), namely those “in which the parts of the parts may be looked upon as parts of the whole” (*WL* I, §84, 400; see also *EG* sec. 3, §§92–93, 153–54; *PU* §5, 42). This feature, appearing to echo the transitivity of the part-whole relation, gives rise to a number of difficulties (Krickel 1995, 179–87). Rusnock has tried to solve them by emphasising that Bolzano should have characterised sums not only by *parts* of parts being parts of the whole but also by *wholes* of parts being parts of the whole. More exactly, Rusnock interprets sums as follows:

[S]ums are collections that retain their identity under three kinds of transformations:

- (1) Rearrangement/permutation of parts (it is this feature that led Bolzano to claim that sums are multitudes).
- (2) Disaggregation/dissolution of proximal parts into their proximal parts.
- (3) Aggregation/fusion of certain proximal parts.

(Rusnock 2013, 161)

While Bolzano explicitly mentions (1) and (2), he should have mentioned (3) as well. Taken together, these clauses state that a sum remains the same “regardless of how it is partitioned” (Rusnock 2013, 162).

An important case of application are sums of natural numbers. In Bolzano’s view, these sums ultimately consist of certain units, namely tokens of the number 1.¹⁴ While early Bolzano has merely

¹⁴See *EG* (sec. 3, §§95–96, 155–56); and Rusnock (2013, 163–64). We do not use Rusnock’s expression “arithmetical sum” because, in early Bolzano’s terminology, it includes not only sums of natural numbers but also sums of seconds, meters and so on.

emphasised that these sums remain unaffected by the type of transformation described in clause (1), his mature self adds the transformations of clauses (2) and (3). Due to (1), it is permitted to rearrange the given units and aggregates of them without affecting the identity of the sum. According to (2), it is allowed to dissolve an aggregate of natural numbers into these units and smaller aggregates. And due to (3), we may fuse both units and aggregates of them. Hence, for example, $7 + 2 = 7 + (1 + 1) = (7 + 1) + 1 = 8 + 1 = 9$.¹⁵

Sums of natural numbers show that the licence to partition is not *all-encompassing*. Clause (2) does not admit of dividing *units* because the result would no longer be a sum of *natural* numbers. More generally, the given transformations are under the proviso that they lead to a collection falling under the concept by which the original collection was picked out. Hence, among other things, they do not allow the sum of natural numbers $7 + 2$ to be transformed into a sum of natural numbers *and fractions*, such as $7 + \frac{3}{2} + \frac{1}{2}$.

This proviso also comes out in the case of so-called *pluralities* (*Vielheiten*). According to Bolzano, pluralities form a subspecies of sums (*EG* sec. 3, §120, 169). A plurality of a certain kind *A*, such as a number of grapefruits, is a sum “in which the parts which are ultimately considered *simple*, i.e., not capable of further division, are *units* of the kind *A*” (*EG* sec. 3, §119, 166; see also *WL* I, §86, 407). Note that, just as young Bolzano, his mature self does not write that the units *are* indivisible but that they are *considered* indivisible. For example, the units within a plurality of grapefruits, viz., grapefruits, are surely capable of further partitioning because they are composed of a skin, pulp etc. But for the plurality to remain a plurality of (whole) *grapefruits*, none of the units must be divided because the result would be a collection of grapefruits *and something else* (*EG* sec. 3, §120,

¹⁵In the appendix to the *Beyträge* (*BM* 147), Bolzano demonstrates that $7 + 2 = 9$ by these steps.

169). This is the sense in which units have to be *considered* simple. Rusnock (2013, 162 n 36) therefore adds to his explication of sums that “this partitioning will be subject to constraints in most cases, determined by the concept of the sum in question”. Just as for sums of natural numbers, fragmentation of pluralities is not completely arbitrary but has to stop once *units* are reached. According to this approach, a plurality of grapefruits is a sum because it retains its identity under any partitioning into grapefruits and multitudes of them.¹⁶

Is there an anticipation of sums or pluralities in early Bolzano’s approach? To substantiate his interpretation of sums, Rusnock cites some passages from Bolzano’s late works (*WLI*, §84, 400; *EG* sec. 3, §93, 154; and *PU* §33, 91). Interestingly enough, however, both of the properties by which Rusnock distinguishes sums from other multitudes may already be found in the *Allgemeine Mathesis*. That is, unlike late Bolzano, his early self explicitly mentions not only disaggregation but also aggregation of parts. Please pay particular attention to the phrase “divided or combined” occurring twice in the following passage:

We call a *discrete quantity* . . . a sum (quantity) consisting of equal parts, where the unit of it (namely those parts) is considered as having *such a property* that it [i.e., the unit] must not be modified (divided or combined) in order to keep this property (remain a thing of this kind). E.g., 100 equestrians are a sum of things, namely *equestrians*, each of which can of course be divided or combined, but then would not remain an equestrian. (*AM* 50; see also *BGA* 2 A 5, 216)

Remember the paradigm of a so-called arithmetic collection, namely a collection determined by a unit and a natural number specifying how many units are contained in the collection. It appears that we are to think of discrete quantities as such col-

¹⁶We are grateful to one of the referees for revising our initial ideas about pluralities.

lections. Discrete quantities consist of a certain number of units, that is, parts belonging to the same kind. In the case of 100 equestrians, the units are equestrians and the collection is composed of 100 units. These units often consist of smaller parts, such as equestrians consist of heads and further body parts; and two or more units can be brought together to form a separate collection, as for instance a pair of equestrians. But since these things do not belong to the kind the units belong to—for neither a head nor a pair of equestrians is an equestrian—they do not count as parts of the discrete quantity.

Is early Bolzano’s discrete quantity of 100 equestrians a plurality in mature Bolzano’s sense? An affirmative answer presupposes that discrete quantities have the same identity conditions as pluralities; but this is unclear. A *plurality* of 100 equestrians retains its identity under any partitioning into equestrians and collections of them. The only transformation not allowed is *disaggregation* of units, i.e., equestrians. By contrast, the passage cited above could be read as saying that a *discrete quantity* of 100 equestrians is subject to a further constraint because it also does not admit of *aggregating* units. That is, while a plurality of 100 equestrians continues to be a plurality of 100 equestrians if the equestrians are grouped into pairs, early Bolzano might have thought that a discrete quantity of 100 equestrians does not stay the same because it is converted into a discrete quantity of 50 pairs of equestrians.

After explicating the concept of a discrete quantity, Bolzano does not carry on by explaining what a continuous quantity is but emphasises that “a continuous quantity may in other respects be called a discrete quantity” (*AM* 50).¹⁷ In the unpublished manuscript “On the Concept of a Quantity and Its Different Kinds” from 1816, he elaborates on this point:

¹⁷Actually, Bolzano uses the expression “concrete quantity” (“konkrete Größe”). But a marginal note makes clear that Bolzano means *continuous* by “concrete”: “Unit for discrete, *measure* for continuous quantities.”

Thus, for example, a *heap of balls*, as such, is a *discrete* quantity because it is a sum (multitude) of things (balls) considered as having a property (namely the shape of a ball) which would not remain once a division or composition of them were carried out. From this one also sees that one and the same thing, in different respects, can be a continuous and a discrete quantity. For example, if we considered only the amount of *matter* [*Stoff*] the balls in the heap together contain, then they would be transformed straight away into a continuous quantity. (*BGA* 2 A 5, 216)

In accordance with what was said before, it seems that Bolzano argues as follows. Conceptualised as a *heap of balls*, the given collection is discrete. For if its units, i.e., the balls of which it consists, are divided or combined, then it is no longer a heap of balls but, say, a heap of balls and two halves of a ball, or a heap of balls and a pair of balls. But conceptualised as a particular *amount of matter*, the heap of balls is a continuous quantity because it remains falling under this concept independently of whether parts of it are divided or combined.

Rusnock (2013, 156–57) points out that the traits of mature Bolzano’s collections are determined by the concept (viz., objective idea) through which they are picked out. The passage above is one of the places displaying that the same holds for collections in early Bolzano. This kind of relativisation will play an important role in Section 8. Here, however, it is more important to note that early Bolzano’s continuous quantities are the clearest case of late Bolzano’s sums because they retain their identity even under completely arbitrary partitions. They thus differ from sums of natural numbers because the latter are affected by disaggregation of units.

7. The General Applicability of “et”

Remember that et-composition “is possible for any two concepts” while cum-composition “is no longer applicable to all things” (*AM* 34–35). The central point here is that composition

via “cum” might result in an idea with an empty extension, or in later Bolzano’s words, an “objectless” idea (“gegenstandlos”, *WL* I, §66, 297; *EG* sec. 2, §4, 51). In the *Allgemeine Mathesis*, Bolzano cites as examples contradictory compounds, such as “geometric figure of two straight lines” and “triangle with three right angles” (*AM* 30, 36). These examples prove that combining satisfied concepts by “cum” may lead to a compound under which nothing can fall. Presumably, Bolzano does not add non-contradictory ideas like “golden mountain” because, although they do not represent real objects, they represent ideal objects in young Bolzano’s sense and therefore do not have an empty extension.

In his later writings, Bolzano is fine with calling contradictory compounds *concepts* (*WL* I, §67, 304; *EG* sec. 1, §2, 48, §4, 51–53). According to this view, any cum-composition of concepts results in a concept, regardless of whether the latter is satisfied or not. Thus, the claim that this type of composition is not applicable to all things would indeed relate to the corresponding things and nothing else. For it would merely state that “cum” might eventuate in a concept representing nothing at all, whether real or ideal. However, in the *Allgemeine Mathesis*, Bolzano takes a different stance:

If *A* cannot be *B*, the expression... *A cum B* does not denote a concept but is just a mere assemblage of words... Thus, in my opinion, ... the words ... ‘a figure of two straight lines’ do not express a concept. (*AM* 30)

Here Bolzano refuses contradictory expressions the status of expressing concepts. On this view, while “golden mountain” gives voice to a concept, “triangle with three right angles” would not. Hence, the claim that “cum” is not universally applicable gets bite not only on the level of things but also on the level of concepts. There are then concepts not combinable in the given way because this would not only result in an *empty* concept but in *no* concept *at all*.

In the case of “et”, however, there is no such restriction: “This composition is possible for any two concepts” (AM 34).¹⁸ The reason for this seems to be that et-composition never leads from satisfied to empty concepts. For example, since there are knives and forks, there are objects falling under “knife et fork”. It is even “possible to combine [the] words [‘geometric figure’ and ‘two straight lines’] with minor changes so that they denote a concept, namely, ‘a geometric figure and two straight lines’, which is the concept of a sum of two things (namely of a figure and of two straight lines)” (AM 30). Analogously, in contrast to “triangle cum three right angles”, “triangle et three right angles” expresses the satisfied concept of a plenty consisting of a triangle and a figure with three right angles.

The crucial question now is which of the four interpretations of “et” comply with its general applicability. For reasons of simplification, let us consider only the general-term variants. Under the set-theoretical reading (S1), “A et B” refers to the set of all A and B; and this set trivially contains elements if there are A or B. (S2) says that “A et B” subsumes all pair sets of an A and a B; and again, there exist such sets if there are A and B. The mereological reading (M1) stipulates that “A et B” denotes the whole of all A and B; and this whole exists if there are A and B. Finally, “A et B” comprises all wholes of an A and a B under (M2); and there are such wholes if there are A and B. Hence, the general applicability of “et” does not allow to distinguish one of the four interpretations because all of them cope with it.

8. Basic Inferences

Part of the better-grounded presentation of mathematics Bolzano wanted to attain through the *Beyträge* is the notion that math-

¹⁸See Leibniz (1666, 169) in his *Dissertatio*: “Licet quotcunque res simul sumere, et tamquam unum Totum supponere”, i.e., “It is permitted to comprehend anything whatsoever and to assume that taken together they form a whole”.

ematical proofs can be ultimately reduced to exactly four *basic* inferences. Two of these inferences appear to be highly relevant to Bolzano’s early views on collections because they contain conclusions with “et” (BM 65–66; for the sake of simplicity, we introduce only the general-term varieties):

$$(1) \quad \begin{array}{l} A \text{ is } C \\ B \text{ is } C \\ \hline (A \text{ et } B) \text{ is } C \end{array} \qquad (2) \quad \begin{array}{l} A \text{ is } B \\ A \text{ is } C \\ \hline A \text{ is } (B \text{ et } C) \end{array}$$

According to the set-theoretical interpretation promoted by Berg, Centrone, Roski and Sebestik, the predicate of the conclusion in inference (2), “B et C”, denotes the set of all things being B or C. However, Roski (2010, 8–9; 2014, 51) and Centrone (2012b, 16) have shown that, although (2) would be valid under this interpretation, one of its premises would be redundant.¹⁹ Briefly, “Apples are red or green” already follows from “Apples are red” or “Apples are green”, respectively.

This problem could be solved by interpreting “et”, even though it links *terms* on the *grammatical* level, as a *sentential* connective on the *logical* level. Then, “(A et B) is C” would amount to “A is C, and B is C”, and “A is (B et C)” would read “A is B, and A is C”. In this manner, both inferences prove valid and non-redundant. But, first, it would not be clear anymore why we have two inference rules instead of just one, namely, conjunction introduction. And secondly, Bolzano glosses the conclusion of (2) as follows: “This sentence is obviously different from the first two, each considered in itself, for it contains a different predicate. It is also not the same as their *sum*, for the latter is not a

¹⁹See BM (87), where Bolzano claims that “every sentence whose subject is a composite concept is a sentence *dependent* on *several* other sentences . . . and therefore can in no way be regarded as an *axiom*” (second emphasis added). But contrast also a diary note from 1809/10 where Bolzano paraphrases “We obtain our knowledge through the senses and the intellect” by “Senses and intellect (A et B = S) are the reason for our knowledge (= P)” and declares it an *axiom*. (The reason for the latter could be the Kantian view that only senses and intellect *together* give us knowledge.)

single sentence but a collection of two" (BM 66; see also WL I, §83, 396–97). Apparently, this means that "A is (B et C)" must not be read in the fashion of "A is B, and A is C" because the latter gives voice to a concatenation of two propositions "A is B" and "A is C", whereas the proposition expressed by the former consists of a subject concept "A" and a predicate concept "B et C".

A more viable solution was proposed by Centrone (2012b, 17–18). In his philosophical diaries, Bolzano himself points out that he made a mistake in the *Beyträge*: "But it actually occurs to me now that the name 'P et Q' in the essay was wrong. I can think of no example where one could not infer with certainty 'S is (P cum Q)' from the two sentences 'S is P' and 'S is Q'" (BGA 2 B 15, 222). Here Bolzano realises that the conclusion of the inference (2) can be strengthened from "A is (B et C)" to "A is (B cum C)" without losing validity. That is, briefly, from "A is B" and "A is C", we are allowed to infer that A is B and C. Probably, Bolzano failed to see this earlier because the German translation of "et" (as much as the English "and") operates differently in the subject and in the predicate. Described set-theoretically, while "Äpfel und Birnen sind grün" is true if and only if the union of apples and pears contains only green elements, "Äpfel sind grün und rund" is true just in case all apples are elements of the intersection of green and round things.

Whatever the reason for Bolzano's previous error may be, on the basis of his revision in the diaries it is possible to stand fast to the union-interpretation of "et" by equipping the inference rule (2) with the stronger conclusion "A is (B cum C)". Against this background, it comes as no surprise that, some pages later in the diaries, Bolzano presents a modified list of his four basic inferences containing such a rule in place of (2) (BGA 2 B 15, 238). The rule to be found in the *Wissenschaftslehre* is equivalent to this stronger variant (WL II, §199, 344, §221, 388, §227, 411). Hence, to find out what et-combinations refer to, it is sufficient to have a closer look at Bolzano's basic inference (1).

Remember the four interpretations introduced in Section 2. According to them, "A et B" refers to

- (S1) the set of all As and all Bs,
- (S2) all sets of an A and a B,
- (M1) the whole of all As and all Bs,
- (M2) all wholes of an A and a B.

Since we are concerned with sentences in this section, the copula "is" gives rise to a further complication. Does it express *subsumption* or *subordination*?²⁰ In other words, do sentences of the form "(A et B) is C" mean that the collection(s) the subject refers to or its constituents have the property of being C?

Early Bolzano's remarks on the copula are of no help because he offers a bewildering variety of disparate paraphrases (BM 65–68, 74, 114–16; Centrone 2012a, 9). The conventional reading of "is" amounts to subsumption. But combining it with the set-theoretical interpretations of "et" leads to the problem that basic inference (1) would be invalid. Under (S1), the conclusion "(A et B) is C" would say that the set of all As and Bs possesses the property of being C. But "Every knife is silver" and "Every fork is silver" do not imply "The set of all knives and forks is silver" for the simple reason that sets are not made of metal. For the same reason, these premises do not entail what they are to entail according to (S2), namely that all sets of a particular knife and a particular fork are silver.

It is thus hardly surprising that Berg, Centrone, Roski and Sebestik, the advocates of set-theoretical interpretation (S1), let the copula "is" express subordination. In their view, "(A et B) is C" states that the set of all A and B is a (proper or improper) subset of the set of all C. If the conclusion of basic inference (1) is understood in this way, the inference is immune to the trouble just mentioned. For, in contrast to the set of all knives and forks, its elements are trivially made of silver if the corresponding premises are true, that is, if every knife and fork is made of silver. Since the same holds for the elements of every pair set of a knife

²⁰One of the *loci classici* for the distinction between subordination and subsumption is Frege (1892, 194–95).

and a fork, both (S1) and (S2) can be maintained by letting “is” express subordination instead of subsumption.

Note, however, that this attempt at salvaging the set-theoretical interpretations is somewhat unsatisfactory because the associated reading of the copula is conducive to sentences dealing with the *elements* of the given sets instead of the sets themselves. Since the subset-relation is defined by the element-relation, at the end of the day the conclusion of Bolzano’s basic inference (1) reads as follows under (S1):

$(A \text{ et } B) \text{ is } C =_{\text{df.}}$ Every element of the set of all A and B is C .
 $(a \text{ et } b) \text{ is } C =_{\text{df.}}$ Every element of the set of a and b is C .

And under (S2):

$(A \text{ et } B) \text{ is } C =_{\text{df.}}$ Every element of every set of an A and
 a B is C .
 $(a \text{ et } b) \text{ is } C =_{\text{df.}}$ Every element of the set of a and b is C .

The set-theoretical interpretations are thus compatible with inference (1), but at the expense of an unconventional reading of “is” resulting in sentences which do not relate to the given sets anymore.

How do the mereological readings of “et” cope with Bolzano’s basic inference (1)? First of all, both (M1) and (M2) come out as nonstarters if we understand “is” as *subsumption*. Just consider the singular-term case of inference (1), which is identical on these interpretations:

$a \text{ is } C$
 $b \text{ is } C$

The whole of a and b is C

If a knife and a fork is made of silver, then the whole of them is also made of silver. But if the knife and fork both weigh 100 g, the whole of them does not weigh 100 g but twice as much. Analogous problems arise for the general-term case.

For this reason, the copula should give voice to *subordination*. Or more exactly, since wholes do not have elements, we have to replace the element-relation by its mereological counterpart, the part-relation. Then interpretation (M1) eventuates in the following renditions of the conclusion:

$(A \text{ et } B) \text{ is } C =_{\text{df.}}$ Every part of the whole of all A and B is C .
 $(a \text{ et } b) \text{ is } C =_{\text{df.}}$ Every part of the whole of a and b is C .

And interpretation (M2) provides:

$(A \text{ et } B) \text{ is } C =_{\text{df.}}$ Every part of every whole of an A and
 a B is C .
 $(a \text{ et } b) \text{ is } C =_{\text{df.}}$ Every part of the whole of a and b is C .

However, quite unlike the corresponding modifications of the set-theoretical readings, these renditions are also in conflict with Bolzano’s basic inference (1) because they make it invalid. Consider again the singular-term variant shared by both interpretations:

$a \text{ is } C$
 $b \text{ is } C$

Every part of the whole of a and b is C

If Kim and Tim both weigh 70 kg, the sum of them surely contains parts weighing 70 kg. Among them are Kim and Tim. But remember that, as per contemporary mereology, an essential characteristic of parthood is that parts of parts are parts of the whole. Hence, further parts of the whole of Kim and Tim are Kim’s and Tim’s arms, noses and so on, with the result that the conclusion falsely claims that these objects weigh 70 kg as well. As to the general-term cases, even if all knives and all forks weighed 100 g, it would not be the case that every part of the whole of them, or every part of a whole of a knife and a fork, weighs 100 g.

Though not concerned with early Bolzano's conception of collections or the problem at hand, Krickel (1995, 123–38, 145–52), Siebel (1996, 41–43), Simons (1997, 94) and Behboud (1997, 110–11) pointed out a feasible solution to this problem (see also Schnieder 2002, 212–13; Krause 2004, 26; Rusnock 2013, 156–57). Sets do not give rise to the problem at hand because elements (or parts) of an element do not have to be elements of the set. The elements of {Kim, Tim} are just Kim and Tim, and not their arms or noses. Bolzano thus needs a notion of parthood assimilating this feature by not letting parts of parts automatically be parts of the whole. Fortunately, both early and late Bolzano's conceptual repertoire contain such a notion.

Simons (1997, 94) claims that Bolzano does not mark this notion terminologically; but this is true only for young Bolzano. In the *Wissenschaftslehre*, Bolzano makes a distinction between “closer and more remote parts” (WL I, §58, 251), and in the *Größenlehre*, he also uses the terms “mediate” and “immediate parts” (EG sec. 3, §9, 103). For example, the expression “creature living on earth” expresses a concept whose closest parts are the concepts expressed by the individual words, such as “creature” and “living”, while the more remote parts are the concepts into which the former can be analysed (WL I, §58, 252). Analogously, the king, his ministers and his subjects are immediate parts of a state, whereas heads, hands and feet of them are only mediate parts (EG sec. 3, §9, 102–03).

This distinction amounts to a relativisation of parthood with respect to the concept picking out the collection. Being an immediate part of a collection is thus not a two-place but a three-place relation. Its relata are not only the part and the collection but also the concept representing the collection. For example, the whole referred to via the concept of a state has as its immediate parts *persons*, but not their limbs, because a state is defined as an ensemble of (among other things) *persons*, but not limbs. Thereby, we get rid of transitivity and thus come close to membership.²¹

²¹Sebestik (1999, 235) identifies the relation of being an immediate part with

Due to parthood becoming a three-place relation, one and the same object can be an *immediate* part of a collection with respect to a particular concept of the collection and a *mediate* part with respect to a different concept of it. For example, with respect to “Kim and Tim”, Tim is an immediate part of the given whole, whereas his nose is a mediate part. But regarding “Kim, Tim's nose and the rest of Tim”, Tim is not an immediate part while his nose is one. Or consider sums of natural numbers, say, $2 + 3$. If we refer to this sum by the concept “ $2 + 3$ ”, the numbers 2 and 3 are the immediate parts, whereas 1 is a mediate part. But if we use the concept “ $1 + 4$ ”, then 1 is an immediate part, while 2 and 3 are mediate parts.

This account blends in well with late Bolzano's remark that, “according to the explanation of a part that I gave . . . , only the individual objects of which we *think* as constituting a certain collection are to be envisaged as parts of it” (WL I, §83, 397; emphasis added). In the same spirit, he says in the *Größenlehre* that a collection is “a complex in which particular things *appear* as parts” (EG sec. 3, §6, 100; emphasis added). Here Bolzano uses “part” in the sense of *immediate part*, and he points out that whether something is an immediate part depends on the concept through which the collection is grasped.

But what about young Bolzano? Although we found no terminologically marked distinction between immediate and mediate parts in his early writings, we came across several indications of it. To begin with a quite general point, remember that collections come into being by acts of thinking objects together. These objects usually have parts themselves, but since these parts are not kept in mind, it is natural to view them as mediate parts, whereas the thought-of objects are immediate parts.

membership (subsumption) and the relation of being a mediate part with the subset-relation (subordination). The latter is questionable for two reasons. First, a part a of b is a mediate part of the whole of b and c , relative to the concept “ b and c ”, but $\{a\}$ is not a subset of $\{b, c\}$. Secondly, $\{b\}$ is a subset of $\{b, c\}$, but b is not a mediate part of the whole of b and c relative to “ b and c ”.

Secondly, remember the following passage, already cited in Section 6, which may be smoothly translated into talk of immediate parts:

We call a *discrete quantity*: . . . a sum (quantity) consisting of equal parts, where the unit of it (namely those parts) is considered as having *such a property* that it [i.e., the unit] must not be modified (divided or combined) in order to keep this property (remain a thing of this kind). E.g., 100 equestrians are a sum of things, namely *equestrians*, each of which can of course be divided or combined, but then would not remain an equestrian. (AM 50)

According to contemporary mereology, a group of equestrians has as its parts not only individual equestrians but also their heads or pairs of equestrians. Bolzano, however, appears to take the latter to be only mediate parts with respect to the concept “this group of equestrians”, while the immediate parts are just the equestrians belonging to the given group. The corresponding quantity is then determined by counting the immediate parts. Note that Bolzano thereby partly anticipates Frege’s insight that, say, the number of evangelists is four because there are four men falling under the concept “evangelist”—and not because this group of men has four parts (Frege 1884, §§22–23, 46).

Thirdly, remember that, in the continuation of the passage just cited, Bolzano emphasises that one and the same collection may be a discrete and a continuous quantity, depending on how we look at it. In the manuscript “On the Concept of a Quantity and Its Different Kinds”, he expands on this with the help of an example. A number of balls, considered as a *heap of balls*, is a discrete quantity. But if we conceptualise it as *the matter of the balls*, we have a continuous quantity (BGA 2 A 5, 216). Two years earlier, the aspect-dependence of the discrete/continuous distinction was even labelled “Very nice” in the diaries (BGA 2 B 4/2, 74). Anyway, Bolzano’s example can be used to illustrate the immediate/mediate distinction and its aspect-dependence. Regarded as a *heap of balls*, the immediate parts of the corresponding collection are the individual balls and not parts or collections of

them. But if we refer to the collection via the concept “the matter of the balls”, then balls, parts of them and collections of them equally deserve the title “immediate parts”.

All in all, there is some evidence for the fact that early Bolzano already utilised the distinction between immediate and mediate parts. It is therefore justified to apply it in order to preserve the mereological interpretations against the problem arising from Bolzano’s basic inference (1). If we understand “part” as referring only to immediate parts, then the singular-term variety of (1) turns out to be valid. For if the whole of Kim and Tim is picked out by the concept “Kim et Tim”, then its immediate parts are Kim and Tim, but not their arms or noses. Hence, “Kim weighs 70 kg” and “Tim weighs 70 kg” entail that the immediate parts of (Kim et Tim) weigh 70 kg. Furthermore, the general-term varieties of (1) would read as follows:

$$\frac{\text{Every } A \text{ is } C \\ \text{Every } B \text{ is } C}{\text{Every immediate part of the whole of all } A \text{ and } B \text{ is } C}$$

$$\frac{\text{Every } A \text{ is } C \\ \text{Every } B \text{ is } C}{\text{Every immediate part of every whole of an } A \text{ and a } B \text{ is } C}$$

The first rule is valid because the immediate parts of the whole of all *A* and *B*, relative to the concept expressed by this description, are exactly those objects falling under the concepts expressed by the general terms “*A*” and “*B*”. Thus, the immediate parts of the whole of all knives and forks are just knives and forks, whereas constituents of knives and forks, or wholes of them, are merely mediate parts. The same holds for the conclusion of the second rule. With respect to “whole of a knife and a fork”, the immediate parts are knives and forks. Since these parts have the property of being *C* if all knives and forks possess it, the conclusion must be true if the premises are true.

All in all, neither the set-theoretical nor the mereological interpretations do justice to Bolzano's basic inference (1) if the copula in the conclusion is taken to express subsumption. If we understand it in the subordinative way, on the other hand, then rule (1) comes out as valid under all interpretations. For the mereological readings to cope with this rule, however, they have to be regarded as being not about parts in the sense of contemporary mereology because this implies transitivity. They rather have to relate to what Bolzano calls "immediate parts" in his mature work. The key advantage of this notion is that it is close to membership insofar as parts of immediate parts are not automatically immediate parts of the whole. Besides, making use of this concept is not anachronistic because there is some indication of it in Bolzano's early writings.

We also need to mention, however, that we thereby end up with interpretations being mereological in a relatively weak sense because they are equivalent to the set-theoretical readings:

Every immediate part of the whole of all A and B is $C \Leftrightarrow$
Every element of the set of all A and B is C .

Every immediate part of every whole of an A and a B is $C \Leftrightarrow$
Every element of every set of an A and a B is C .

Still more, all four formulations are equivalent because, if we collect the elements of every set of an A and a B , we obtain the set of all A and B . For example, the set of all knives and forks is identical with the set containing all elements of the sets whose members are a knife and a fork. Hence, these formulations eventually boil down to:

Every object which is A or B is C .

It might thus come as no surprise that the conclusion Bolzano draws from " A is C " and " B is C " in the *Wissenschaftslehre* reads as follows (WL I, §§227, 412):

Every object represented by any of the objective ideas ' A ' and ' B ' is C .

Mature Bolzano thereby avoids the pitfalls arising from the question whether " A et B " (as well as " a et b ") refers to a set or a whole.

9. Conclusion

Most of the textual evidence speaks for a mereological and against a set-theoretical interpretation of collections in early Bolzano. This holds for the examples he offers (Section 4), his reluctance to collections having just one constituent and collections containing the same thing twice (Section 5), as well as his conception of quantities as a particular kind of collection (Section 6). The ideality of collections (Section 3) and the general applicability of "et" (Section 7) constitute an exception only insofar as they do not speak against a set-theoretical reading but are compatible with both kinds of interpretation.

Similarly, all interpretations could be attuned to basic inference (1) from the *Beyträge* (Section 8). This was possible, however, only at the expense of understanding the copula "is" as expressing subordination, entailing that the relevant sentences are basically not about the corresponding plenties anymore but about their constituents. For example, under interpretation (S1), "(A et B) is C " means "Every element of the set of all A and B is C ". There is also the fact that the mereological interpretations needed further tuning. For inference (1) to turn out valid, we had to deploy a restricted part-relation. Since the relation of being an immediate part is extremely close to membership, we end up with readings being equivalent to their set-theoretical counterparts.

There is an explanation for the complications surrounding basic inference (1). In Section 2, we pointed out that mereology is concerned with "the one as many", while set theory focuses on "the many as one". The paradigm of mereology is a given individual which can be divided into something smaller, namely its parts. Since the parts resulting from a division may themselves consist of still smaller parts, there is usually more than

one possible division. Hence, mereology cannot ignore the inner structure of parts. Furthermore, since all parts belong to the same given thing in the paradigm case, they form a natural whole. By contrast, set theory collects individuals in order to create something larger, namely the set of them. Here the inner structure of the elements is irrelevant; and there need not be a natural unity because one is allowed to combine any number of things.

The challenge for early Bolzano seems to be that he is after “the many as one” but approaches it with the notion of wholes and parts. That is, while the conclusion of basic inference (1) calls for set theory, he relies on mereological tools when examining et-composition. Bolzano is thus forced to enlarge the mereological inventory in order to award collections the aforementioned characteristics of sets. First, he exceeds mereology’s paradigm of a unified object by allowing a collection to consist of things of any kind whatsoever, regardless of whether they form a natural whole or not. Secondly, he distinguishes mediate from immediate parts to obtain collections built up of parts whose inner structure is irrelevant. Simons (1997, 87) wrote that Bolzano’s theory of collections “is best interpreted neither as a theory of sets nor a mereology but as a distinct and distinctive theory”. If we are right about thinking that Bolzano tried to implant set-theoretical features into mereology, then his particular blend of set-theoretical and mereological features is more exactly described as a set-theoretically enhanced mereology.

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