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The Logical Significance of Assertion

Walter B. Pedriali

Assertion plays a crucial dual role in Frege's conception of logic, a formal and a transcendental one. A recurrent complaint is that Frege's inclusion of the judgement-stroke (the formal counterpart of assertion) in the *Begriffsschrift* is either in tension with his anti-psychologism or wholly superfluous. Assertion, the objection goes, is at best of merely psychological significance. In this paper, I defend Frege against the objection by giving reasons for recognising the central logical significance of assertion in both its formal and its transcendental role.

The Logical Significance of Assertion

Frege on the Essence of Logic

Walter B. Pedriali

1. Introduction

The notion of assertion plays a crucial dual role in Frege's mature conception of logic, a formal and a transcendental one. In its formal role, assertion provides a criterion for sentential well-formedness in Frege's *Begriffsschrift*, since only formulas prefixed by the judgement-stroke (the formal counterpart of assertion) count as sentences in the language.¹ In its transcendental role, assertion is instead assigned the task of revealing the essence of logic, the task of characterising logic as the privileged doctrine that Frege takes it to be (NS 272/252).

Since the early complaints by Russell, Wittgenstein, and Jourdain, however, a recurrent objection has been raised against the idea that assertion should have anything to do with logic. Assertion, the objection goes, is a "merely psychological" notion (NB 95; BW 126/78). Moreover, the objection continues, even if it could be shown that assertion is non-psychological, its putative formal role would in any case lack logical significance, since no sign could possibly express assertoric force. Any sign that attempted to do so would be superfluous because semantically idle, and its inclusion in a language would represent a gross category mistake, a bad confusion between illocutionary and

¹See (GG I §5, §26). In this purely grammatical role, the judgement-stroke marks out a syntactic distinction between terms and sentences. See also (GG §32): "in every well-formed (*rechtmässig gebildeten*) sentence of the *Begriffsschrift* we have a judgement that a thought is true".

locutionary features of expressions and their use. The Russell-Wittgenstein-Jourdain (RWJ) objection, if sustained, visits an uncomfortable dilemma on Frege: either the judgement-stroke has psychological content or it has no content at all. On either horn, assertion lacks logical significance.²

Despite some vigorous attempts to debunk the RWJ objection, most notably Greimann (2000; 2014), Smith (2000; 2009) and Textor (2010), it seems to me that we still lack a wholly satisfactory answer to the question of why the judgement-stroke should play such a prominent part in Frege's system, to the question of why "a thinker as rigorous as Frege" (TLP §6.1271) should have insisted on its presence in his formal language. In this paper, I provide a new defence against the RWJ objection, one that includes a fresh reading of a crucial passage in Frege's 1915 draft "My Basic Logical Insights" concerning the transcendental role of the judgement-stroke, a reading that takes forward and significantly modifies the readings proposed in Taschek (2008) and Greimann (2014). Throughout the paper, I shall give a battery of arguments in favour of the view that Frege's notion of assertion, far from being merely psychological, is of deep logical and philosophical significance in both of its roles. The two main elements of novelty in the account are the stress on the role of reasoning acts in Frege's conception of logic, and on the role of the judgement-stroke in signalling the logical status of a language within the hierarchy of languages that tend towards that logically perfect language which Frege was striving to approximate (§4). The overall story I tell is, I think, coherently Fregean, if not quite the story that Frege himself tells.

To make clear the scope of my discussion: I'm taking Frege not to be giving a general theory of judgement or assertion—many of the objections against his account overlook this restriction which

²I shall be switching between talk of the significance of assertion, of the judgement-stroke, and of assertoric force, since in Frege's conception of logic their role is inextricably connected.

I think is instead fairly explicit in Frege's work.³ I thus take the question at the heart of the paper to be why Frege insisted that judgement and assertoric force, in his restricted sense, should be part of his formal language, and whether the reasons that could be given for their inclusion stand up to scrutiny.⁴

The structure of the paper is as follows. In §2, I discuss and motivate the formal role of assertion within Frege's system. In §3, I state the RWJ objection and block the first horn of the dilemma. In §4, I turn to the transcendental role of assertion and block the second horn of the dilemma too.

2. A Calculus of Asserted Truths

For Frege, the *Begriffsschrift*, the formal language of pure thought, is and must be a calculus of asserted truths, a calculus of judgements, rather than of propositions.⁵ Only formulae prefixed by the judgement-stroke can occur on a line of proof. The main aim of this section is to show why Frege assigned this pivotal formal role to assertion.

I begin by noting that Frege uses a bunch of ordinary notions in a pointedly non-standard (*ungewöhnlich*) way (NS 147/135–36), a way that inevitably strikes modern readers as rather alien. The notions in question are those of *sentence*, *concept*, *thought*, *judgement*, *assertion*, and *inference*. The ordinary notions incor-

³For a similar, but distinct, recent attempt to draw a distinction between the logical and the empirical notion of judgement, see van der Schaar (2017).

⁴The restriction does not trivialise the claim that assertion has logical significance. It is one thing to claim that Frege had a technical notion in mind. It is quite another to show that the notion has logical significance.

⁵Although one proves thoughts (NS 223/206), the laws of logic concern judgements (NS 190/175). Bell's (1979, 98) claim that Frege's calculus is one of "interiorized speech acts" is not correct. (KS 372/383) clearly tells us that "the assertoric force in the language... answers to the act of judging". Accordingly, the normative bedrock is provided by judgement. If so, Fregean assertion must not be seen as a linguistic phenomenon (*contra* Currie 1987, 58), nor can it be dissociated from judgement as Textor (2010, 642) does in claiming that "asserting is not making a judgement".

porate both logical and non-logical uses of those words. Frege, however, distinguishes between their restricted, logical usage, and their looser, non-logical usage. In the latter case, that of non-nonstandard uses, Frege silently (or explicitly) qualifies the occurrence of those words with the "pseudo" (*Schein*) prefix in contrast with the properly logical (*eigentlich*) notion. As far as logic is concerned, the only legitimate notions are those on the non-pseudo side of the divide. I'll concentrate here on the notions of judgement, inference and assertion only.⁶

Judgement is the acknowledgement of the truth of a thought (BS §2; KS 149 note 7/164 note 10; 346/355–56; GG I §5). Although Fregean judgement is an act carried out by a reasoner, it remains a firmly objective notion:

Whoever has once acknowledged a law of being true has thereby also acknowledged a law that prescribes what ought to be judged, wherever, whenever and *by whomsoever* the judgement may be made. (GG I xvii; my emphasis)⁷

⁶By *Satz*, Frege meant a sentence (an inscription, or an utterance), and not a proposition in Russell's sense (NS 222/206, 280/260; BW 231/149). Empirical sentences may need to incorporate extra-linguistic elements too (KS 348/357–58; NS 146/134–35, 230/213). See e.g. Kühne (1992) and Textor (2007, 2015) for discussion. (NS 163/150–51) shows that Frege countenanced cases of ellipsis (e.g. "Yes") as assertoric too, which suggests he had in mind a broader conception of form than surface linguistic form. For the pseudo/authentic contrast see e.g. (KS 149 note 6/163 note 8; BW 34–35/21–22). For pseudo-thoughts, see (NS 141–42/130). On concepts see (GG I 9, note 3; NS 248/241); on the logical character of thought see e.g. (NS 5/5, 154–55/142–43); for pseudo-assertion, see (NS 142/130, 347/356). (KS 273/253) is often taken to indicate that Frege thought of judgement as psychological: "grasping a thought and making a judgement are acts of a knowing subject, and are thus to be assigned to psychology". I disagree. Here Frege was contrasting ordinary use (*Urteilen als seelischer Vorgang*, KS 372/383) with the non-standard notions of thought and judgement he was defending. Finally, (KS 317/333; NS 266–68/247–49) discuss *pseudo-axioms*, sentences that are presented as if they "might be" axioms but that do not express a thought and hence are not axioms either (since for Frege genuine axioms are true thoughts).

⁷Occasionally, Frege himself slips into thinking of judgement as reasoner-relative (see the already quoted KS 273/253) or as something in need of an

In other words, the reasoner-neutral character of the judged content and of the laws of judgement determines the reasoner-neutral character of the act of judging. And if judgement is reasoner- and speaker-insensitive, assertion is so too, since in making judgement manifest (NS 150/139; BW 33/20) assertion does not affect the objective status of what is made manifest. For Frege, assertion and judgement are regulated by exactly the same laws, they are justified in exactly the same circumstances.

I now turn to one highly contentious issue, namely, the question whether Fregean judgement is meant to be factive. Opinion on the matter is deeply divided.⁸ I take Frege's notion to be factive.⁹ To judge, as we just saw, is to acknowledge the truth of a thought. Since Frege gives a factive reading of "acknowledge" ("We can only acknowledge what is there" KS 367/377),¹⁰

owner (KS 372/383). I take the quote in the text to represent more accurately Frege's austere conception of judgement as reasoner-neutral. Gabriel's (1996, 337, note 27) insightful remarks against Kitcher (1979) are useful here.

⁸See e.g. Ricketts (1996, 131), Kremer (2000), Textor (2010, §§5–8), Künne (2010, 432–37, 2013, 61–67).

⁹Factivity is fully compatible with the fallibilist streak in Frege rightly identified in Burge (1998, 340ff.), Reck (2007, §V, §XII), Macbeth (2014, §8.2). Frege's fallibilism in fact *requires* factivity, since one can be wrong only if one takes the "risky step" to the realm of reference (KS 149/164, 162/177, 358/367; NS 143/132), only if normative standards grounded in that realm are in place. There are of course normative standards for purely schematic reasoning too. But their very formulation is entirely parasitic on the instantiation of the schemata. Indeed, appeal to putative instantiation is what allows one to distinguish between valid and invalid schemata (the latter, but not the former, have invalid instances).

¹⁰Kremer (2000) and Künne (2013, 60) reject the factive reading of *anerkennen*. Künne's interpretation is vitiated by a misreading of a passage in (KS 365/376) where Frege speaks of "*einen falschen Gedanken als seiend anerkennen*". Künne takes this to support the view that Frege granted that we can acknowledge false thoughts. But clearly here the recognition regards the *existence* of a false thought, not its (assumed) truth. The factive sense of judgement at issue, however, concerns exclusively *anerkennen als wahr* and its cognates. Frege is also very careful to distinguish between the non-factive *für wahr annehmen* (NS 3/3) and *anerkennen als wahr*, never glossing judgement in the former sense.

it follows that non-pseudo judgement only takes place when what is there is a thought that is true. If the intended target is not a thought, or if it is a false thought, then by Frege's lights no judgement properly so called occurs.¹¹ Any attempt to acknowledge the truth of a false thought will misfire because although the thought is there all right, the fact that it isn't true means that the thought is not present *as* a true thought to be acknowledged as such in that specific judgement.¹² In other words, acknowledgement in Frege's sense is acknowledgement-as-true, acknowledgement with a very special sortal character built into its constitutive features.¹³ Put slightly differently still: while Fregean thoughts are ways of thinking of truth-values, judgement is a *specific* way of thinking a thought, of thinking it as-true, of thinking the True *through* it.¹⁴ If the thought is not true, the act of acknowledgement will fall short of its purported target, producing, at best, an illusion of understanding.¹⁵

¹¹On the failure of factivity with respect to the first case, see the already discussed case of pseudo-axioms (note 6 above) and (BW 34/21).

¹²The thought may of course *seem* to be true to the reasoner, but there is no true thought there for the taking. The proper way to acknowledge its existence as a false thought is to judge that its negation is true (NS 201/185).

¹³Textor (2010, 628) defends an ontic reading of *anerkennen*, whereby in judging we acknowledge the True as an object. My view is that, in judging, what we acknowledge is not the existence of the True, but rather the truth of a specific thought, the specific way in which the conditions contained in the thought determine the True. The ontic reading of acknowledging *das Wahrsein des Sinnes* stresses the acknowledgement of the True; the reading I propose, by contrast, stresses the acknowledgement of the indissoluble unity of the thought-as-true—in *Grundgesetze*, the asserted lines of proofs are necessary truths (*modulo* Frege's little slip over Basic Law Vb, of course).

¹⁴Here, Frege's suggestive talk (KS 144/158) of an all-encompassing (*allseitig*) knowledge of the reference as yielding immediate grasp of all its possible modes of presentation is highly relevant. Note also that the notion of thinking-as-true that I'm proposing here is in fact a consequence of Frege's denial that "true" is a predicate. In thinking of the True, we do not bring a thought under the concept TRUE. We think a thought-as-true.

¹⁵If we read *anerkennen* in this way, Frege's otherwise "astonishing" (Currie 1987, 66, note 9) or "surprising" (Kremer 2000, 568) claim that asserting a false

Judgement and assertion are represented in Frege's *Begriffsschrift* by means of the judgement-stroke, a sign that is ambiguous between two senses of judgement. In the first sense, judgement is the unmediated acknowledgement of the truth of axioms (NS 3/3), based solely on direct grasp of the content of the axioms.¹⁶ The second sense of judgement, the one with which logic is most directly concerned, is judgement as a movement between true thoughts by means of logical laws and definitions (NS 3/3, 190/175; KS 303–04/318).

It is this second sense of judgement that coincides with inference in Frege's strict sense:

To make a judgement because we are cognisant of other truths as providing a justification for it is known as *inferring*. (NS 3/3)

Frege's requirement that truths be the starting points for inference is in stark contrast to the modern conception that allows deductive rules to operate on assumptions too. Moreover, the truths must be acknowledged as such; in Fregean inference, that is, we move from the acknowledgement of true thoughts to the acknowledgement of further true thoughts.¹⁷ It follows that any inference-like move that starts from false thoughts is but a pseudo-inference:

thought is "incomprehensible" (*unverständlich*) (BW 127/79) becomes perfectly clear. To (seem to) judge that a necessarily false thought is true is to attempt to think an illogical (and therefore unthinkable) thought.

¹⁶For recent discussion of this kind of judgement, see Leech (2015) and my (forthcoming). Some classic accounts are Gabriel (1996), Burge (1998), Reck (2007). Wittgenstein (TLP §6.1271, §4.442) criticised both varieties of Fregean judgement as allegedly importing psychological notions into logic (self-evidence with respect to the axioms, assertion with respect to theorems).

¹⁷In §2 of BS, Frege says that we can draw consequences (*Folgerungen*) from unasserted propositions "so as to test the correctness of the thought". This doesn't contradict the later statements disqualifying unasserted propositions from figuring in inference. There's an *epistemological* difference between inference proper and pseudo-inference, a difference that mirrors the one between evaluating the *putative* consequences of a thought and assigning a truth-value to the thought on the basis of that evaluation.

It is necessary to recognize the truth of the premises. When we infer, we recognize a truth on the basis of previously recognized truths according to a logical law. Suppose we have arbitrarily formed the propositions

'2 < 1'

and

'If something is smaller than 1, then it is greater than 2'

without knowing whether these propositions are true. We could derive [*ableiten*]

'2 > 2'

from them in a purely formal way; but this would not be an inference [*Schluss*] because the truth of the premises is lacking. And the truth of the conclusion is no better grounded by means of this pseudo-inference than without it. (BW 30/17)¹⁸

Frege's stress on grounding shows that it is due to the epistemic role of proof that "mere hypotheses cannot be used as premises" (BW 118/182). Proof can fulfil its justification- and knowledge-generating function (of which more later) only if it starts from acknowledged truths in the sense just rehearsed.¹⁹

Finally, we should note that inference, for Frege, is not to be distinguished from consequence:

[That] one proposition follows (*folgt*) from certain others is something objective, something independent of the laws that govern the

¹⁸See also (KS 319/335): "Only true thoughts can be premises of inference"; (KS 364/375): "We cannot infer anything from a false thought"; (NS 264/244): "We can draw no conclusion from something false"; (BW 34/20): "We cannot infer anything as long as we do not know that it is true"; (BW 118/182): "From false premises nothing at all can be inferred". See also the discussion of indirect proof at (NS 264–67/245–47) and the careful way in which Frege discusses the case of a *Fehlschluss* in which it only *seems* as if something has been inferred (NS 239/221).

¹⁹For Frege, justification comes in two forms: the self-evidence of the basic laws, or a proof from those laws and definitions. Neither case applies to mere hypotheses. Hence, anything that we would derive from those hypotheses would itself lack justification.

movements of our attention, and something to which it is immaterial whether we actually draw the conclusion (*den Schluss wirklich machen*) or not. (GL §80)

[P]roof reveals logical relations between truths. (NS 220/204)²⁰

As these passages make clear, Frege's objectivism about inference perfectly mirrors his objectivism about geometrical lines:

What is it that we are doing when we correlate objects for the purpose of a proof? Seemingly something similar to drawing an auxiliary line in geometry. . . . [T]he drawing of a line should no more be regarded as a creation, than the specification of a point of intersection. Rather, in both cases we merely bring to attention, apprehend, what is already there. (GG I §66)²¹

In tracing inferential lines, then, the judgement-stroke does not mark a subjective piece of reasoning, but rather the making manifest of pre-existing, eternal relations between thoughts.²² By considering a passage from *Über die Grundlagen der Geometrie* (KS 320/336) where Frege ostensibly treats inference and consequence as on a par:

If the thought *G* follows (*folgt*) from the thoughts *A, B, C* by a logical inference (*durch einen logischen Schluß*), then *G* is true.

we can now identify a further function of the judgement-stroke. The sign is also intended to obliterate the (psychologism-prone) distinction between inference and consequence, between subjective acts and objective relations. In this role, the judgement-stroke marks out those cases in which the reasons grounding a reasoning move are "inherently related" to truth and can only

²⁰This directly contradicts Smith's (2009, 646) claim that for Frege logic is not concerned with "eternal relations amongst propositions" but rather with relations between judgements, as well as Sundholm's (2012, 954, note 4) claim that Frege had no interest in consequence.

²¹See also (KS 253/264) and (NS 223/207).

²²The suggestive notion of an inferential trace is Martin-Löf's. See Sundholm (2012, 948).

lead from already established truths to a further truth (NS 2/2). In a slogan, Fregean inference is consequence made manifest.²³ More precisely, Frege's inference relation is the subset of ordered pairs of (sets of) thoughts related by the (modern) consequence relation where both elements of the pair are true.

We can now turn to the details of Frege's conception of assertion. First, let's examine how the well-formed sentences in the *Begriffsschrift* are constructed. We start with a sign of this form:

(1) A

(1) is the formal representation of a (non-pseudo) declarative sentence, a sentence that predicates, in Frege's extended sense, properties of objects, sequences of objects, or classes. The content represented by "*A*" is that kernel of thought that remains invariant when it is asserted and when it occurs unasserted (e.g. in the antecedent of a conditional). In order to ensure this homogeneity of content-individuation—a homogeneity that, as Geach (1965, 258) noted, is crucial to the validity of his rule of proof, *modus ponens*—Frege needed a symbol that would extract and remove assertoric force from the sentence:

We must deprive the relation sign [and by extension, the sentence as a whole] of the assertoric force with which it has been unintentionally invested. (KS 232/247)

We must, that is, ensure "the dissociation of assertoric force from the predicate" (NS 201/185).²⁴ To that purpose, Frege in-

²³This follows from the factivity of Fregean judgement.

²⁴Despite Künnle's (2013, 56) criticism, Frege's point here is surely the counterpart of the remark in (KS 346/355) that thought and assertoric force (putting forward as true) are so closely bound that the possibility of their separability can easily be overlooked, a separability which in Frege's mind clearly corresponds to the separability of grasp and judgement of thought (KS 364/374; NS/11). Klement (2002, 29, 62) takes the need to separate the two naturally bound elements to be the main motivation and justification for Frege's introduction of the judgement-stroke. A related task for the judgement-stroke is to mark out the disambiguation between two possible uses of relational signs, referring to a relation versus asserting that it holds (KS 232/247; NS 201/184; BW 123/193).

roduced what he called *the content-stroke* in BS and *the horizontal* in GG:²⁵

(2) —

Many, including Frege himself, have been tempted to provide a natural language gloss for the expression formed by prefixing a formula with the content-stroke.²⁶ However, no such gloss could do justice to the logical purpose of the content-stroke and of the judgement-stroke. But this is no defect at all (or else, why would we *need* a regimented language at all?).²⁷ Accordingly, the role of the content-stroke in e.g. (3):

(3) — A

is best seen as that of a purely formal device that strips away or neutralises the assertoric force of the thought expressed by “A”.²⁸

²⁵There is an important difference between BS and GG. In BS, the force-stripping task is assigned to the content-stroke. In GG, the fact that a formula in predicational form is treated as a name of a truth-value has already deprived the expression of its illocutionary force. The horizontal merely preserves the force-neutral character of the formula by transforming it into a function-name, taking any object other than the True to the False. The final transformation into an asserted proposition is carried out by the judgement-stroke. The syntactic category of the judgement-stroke in GG is therefore that of a (*Begriffsschrift*)-sentence-forming operator taking terms as arguments. See e.g. Klement (2002, 28–30) for discussion.

²⁶In BS §3, Frege had suggested the reading: “the circumstance that A is a fact”. See Bell (1979, §1.3), Künne (2010, §7), Kanterian (2012, §§3.1–3.2) for extended discussion of this and other equally unsatisfactory suggestions.

²⁷Contra Kanterian (2012, 75). It is highly significant that Frege refrained from giving a linguistic gloss to the horizontal in GG. In my view, at that time he’d realised that a linguistic gloss would force a similar gloss for the judgement-stroke. The only plausible candidate for that could be “it is true that”, or, equivalently, “it is asserted that”. Both glosses would make the sign, by Frege’s own lights, *terminally* redundant. See also note 74 below.

²⁸See note 25 for a crucial proviso. In my view, the best natural language reading of (3), the best of a bad lot, would be “whether A”. Contra Dummett

Finally, to enable a formula to be eligible to occur on a line of proof, we need to add the judgement-stroke, the vertical sign attached to the horizontal/content-stroke. What we thus get is:

(4) ⊢— A

By neutralising the assertion-neutralising effect of the content-stroke, the addition of the judgement-stroke restores assertoric force, it transforms the bare *Aussage* into a proper *Behauptung*.²⁹ I claim that Frege’s insistence on the logical need for a separation between predication and assertion is based not just on the need to ensure the validity of his rule of proof. Ultimately, his insistence is based on the idea that thoughts are, as Bell (1979, 127) usefully put it, “naturally assertive”, that the concept-words occurring in copula position inherit the essential “predicative (*prädikative*) nature” of concepts (NS 133/122, 231/214).³⁰ Many find this claim questionable, thereby rejecting the need for the separation of assertoric force from the copula, since, on their view, the copula had no assertoric force in the first place.³¹

(1973, 315) and Potter (2009, 87), the content-stroke is not conferring propositional unity to something that lacks it. When Frege (1882/1964a, 101) says that “in the content stroke I think to myself the following content as unified”, he’s simply taking the content-stroke to mark the completed formation (*Bildung*) of the judgeable content that follows, the intermediate stage between formula-completion and its normative assessment. In this regard, its role is that of a pair of braces, corraling in the “complex of ideas” under consideration.

²⁹On the contrast between the two German terms, see the useful note in Ebert and Rossberg (2013, xvii). With regard to terminology, it has been customary since Whitehead and Russell (1910/1997, 8, 92) to refer to the sign as assertion-sign rather than as judgement-stroke (*Urtheilsstrich*)—Wittgenstein even re-translated it back into German as *Behauptungszeichen* (PI §22).

³⁰That’s what Russell called the “ultimate notion of assertion, given by the verb” (Russell 1903 §52). For some antecedents, see Baker and Hacker (1984, 77). For a recent defence, see Hanks (2015, chap. 3). Note that the claim applies only to sentences that are used, not mentioned, and not embedded under intensional operators. For some further qualifications, see Dummett (1973, 302–04).

³¹See e.g. Baker and Hacker (1984, 78).

In response, I want to insist that although, as Frege himself granted (NS 288/269), we can imagine reasoners that would be able to grasp thoughts without linguistic mediation, for any reasoners who instead have to rely on language to grasp and express thoughts, Frege's remarks about assertoric force stand. That assertoric sentences carry assertoric force is not a matter of convention.³² Rather, it is of the essence of language (and of linguistically-clothed thoughts) that when the copula is attached to an expression of the appropriate type the corresponding property is assertorically predicated of the logical subject. In their linguistic guise, that is, thoughts present themselves to us not just as truth-apt, but as true as well. Logic, however, demands that we separate true thoughts from their misleading companions (the false, and the fictional ones). The introduction of the content-/judgement-stroke pair is thus one more element in that fight against language that occupied Frege throughout his career.³³ Given that constitutive feature of concept-words, there is a logical need to visibly separate apprehension and evaluation of the thought, there is a logical need to "protect our thinking from error" (Frege 1882/1964b, 156) by marking out the presence of truth. This in turn allows for the separation between true thoughts and false (or pseudo) ones, a task that language cannot carry out unaided because of its essentially assertoric nature.³⁴ Accordingly, logic requires a sign whose sole purpose is

³²Textor (2010, 622) suggests a default condition: "If one utters a sentence in the indicative in the absence of factors that defeat its force, one makes an assertion". Dummett (1973, 311) too implicitly suggested something similar with his remark that the actor has to do something more, namely, suspending the default assertoric force by acting it out. See also Künne (2013, 67).

³³From the BS preface to (NS 289/270). In this sense, it is misleading to think of Fregean judgement as grounded in linguistic practice, as Ricketts (1986, 72) and Reck (2007, §XI) seem to do. On the contrary, Fregean judgement is grounded *against* linguistic practice.

³⁴Just as it is "not possible on the basis of linguistic form to distinguish a flawless argumentation from one in which steps have been left out" (Frege 1882/1964b, 157), it isn't possible, on the same basis, to distinguish be-

to embody assertoric force (NS 201/185), a sign that is lacking from the vernacular precisely because it goes against the naturally assertoric grain of language. But without such a sign there would always be a doubt as to whether the *Begriffsschrift* really is in the truth-tracking business.

With the judgement-stroke in place, Frege's calculus becomes a calculus of provability, a system of proof as opposed to a system of derivations that would instead allow for the occurrence of (unembedded) unasserted thoughts.³⁵ Each line of proof either expresses a thought for which we have a proof from previous lines, or it expresses a thought that stands in no need of proof (i.e., an axiom). Given these constraints, there cannot be a proof of a false thought, since the calculus, at least at the time of BS, is sound and the axioms are true. And since each line of proof is itself the end of a proof, only truths occur in the calculus. In effect, any asserted line in the *Begriffsschrift* reaffirms that the axioms are true and that the calculus is sound, since each line is labelled with the justificatory grounds that led us there, thus providing a renewed justification of the rules themselves (as required by GG II §89).³⁶ So conceived, the calculus also implicitly charac-

tween true thoughts and their truth-disengaged counterparts. Since for Frege (1882/1964b, 156) the "first demand" made on language is that it be "un-equivocal", and since at the time of *Grundgesetze* any indeterminacy regarding the presence of assertoric force entails that one element of judgeable content, namely, truth-value (GG I x, 9 note 2; KS 370, note 10/381, note 13), is left underdetermined too, it follows that the judgement-stroke fulfils an essential disambiguating role.

³⁵Here I disagree with Sundholm's (2012, 946) claim that Gentzen and Frege are in agreement in their conception of inference. Even if systems of assumptions can be transformed into systems of proof, the very fact that we disengage from truth in making an assumption would, for Frege, amount to a disengagement, albeit temporary, from serious scientific discourse.

³⁶This is not to say that Frege's judgement-stroke is to be read as the modern turnstile. The latter sign merely claims derivability in a given deductive system, it is merely committed to the conditional truth of the theorems. Frege's sign, by contrast, embodies an explicit commitment to the truth of the axioms as well. Accordingly, each asserted line manifests the required "reflective ap-

terises the good rational standing of the reasoner, her justified engagement with truth, her full understanding of the content expressed on that line—to properly understand the content of a necessarily true thought is to understand that the thought is true, on the basis of proof, whenever proof is possible.³⁷

I now turn to two further and closely connected reasons why the formal role of assertion must be represented in the *Begriffsschrift*. First, assertion ensures that logic is contentful, *inhaltlich*, (GG II §§91–104). Secondly, assertion thereby enables the *Begriffsschrift* to be a justification- and knowledge-generating system.

In regard to the first reason, by ensuring that logic is contentful, assertion enforces Frege’s anti-formalist constraints in a way that is not purely tautologous. As is familiar, for Frege, proofs are not a mere play with symbols (GG II §95).³⁸ The aim of inference is therefore to establish and make manifest the epistemic pedigree of truths, their ultimate grounding in the basic laws and definitions that undergird his system (GL §3). But, as we saw, unless we start from truths, and stick with truths all along, there’s no way in which proof can fulfil its epistemological duties.

Accordingly, Frege’s calculus has to be one of asserted truths, one that implements that distinction (e.g. NS 264/244) between derivation (*Ableitung*) and proof or inference proper (*Schluss*) that

preciation of the adequacy of a completed structure of justification” (Sullivan 2004, 738).

³⁷It is understanding in this robustly epistemic sense that is the constitutive norm of assertion for Frege. See Sundholm (2012, 949) and Macbeth (2014, 369, 382) for discussion.

³⁸See also (KS 302 note 3/316 note 3): “A mere wording without a thought-content can never be proved”. As Peter Sullivan pointed out, in personal conversation, in (NS 67/60), Frege says that “the rules of logic always presuppose . . . that our sentences express judgements, that one is not playing a mere game with words”. (BW 127/79) is however very clear about the need to make this logically essential presupposition explicit. Whether, and in what ways, this need can be fulfilled is a question I discuss in §4.

alone can mark logic as a contentful discipline.³⁹ In particular, while it is true that Hilbert’s axiomatic system, in contrast to Jaśkowski-Gentzen-Prawitz systems, purported to be a system of proof, it nevertheless lacked that anterior commitment to the truth of the axioms which is distinctive of Frege’s approach.⁴⁰ Only a system of proof that is committed to the truth-entailing assertion of *every* line of proof can satisfy the anti-formalist requirement, the requirement that, in each line, the truth of a thought be established.⁴¹

This takes us neatly to the second reason. As Frege’s discussion at (GG II §§91–94, §104, §140, §158) makes clear, a language can give rise to genuine knowledge only if its rules are grounded in the reference of the signs.⁴² Since natural language does not distinguish between mere expression of a thought and its outright assertion, and since it cannot distinguish, purely on the basis of overt linguistic form, between sentences that make genuine claims about a scientific subject-matter and those that

³⁹See also the quote I gave on page 4. Significantly, when Frege withdraws the use of the judgement-stroke in the derivations from Basic Law Vb “because of [its] doubtful truth” in the Afterword to *Grundgesetze*, he does not use *schließen* but *folgen* to mark out the various reasoning steps as he tests the consequences of that defective law.

⁴⁰Following Kant, Hilbert (1925, 376) argued that mathematics relies on anterior availability of content provided via intuition. Frege’s judgement-based metaphysics (Hylton 1990, 223) instead insists that content must be generated *within* a system of logic—ontologically speaking, logic is a self-supporting enterprise. On the “purely logical” derivation of objects from concepts, see (BW 121/191 note 69, 223/141).

⁴¹As Frege notes, the genuine discharge of assumptions can only occur when their *truth* is established (NS 264/245; BW 36/23). And as (NS 2/2) tells us, logic can only concern itself with its objects “insofar as they are true”. The presence of the judgement-stroke is thus meant to signal that logic is neither fictionalist nor formalist.

⁴²Here, I am heavily indebted to the discussion in Sundholm (1998, 183), Ricketts (1996, 131), and Heck (2012, 40–45). See also (NS 220/204): “we make advances in mathematics by choosing as the premises of an inference one or two propositions that have already been recognised as true. The conclusion obtained from these is a new truth of mathematics”.

concern fictional domains only, we need the judgement-stroke to mark out and enable the knowledge-generating character of Frege's system. The "mere thought alone", Frege (KS 150/164) tells us, "yields no knowledge", for the simple reason that a thought is insensitive to any lapse from the realm of truth into that of fiction (KS 149/163; NS 208/191). The step to the truth-value, the step from the realm of sense to that of reference, must thus be taken and recorded in the *Begriffsschrift*, to signal the fact that one could not acquire new knowledge via guesswork or happenstance. In fact, the "peculiar and incomparable" character of judgement derives from its epistemic and ontological journey across the two realms. In judgement we determine the True in ways that are constrained by the features of the thought involved.⁴³ The anti-formalist constraint is thus further reinforced, since the applicability of mathematics, as captured in the *Begriffsschrift*, is only ensured if the language is expressing thoughts that are capable of gaining us genuinely new knowledge, rather than the mere "illusion of knowledge" (GG II §§92, 104, 140).⁴⁴

Finally, it should now be clear why assertion in this strict sense is not to be equated with taking-to-be-true. The latter

⁴³(KS 150/165) (in)famously speak of judgement detecting "parts within truth-values", a statement Frege later retracted (NS 275/255). But it is of interest that in that passage Frege says that the detection of parts is done by means of a "return to the thought". The discussion in Textor (2010, 645–46) is useful here. There is a further metatheoretical role for the judgement-stroke. The *Begriffsschrift* signs *must* refer (GG II §71). In merely entertaining thoughts, we leave it open that the terms occurring in the unasserted sentences may lack reference.

⁴⁴Along the same lines, the judgement-stroke also plays a constitutive role with respect to content. Contentful arithmetic is possible only if expressions do not just aim at truth. Truth must be incorporated into each and every line of proof by means of the judgement-stroke. Künne (2013, 61, 66) takes truth-commitment to be the strongest view attributable to Frege. Given the factivity of judgement, however, the judgement-stroke marks not (just) the presence of the commitment but the presence of truth; it signals that the reasons listed are not just favouring our taking the relevant thought to be true, but rather that they are the objective reasons why the thought *is* true.

stance would indeed be merely psychological. The fact that each asserted truth is instead firmly grounded in the axioms ensures that the *Begriffsschrift* is, at all times, the language of genuine truths, the language of *Wahrsein*, rather than of *Fürwahrhalten*, that it is a language of reasons, and not of causes (NS 159/147).

If the considerations I canvassed in this section are sound, assertion has a formal and logically significant role to play in Frege's *Begriffsschrift*. The question we now need to consider is whether the judgement-stroke could unproblematically play its required formal role.

3. The RWJ Objection

Let me now discuss in some detail the RWJ dilemma that I outlined at the beginning. The dilemma is triggered by the following, perfectly legitimate question: since assertion is formally represented by the judgement-stroke, what kind of content properly belongs to that sign? On one horn of the dilemma, the answer is that the only content it could have would be psychological in character and thus in tension with Frege's lifelong commitment to anti-psychologism. On the other horn, the answer is that the judgement-stroke could have no content at all, because no sign can carry, or guarantee, assertoric force; its presence in the *Begriffsschrift* therefore "serves no purpose" (Black 1964, 227), is "futile" (Kenny 1995, 132), "otiose" (Dudman 1971, 26), "inane", and "superfluous" (Baker and Hacker 1984, 80, 83).⁴⁵

⁴⁵For some other examples, see (TLP 4.442), (NB 95, 103), (PI §22), Bell (1979, 95), Kenny (1995, 36), Kanterian (2012, 62). Proops (1997, 124) finds the idea of logical assertion "ultimately incoherent". By contrast, Geach (1976, 63), Dummett (1973, 311–12), Smith (2000; 2009), and Textor (2010) dismiss the RWJ objection. Cohen's (1964, 121) discussion of Austin's speech act theory raises an interesting version of the RWJ objection. By Frege's own lights (KS 153/167), the occurrence of "*p*" in "I assert that *p*" is such that its own assertoric force has been removed. Moreover, "*p*" and "I assert that *p*" have different content and truth-conditions (in the latter case, the truth of

If the dilemma is genuine, it straightforwardly follows that assertion is “logically quite without significance” (NB 103), since it either has psychological content, or it has no content at all.

In the previous section, I’ve argued that assertion has logical significance in its formal role. My aim in this section is to block the first horn of the RWJ dilemma by showing that assertion can carry out that role without importing psychologistic elements into logic. I’ll deal with the second horn in §4 instead.

The worry on the first horn is that, by introducing the judgement-stroke in the *Begriffsschrift*, Frege had allowed “psychological elements” to “intrude in describing judgement as recognition of truth” (Russell 1903, §478).⁴⁶ But why should we think that assertion inevitably introduces psychological elements into logic? What, exactly, motivates the RWJ objection? One reason behind it might be that Frege himself had famously dismissed the logical significance of modal operators by saying that they only express the doxastic strength of one’s beliefs, something that, he added, has nothing to do with logic (BS §2).⁴⁷ Surely, the complaint goes, this dismissal should apply just as much to the judgement-stroke. After all, any plausible norm governing assertion will make essential evaluative reference to

the sentence does not depend on the truth of p , since the reference of “ p ” is a thought, rather than a truth value). In itself, this equivalence failure is no objection to Frege, since he distinguished force from content. But it is a potential objection to his *use* of the assertion sign. Far from being redundant, the sign, if equivalent to the performance of an act of assertion made explicit, would alter the content of p . Any content the sign *would* have would alter the content of the embedded sentence. I return to this point in §4.

⁴⁶See also Russell’s letter to Frege in (BW 250–51/169): “We do not assert the thought, for this is a private psychological matter”. It’s likely that during his conversations with Frege in 1911 Wittgenstein raised the same worries later put to print. Jourdain repeated the complaints in a letter to Frege (BW 126/78). As Peter Sullivan noted, in personal conversation, Russell’s objection is weaker than Wittgenstein’s. For Russell, assertion imports psychological elements, but is not merely psychological. I argue against this weaker reading of the RWJ objection too.

⁴⁷See Fitting and Mendelsohn (1998, 4–5) for criticism.

the doxastic state of the assertor. If so, assertion will involve reference to a specific assertor, something that Frege wants instead excluded from logic.

A further reason to think that assertion is psychological by Frege’s own lights is a much-quoted remark in “*Der Gedanke*”, where assertion is said to be the outward form of an internal act of judgement. Here, the RWJ objector seems to be reasoning along the following lines. Since judgement is internal, it contains mental elements. Since assertion is nothing but the outer manifestation of judgement, it too must incorporate mental elements. But mental elements are psychological. And so assertion is psychological too.⁴⁸

Both reasons, then, take assertion to be indexed to the doxastic or mental states of particular speakers, and thereby to fall foul of Frege’s anti-psychologistic strictures. If these are the reasons supporting the RWJ objection, the objection relies on three main assumptions: 1) that the mental is wholly contained within the psychological; 2) that doxastic and mental states make essential reference to specific reasoners, and 3) that *any* reference to reasoners imports psychological considerations into logic. I’ll now be arguing that these three assumptions are false. Since the accusation that Frege’s notion of assertion is psychological cannot be supported without showing that all three assumptions are true, I take it that the arguments below, if sound, will more than suffice to rebut the RWJ objection.

⁴⁸There is a further complaint, specific to Wittgenstein, namely, that assertion, like sense, imports arbitrary, conventional, and therefore non-logical factors into the sign-referent relation. Potter (2009, 100) concludes that assertion is not psychological in Russell’s sense but that it is so in this latter sense (i.e., non-logical). For Frege, however, the naming relation between signs and referents is non-arbitrarily determined through the compositionality of sense—the arbitrariness in the choice of primitive names is wholly irrelevant, precisely because it is arbitrary (any choice of primitive names would have done just as well). See e.g. the end of §158 in (GG II): “the rules here follow necessarily from the reference of the signs, and this reference is to the proper objects of arithmetic; what is arbitrary is only the notation”. See Heck (2012, 44ff.) for discussion.

With regard to 1), the mental, at least in Frege's sense, is clearly wider than the psychological, rather than being contained in it. In particular, the realm of sense, in the austere logical sense of "sense" that Frege was committed to, belongs to the mental, but not to the psychological, since it constitutes the logical portion of the mental, the portion that would remain invariant, no matter what alterations to the psychological make-up of reasoners would occur (NS 160/148). Sense is thus resolutely non-psychological.⁴⁹ It is the portion that is both publicly accessible and free from any contingent, locally-inflected features. It has no features that concern parochial, psychological modalities through which particular pieces of content are apprehended. It is thus a notion that is individuated solely in terms of reasons, *never* in terms of causes. Crucially, Fregean assertion only involves the mental in this restricted sense. It is thus a purely normative, wholly reasons-based notion.

We can now also respond to 2), and point out that Frege's use of the judgement-stroke leaves the logical status of the portion of the mental that is involved in acts of judgement and assertion untouched by any local consideration. Contrary to the RWJ objection, assertion introduces "no mention at all of the class of beings for whose judgements" a particular line of proof would be taken to be properly established (NS 159/148). As Frege (BW 126–27/79) makes clear, that is, assertion and judgement, in his *ungewöhnlich* sense, have nothing to do with the psychological make-up of reasoners. In asserting that p , I am saying nothing about myself (I am not speaking *of* me); the assertion does not mention, involve, or mobilise any goings-on internal to my soul,

⁴⁹See my (2017, §2) for arguments to that effect. Kremer (2010, 281–86) defends a contrary view. Taking sense to be non-psychological, however, is the only way to make sense of Frege's remarks that "a thought, as I understand the word, is in no way to be identified with the content of my consciousness" (NS 226/209), and that "[w]hat is a content of my consciousness, my idea, should be sharply distinguished from what is an object of my thought" (KS 357/366).

as he put it. Above all, the mental, in this restricted reading, is invariant, in that it is perfectly robust under variations in psychological make-up.

As for 3), this assumption presupposes two things which are both false. First, that any reference to reasoners would import psychological considerations. Secondly, that logic can make no reference to reasoners, on pain of psychological contamination. Against both assumptions, we must note that logic is inherently but harmlessly perspectival, since the implication relations it deals with are constitutively perspectival.⁵⁰ They start from a bunch of established premises, and list their immediate consequences. The perspective, however, is not psychological; it is, again, not formulated in terms of causes. It is wholly rational, solely formulated in terms of reasons, of objective justifiers. Because of this, both sense and assertion are non-subjective notions without which one can form no conception at all of logic as a science. Logic is about rational movements between thoughts, movements made under rational licence. movements originating from reasons, not from causes—this is indeed where the normative character of logic that Frege (NS 139/128) insisted upon derives from.⁵¹ When φ is a logical consequence of Γ , a rational thinker who knows this could not hold that all of Γ is true and φ false. A basic bridge principle of this kind is logical if anything is, and it is what gives content to the claim that logic is normative. And yet bridge principles of this kind must contain a perspective, or else it would be impossible to state them, or claim that logic is a normative discipline. The point remains, however, that the perspective makes no mention of the cogni-

⁵⁰By "perspectival" I mean: related to the reasons one owns (in Broome's (2013, 13) sense). As (GL §26) tells us, we could form no conception of what things are independent of Reason. And reasons (or reasoning, if, like Broome, you think this is the primary notion) are one's way into Reason.

⁵¹See Taschek (2008) for discussion. Clearly, this is a point on which Frege's and Wittgenstein's conception of logic differs, since for Wittgenstein logic has no prescriptive role, only a transcendental one (TLP §6.13).

tive processes whereby one comes to know either the logical principles themselves or the truth of the sentences in Γ . The perspectival point is individuated solely in terms of reasons, of what is true, and proven to be true.

If these considerations are correct, the three assumptions on which the RWJ rests are to be dismissed as false, as having arisen out of a deep misreading of Frege's conception of logic. Contrary to the RWJ objection, there's nothing psychological to assertion in Frege's sense. As he makes quite clear (KS 157/145), what matters to assertion is that one grasp a thought and acknowledge its truth: "how one does this is a completely separate question" (my emphasis).

I take myself to have now shown that Fregean assertion in its formal role has a logical significance that remains untainted by psychological contaminants. My next task is to articulate and defend the transcendental role of assertion.

4. The Essence of Logic

It's time to face the second horn of the RWJ dilemma. Earlier on, I rehearsed the accusation that the judgement-stroke is superfluous because "no sign . . . can guarantee assertoric force" (Baker and Hacker 1984, 803).⁵² Since the latter point is surely sound, it might then seem that even if I'm right to claim that the judgement-stroke imports no psychological elements into logic the further claim that the judgement-stroke has logical significance is in jeopardy anyway. What use could logic possibly have for a superfluous sign?

Frege, however, made that very superfluousness into the lynchpin of his striking claim in a 1915 note called "My Basic Logical Insights" that it is assertion, rather than truth, that gives the essence of logic. Without question, that note is crucial to un-

⁵²Peano must have been the first to raise the issue in his 1895 review of GG. See Dudman (1971).

derstanding why a thinker as rigorous as Frege insisted on the presence of the judgement-stroke in his *Begriffsschrift*. It is a note, though, that is still owed a wholly convincing reading, a reading that would reconcile Frege's acknowledgement of the illocutionary superfluousness of the judgement-stroke with his insistence that the presence of the judgement-stroke in the *Begriffsschrift* is nonetheless a logically "essential thing" (BW 127/79) to be explicitly recorded for logical book-keeping purposes and not left to "tacit convention" or, worse, guesswork.

I will make two novel suggestions as to how we should interpret this passage, two suggestions that, I think, neatly effect the needed reconciliation of those two strands in Frege's thought. I shall propose that assertion gives the essence of logic because assertoric acts are essential to the full exemplification of the logical laws, and that the judgement-stroke gives the essence of logic because it marks out those languages that are both sensitive to the requirements of logic and at the service of reasoners still in need of logic.

Here, then, is the 1915 passage (nearly) in full (NS 271–72/251–52):⁵³

[A]ssertion is not to be found in the word 'true', but in the assertoric force with which the sentence is uttered. This may lead us to think that the word 'true' has no sense at all. But in that case a sentence in which 'true' occurred as a predicate would have no sense either. All one can say is: the word 'true' has a sense that contributes nothing to the sense of the whole sense in which it occurs as a predicate. But it is precisely for this reason that this word seems fitted to indicate the essence of logic. . . . [T]he word 'true' seems to make [alternative

⁵³For some analyses of this passage see e.g. Ricketts (1996, 133), Sluga (2002, 87–89), Shieh (2002, 110), Taschek (2008, 382–83). One common conclusion is that the passage is in tension with the stress on truth elsewhere in Frege's work. I do not think that there is any tension between the 1915 passage and earlier passages where Frege (NS 137/126) claimed that logic is characterized by the word "true". In 1915, Frege finally understood what he meant by that claim, just as only in GG did he understand completely what he'd meant at the time of BS by "judgeable content".

version: “to be trying to make”) the impossible possible: it allows what corresponds to the assertoric force to assume the form of a contribution to the thought. And although this attempt miscarries, or rather through the very fact that it miscarries, it indicates what is characteristic of logic. . . . ‘[T]rue’ only makes an abortive attempt to indicate the essence of logic, since what logic is really concerned with is not contained in the word ‘true’ at all but in the assertoric force with which a sentence is uttered. . . . [T]he thing that indicates most clearly the essence of logic is the assertoric force with which a sentence is uttered. But no word, or part of a sentence, corresponds to this.

The passage is very rich. It is also pivotal to Frege’s project, because, as he tells us in (GG I xiv–xv), the key anti-psychologistic move consists in giving a proper account of our grasp of logical laws, which in turn depends on “how one understands the word ‘true’”, precisely what this passage is all about.⁵⁴ At first blush, it is hard to know what to make of it, however, since the passage seems to be in tension with the otherwise “pure” conception of logic that is traditionally associated with, and traced back to, Frege’s work. How can *signs* make attempts, abortive ones at that, to indicate the essence of logic?⁵⁵ And why should their expressive failure with regard to assertoric force indicate the characteristic marks of logic?

My answer to these questions is intended to take forward and complement the insightful discussion in [Taschek \(2008, 389–98\)](#) and [Greimann \(2014, §4\)](#). I agree that the 1915 passage highlights the effort by Frege to secure a privileged normative

⁵⁴See also (NS 3/3, translation modified): “Anyone who has failed to grasp the peculiar meaning of [the word ‘true’] cannot attain any clear idea of what the task of logic is”.

⁵⁵One may question the need for logic to give its own essence. In response, we should note that for Frege (NS 139/128) any science is under a duty to enquire into its own essence, into what it is about, how it goes about pursuing its aims, and so on. Logic is no exception, and getting to the essence of logic requires becoming clear about the “quite peculiar way” in which logic deals with the truth predicate.

status for logic (Taschek) and that the presence of the judgement-stroke is meant to mark the required step “from the level of sense to the level of *Bedeutung*” (Greimann). I do not however think that these two important suggestions give the complete story of what Frege is up to in that passage. Specifically, I take issue with Taschek’s claim that this passage shows that for Frege assertoric force reveals the essence of logic because it discloses the role of truth as “the constitutive aim of judgement” (390) and with Greimann’s claim that “if we already had a logically perfect language, we could use the assertion sign to explain the essence of logic” (285). Against those remarks, I want to claim that the transcendental role of assertion is not just that of signalling the presence of truth, but also, and more importantly still, that of revealing the sense in which logic constitutively requires acts of judgement, as well as flagging the sense in which language-bound reasoners have a continuing need for logic.

In my view, then, the presence of the judgement-stroke transcendently marks two logical needs: the need that the laws of logic be *fully* exemplified, and the need that our distinctive condition as thinkers and, correlatively, the precise location of Frege’s formal language in the hierarchy of languages that tend towards the logically perfect language, be formally marked. In this latter respect, I think, *contra* Greimann, that for Frege the logically perfect language would have no need for the judgement-stroke. The *Begriffsschrift*, however, inasmuch as it is a mere approximation to that logically perfect but unachievable language, still requires its presence, precisely to mark both its superiority with respect to those languages that, like ordinary language, draw no distinction between asserted and unasserted propositions, and its inferiority with respect to the logically perfect language that would instead have no need for that distinction.

Arguably, the reading I propose, however speculative it might be taken to be, is grounded in what Frege says at the end of the passage in question:

If our language were logically more perfect, we would perhaps have no further need of logic, or we might read it off the language. (NS 272/252)

I take Frege here to be referring to the *Begriffsschrift*, not to ordinary language. I therefore take one of his basic logical insights to be that there is a crucial logical imperfection to language *in general*, that a formal language can at best be “more perfect” than ordinary language, and that, at the limit, an increase in perfection would remove the need for logic precisely by making logic and all that is essential to it a genuinely integral part of the language. As we shall see, no language known to us can do that, however. In fact, we have no idea what a language of that kind would even look like and whether it would still count as a language.⁵⁶ What Frege is grappling with in the 1915 passage is therefore one more limitative result regarding the expressive power of language. Assertoric force is both essential to inference and inexpressible by language, in exactly the same way that truth is essential to logic but formally undefinable. For all that, there is a standing normative duty to *try* and express all that is essential to logic, including those constitutively ineffable aspects attached to assertoric force and truth, because logic is (transcendentally) characterised precisely in terms of those unrepresentable aspects.⁵⁷

To see how we obtain this limitative result with respect to assertion, let’s first ask what it would mean to write logic into a language. *What*, exactly, would need to be written into language for logic to be transparent to its users, for logic to be read off

⁵⁶This is another version of the logical aliens conundrum that Frege discusses at (GG I xvii).

⁵⁷Assertion is thus transcendental twice over: it provides the conditions for the possibility of logic as the science of reasoning (it *tries* to say why *any* language-bound reasoner needs logic and how that need is manifested), and it is “itself unsayable” (to borrow Anscombe’s (1959, 166) gloss on Wittgenstein’s claim that logic is transcendental). On the first sense of transcendental, the discussion in Sullivan (2011, §§7–8) is illuminating.

the language? The answer seems obvious: the laws of logic. In this regard, the contrast between a logically informed language and natural language is marked as clearly as it can be marked, since in that constitutively defective language the laws of logic can be no more than “externally hung plumb lines” (Frege 1882/1964b, 157), rather than being, as they are *intended* to be in the *Begriffsschrift*, a fully integral part of the language. A wholly non-defective language, a language that treated the laws of logic as intrinsically constitutive of its structure, *would* be one where both language and reasoners are constitutively and inextricably bound by truth, so that the presence of truth could go unmarked.⁵⁸ Any non-nondefective language that does not contain a judgement-stroke marking the constitutive presence of truth both with respect to the axioms and to the derivation of a given theorem betrays instead a deep misunderstanding about its own defective nature, a misunderstanding that *requires* correction via the detour (a *diagnostic* detour) through the content- and judgement-stroke epicycles—with those two strokes absent, the misunderstanding goes unnoticed and, *a fortiori*, undiagnosed.

The next question is whether the laws of logic could be written into a language. More generally, could all that is essential to inference be written into language? There are, I think, two obstacles to the achievement of a logically perfect language in this sense, and Frege had both obstacles clearly in his sight. First, the Carroll Regress. Secondly, the ineliminable role of circumstances, of actual acts of assertion by truth-bound reasoners.

⁵⁸The languages at the core of our logical systems are compatible with a variety of logical laws. The logically perfect language that Frege had in mind would only be compatible with his logic of choice because there would be no need to *state* its laws as axioms. In fact, and in the light of the discussion to follow, the very distinction between axioms and rules would be obliterated. That’s why the presence of truth would not need marking. Frege’s insistence on the advantages of his multi-dimensional language was of a piece with his striving towards a language that would effectively, that is, performatively, represent its laws, truth and assertoric force.

Given these two obstacles, no language could fully represent movements between true thoughts *as* reasoning steps that are justified in the light of the laws of logic, all of those laws and nothing but those laws. Let me now discuss these obstacles in detail.

Earlier, I said that for Frege inference is consequence made manifest. Not every justificatory aspect of consequence, however, can be made manifest, since not everything that is needed to justify a reasoning step, in particular, recognition of validity, can feature as a premise in that step. This is, in effect, the familiar lesson of the Carroll Regress, a lesson that Frege had very much in mind (see the opening of BS §13).⁵⁹ As Frege notes, some laws essential to inference must form the external basis to a logical system, but they cannot be represented therein. Those justificatory elements that cannot take the form of a premise, those laws of justified reasoning that must remain rule-like on pain of regress, can only be made manifest in actual acts of reasoning, they can only be exemplified *as* acts of reasoning, as actual takings of a reasoning step. Without such acts, logic cannot get going as a justificatory enterprise (which is what logic *is* according to Frege). For all that, inferential acts do not give rise to fresh determinations of the logical facts but merely exemplify them.⁶⁰ That is to say, while the act of drawing inferential lines is essential to logic because without that act the justificatory reasons would not be represented in their entirety, the act itself merely unveils logical relations between truths that had already been determined ahead of the act (NS 220/204, 223/207).

⁵⁹Reactions to the regress are many and varied. See my (2012) for discussion. I agree with Reck (2007, 159) that the *Begriffsschrift* aims to articulate “internal standards for judgement and inference”. I add that, due to regress worries (see also Ricketts 1986, 82–83), that articulation must be complemented by acts of judgement. Incidentally, if I’m right about Frege’s attitude towards the regress, he’s here agreeing with Kant, (1787 A 132–133/B171–B172), that logic cannot legislate over the power of judgement precisely because of that regress. The only way out for logic is to *incorporate* judgement into its structure.

⁶⁰This is the realist aspect of Frege’s conception of logic that I discussed back on page 5 coming to the fore again.

I now turn to the other obstacle. As Frege was at pains to stress, no language can fully represent, or perform, those essential reasoning acts not just because of Carroll Regress worries but also because without a suitable background of “serious” circumstances, circumstances where truth is being factively pursued, we could always raise doubts as to the seriousness of the assertion (a seriousness purportedly represented *in* the sign) and of the surrounding practice (KS 347/356).⁶¹ This is in fact the point of the remarks about the actor on stage (NS 211/194, 252/233–34, 271–72/251–52) that have been misunderstood so radically by e.g. Anscombe (1959, 113–14). Of course, an actor too could engage in genuine assertoric acts, precisely because, in general, the fact that an utterance takes place cannot by itself ever settle whether genuine assertoric force attaches to it.⁶² For Frege, what that shows is that only circumstances can ensure that the required seriousness is present, that logic is genuinely being done, that its law-governed movements between thoughts are enacted, exemplified.⁶³

In this regard, I think we should extend to the notion of force those externalist components undoubtedly present in Frege’s

⁶¹(BW 33–34/20) is very clear too: assertion is determined by and recognised on the basis of “outer circumstances”. Frege is here very close to e.g. Wittgenstein’s (1939/1975, 93) remark in discussion with Turing concerning the content-fixing role of “surrounding circumstances”. Frege’s stress on outer circumstances shows, it seems to me, that his view of language was much closer to some aspects of the Wittgenstein of the *Philosophical Investigations* than that of the author of the *Tractatus* ever was—the boxer example in PI §22 is an indictment of the picture theory of meaning, not of Frege’s *actual* conception of assertion. I think this further supports Reck’s (1997; 2002b) already convincing and extensive case in favour of Frege’s lasting influence on Wittgenstein’s thought.

⁶²The point against Anscombe was made in Smith (2000, 172), but is worth repeating.

⁶³Ricketts (1996, 124) says that “[Frege] tells us what logic is by identifying specific laws and inferences as logical”. Quite. But that can only happen with assertion in place.

conception of content.⁶⁴ Here's a quick argument to that effect. Frege says two things concerning the location of assertoric force. First, that it is to be traced back to the linguistic form of a sentence taken as a whole (there is no specifiable constituent to which it attaches, although the predicate is the most likely candidate).⁶⁵ Secondly, that two sentences of the same overt linguistic form can nonetheless differ in assertoric force.⁶⁶ Frege accounts for the difference in terms of a difference in circumstances. If we assume that some kind of compositionality principle for assertoric force holds,⁶⁷ Frege's view then seems to be that assertoric force is given by (or contained in) the union of linguistic form and circumstances, with both constituents singly necessary and jointly sufficient for assertoric force to be in place.⁶⁸ Note that

⁶⁴For some externalist readings, see Burge (2005, 57–58), Wiggins (1994), and my (forthcoming, §6). In a sense, I'm proposing that we expand what Reck (1997, §IV) aptly called Frege's *contextual platonism* to include extra-linguistic elements in the determination of content. I think Frege was divided between internalist and externalist requirements. His take on judgement, requiring both recognition and factivity, was meant to reconcile these two strands in his thought.

⁶⁵See (KS 150/164), (NS 140/129), (KS 372/383): "assertive force is supplied by the form of the assertoric sentence, which is specially well-marked in the predicate" or therein "presented" (KS 347/356). But we cannot locate any "particular part in the assertive sentence which corresponds to the assertion; that something is being asserted lies rather in the form of the assertive sentence" (KS 346 note 3/356 note 5). In another passage, however, Frege traces assertoric force to the copula or to the predicate (KS 192/177).

⁶⁶(NS 138/127): "[i]n an assertoric sentence the expression of a thought and the recognition of its truth usually go hand in hand", this, however, "does not have to be so" (the connection is not necessary); "[a]n assertoric sentence does not always contain an assertion". Similarly, in (KS 347/356; BW 33/20) we read that utterances of sentences having the same overt form as genuinely asserted sentences do not constitute assertion because "seriousness", that is, concern with truth, is lacking.

⁶⁷Frege's compositionality principles operate on content, not force. It does not seem far-fetched to think they can have applicability in helping locate the carriers of assertoric force too.

⁶⁸See Sher and Wright (2007, 283) for a similar suggestion.

circumstances so conceived also include the actual performance of acts of assertion.⁶⁹ In going proxy for circumstances, then, the judgement-stroke marks the execution of genuine assertoric acts, and it thus completes the system of logic by supplying a component that cannot be overtly expressed but only enacted.⁷⁰

Let me take stock of the discussion so far. I have insisted that linguistically represented thoughts are intrinsically assertive, that they constitutively put themselves forward as true. Hence, on the basis of linguistic form alone we cannot discriminate between true, false, and fictional thoughts. The rules of formation of the *Begriffsschrift* ensure that all terms refer (GG I §§10, 31–32), thereby screening off fictional thoughts. We still need to distinguish between true and false thoughts, however, and between true thoughts that are asserted on the basis of demonstrative reasons (and hence known to be true) and those that are instead (unwarrantedly) reached (even partly) via guesswork. This is where assertion comes in. First, circumstances help discriminate between genuine and pseudo-assertion by providing enabling conditions for proper assertion. Secondly, the judgement-stroke and the *Begriffsschrift* as a whole ensure that only properly justi-

⁶⁹Circumstances count as (enabling) reasons and not as causes. They partly determine whether assertion occurs, and they are necessary to the removal of guesswork exactly in the same way in which appeal to circumstances is required whenever "the mere wording . . . does not suffice for the expression of the thought" (NS 230/213) and we therefore need to take expressions as requiring integration by "accompanying gestures" and the "knowledge of certain conditions accompanying the utterance" (KS 349/358). I'm thus suggesting that for Frege both content and force are partly determined by the environment (including the availability of and recourse to a system of proof), and that there are unarticulated constituents at the level of logical form whose expression is entrusted to circumstances so conceived. Incidentally, it is the system that rescues Frege from the dilemma posed in Kremer (2000, 580–81), since it provides standard of correctness that are neither coherentist nor enforced by some correspondence theory story.

⁷⁰If I'm right, the judgement-stroke, far from being superfluous, is in fact overloaded with tasks, both formal and transcendental, to be fulfilled simultaneously.

fied true thoughts figure on a line of proof.⁷¹ The specific contribution of the judgement-stroke is to mark both the presence of truth through the enabling co-operation of circumstances, and the ascent to the level of reference exemplified through reasoning acts. It marks, that is, those justifying grounds that cannot be represented in the side annotations of a proof.

At this point, however, the superfluousness objection will resurface again under a new guise, reminding us that no sign could go proxy for circumstances either. And indeed, that stubborn difficulty is exactly what the 1915 passage faces up to. The judgement-stroke, *qua* sign, can do no more than engage in a representational attempt, inevitably doomed to failure. The failure is nonetheless revealing, precisely because it shows the ineliminable role of circumstances in fleshing out the essence of logic, precisely because it embodies the requirement that the step to the realm of reference be actually taken.

Now, reflection on the peculiar linguistic role of the judgement-stroke brought to prominence limitative results concerning the expressive powers of language. Familiarly, those results are also encountered through reflection on the semantics of a single, highly peculiar word, namely, the word “true”. Let me close by looking at what Frege says about that word in the 1915 passage. Here, the crucial thing to note is that Frege does not say that “true” has no content, that no sense attaches to it. His point is rather that the sense of “true” is compositionally inert, that it makes no truth-conditionally relevant contribution to the sense of the containing sentence.⁷² It is not empty, though,

⁷¹This is true only of the consistent fragments of GG. See e.g. Wehmeier (1999), Heck (1996).

⁷²As made clear in (NS 251/233): “nothing at all is added to the sense [of a sentence] by this predicate”. We can think of a compositionally inert sense as a function that intervenes at the end of the compositional sense-calculation and returns as value whatever value the preceding compositional operations had determined. The contrast is with cases such as “although”, which according to Frege has no sense of its own at all but simply inherits the sense of “and”

or else, as he notes, the whole sentence would lack a sense.⁷³ Furthermore, “true” must have a sense precisely so that it can allow “what corresponds to the assertoric force to assume the form of a contribution to the thought”. And it is “through the very fact that [the attempt] miscarries” that the word “indicates what is characteristic of logic”.

It seems to me that Frege’s focus on the attempted shift from illocutionary to locutionary content with respect to *both* truth and assertion is the subtlest and deepest point he makes in that passage.⁷⁴ For Frege, that is, the essence of logic is given precisely by this forced, dual category mistake, by the two mandatory attempts to make truth-predication and assertoric force into thought-constituents, since the judgement-stroke too allows us to attempt to represent the contribution from reasoning acts as if it could be a contribution that could, *per impossibile*, be written, in all of its logical aspects, into the language.⁷⁵

(KS 158/172). It is the very ubiquity of the truth predicate (NS 140/129) that deprives it of compositional content, just as Frege’s theory of meaning predicts: the wider the extension of a concept, the emptier its content (GL §29; NS 71/63).

⁷³See also (GG I xvii; NS 144/133).

⁷⁴For Frege, the judgement-stroke is the formal counterpart of “true”, since the predicative role of “true” is constitutively inert (NS 140/129), and the only way of “predicating” truth of a thought is therefore by asserting the thought: “it is really by using the form of an assertoric sentence that we assert truth, and to do this we do not need the word ‘true’”. This is one more reason why it is a bad idea to try to translate a sentence prefixed by the content-stroke into natural language. By the same token, we would now be tempted to translate the judgement-stroke as “it is asserted that” or to treat it as “an illocutionary operator” (Mendelsohn 2001, 45), a truth predicate (e.g. Cook 2013, A4), an alethic operator (Sher and Wright 2007, 291), or a truth-operator that represents “the attribution of truth in scientific judgement” (Greimann 2000, 214, 216–17). All such interpretations miss out Frege’s insistence, in the 1915 passage, on the representational-performative failure of the judgement-stroke, as do those interpretations that treat the judgement-stroke as a performative (Dummett 1973, 316, 335, Bell 1979, 98 Smith 2000, 164).

⁷⁵Does the judgement-stroke represent the attempt to represent assertoric force, or does it directly attempt to represent assertoric force? I’m not really

I think my insistence on the role of reasoning acts and on its essential connection to the semantic inertness of the word “true” is further supported by the fact that it explains and solves an otherwise inconvenient paradox. At the time of BS, Frege had said that the laws of logic unfold the content of the word “true” (NS 3/3). In 1915, however, we are told that “true” has a semantically inert content. That there is no contradiction between the two statements should now be clear. Although the content of “true” is semantically inert because transcendental, it does not follow that there is no content to be unfolded. It is just that what is unfolded by the laws of logic is something that could only be unfolded in acts of assertion, something to be exemplified in actual movements from known truths to truths known on the basis of those movements. Given our two limitative results, there is no way to capture those movements solely through propositional means and pack them into the content of the word “true”. That content must therefore remain propositionally and compositionally empty.⁷⁶

If these suggestions are along the right lines, we then have the required purchase on the sense in which assertion, in its transcendental role, gives the essence of logic. Assertion makes logic possible by providing the condition of possibility for its laws to be exemplified in their entirety, for the bafflingly empty content of the word “true” to be unfolded. And precisely because that content is itself transcendental and can only be fully unfolded

sure.

⁷⁶I say this with some trepidation, but the conception of logic that I attribute to Frege has an Aristotelian flavour, in that the aim of logic is to determine and be constituted by an *activity*, the activity of reasoning only in truth-bound ways. Here, Gödel’s (1961, 383) remark that the “clarification of meaning [of the axioms] consists in focusing more sharply on the concepts concerned by directing our attention . . . onto our own acts in the use of these concepts, onto our powers in carrying out our acts” is relevant. What needs doing is replacing “clarification” with “determination”. Note that the role of reasoning acts I envisage is much narrower than in Macbeth (2014), in that I restrict it to those exemplified through use of the *Begriffsschrift*.

in assertoric acts, we must attempt to represent *as an act* the (transcendental) unrepresentability of assertoric force—which is to say, the unrepresentability of truth. It is only through that very failure that we can signal the standing need for corroboration by the presence of serious, truth-bound enactors of the laws of logic. With this, we have defused the second horn of the dilemma too. The judgement-stroke has indeed no content at all, but it is precisely in virtue of its contentlessness that it has logical significance.

The still deeper lesson from the 1915 note, then, is that in a fully non-defective language assertoric force could *not* be removed. The presence of the judgement-stroke in the *Begriffsschrift* is thus a reminder that Frege’s language, although maximally non-defective, is not yet perfect, that it can only try to incorporate the laws of logic right into its fabric. The constitutive failure to incorporate those laws is due to the fact that it is still a language, barring, like all languages, unimpeded access to pure thought.⁷⁷ *Qua* language, it still requires enactors and co-operative circumstances. Logic nevertheless demands that the *Begriffsschrift* mark the awareness of its own inadequacy through the presence of the judgement-stroke.⁷⁸ And the overt inclusion of the judgement-stroke characterises the essence of logic as residing in the unavoidably frustrated *aspiration* to tend towards that non-defective, but forever elusive, language of pure thought that would make logic itself redundant. Otherwise put: the presence of the judgement-stroke signals, as it must, our continuing *need* for logic.

⁷⁷Here, I generalise Gabriel’s (1996, 332) extremely apt remark that Frege begrudges language the fact that it “denies him pure access to thought”.

⁷⁸If the main purpose of the *Begriffsschrift* was to fill in all the gaps that *could* be filled in proof, the transcendental role of the judgement-stroke is to make visible the one unfillable gap, the gap that only acts and circumstances can properly fill.

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GL: *Die Grundlagen der Arithmetik*. Breslau: W. Koebner, 1884. Translated as *The Foundations of Arithmetic*, translated by J. L. Austin. Oxford: Blackwell, 1953.

GG: *Grundgesetze der Arithmetik*, 2 vols., Jena: H. Pohle, 1893 and 1902. Translated as [Frege \(1893/2013\)](#).

BW: *Wissenschaftlicher Briefwechsel* Hamburg: Verlag, 1976. Translated as *Philosophical and Mathematical Correspondence*, translated by H. Kaal. Oxford: Blackwell, 1980.

NS: *Nachgelassene Schriften*. Hamburg: Verlag, 1983. Translated

as *Posthumous Writings*, translated by Peter Long and Roger White. Oxford: Blackwell, 1979.

KS: *Kleine Schriften*. Hildesheim: Verlag, 2011. Translated as *Collected Papers on Mathematics, Logic, and Philosophy*, edited by B. McGuinness. Oxford: Blackwell, 1980.

Other abbreviations:

TLP: Wittgenstein, Ludwig. *Tractatus Logicus Philosophicus*, translated by C. K. Ogden. London: Routledge and Kegan Paul, 1922.

PI: Wittgenstein, Ludwig. *Philosophical Investigations*, edited and translated by G. E. M. Anscombe. Oxford: Blackwell, 1953.

NB: Wittgenstein, Ludwig. *Notebooks 1914–1916*, edited by G. H. von Wright and G. E. M. Anscombe. Oxford: Blackwell, 1961.

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