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What is the Task of Frege's Criteria of Referentiality?

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ABSTRACT

It is generally assumed that Frege's solution to the problem of referential indeterminacy in the first volume of *Grundgesetze* is based on a contextualist conception of reference that is spelt out by his criteria of referentiality. The present paper argues that this assumption is hardly consistent with Frege's remarks on the task of the criteria in the second volume. They leave no doubt that the criteria are not to be understood as explanations of the concept of reference, but as explanations of the requirement that in the exact sciences all concepts must have sharp boundaries. The crux of the problem of referential indeterminacy is that, to sharply delimit the boundaries of a vague or ambiguous concept, we have almost only concepts at our disposal that also lack sharp boundaries. To solve it, we need a non-inductive method for the complete determination of concepts. It is provided by Frege's criteria of referentiality. They define a kind of "holistic" method for the complete determination of concepts according to which vague and ambiguous concepts must be completely determined in a reciprocal manner.

1. Introduction

The famous problem of referential indeterminacy in the first volume of Frege's *Grundgesetze* (1893) is that axiom V, which can be transcribed as

> (V) The value-range of the function f = the value-range of the function g iff for all x: f(x) = g(x),

does not determine the reference of the names of a value-range completely. For instance, (V) does not determine whether the True and the False are value-ranges and, if so, which ones. To solve this problem, Frege makes stipulations that determine the truth values of all mixed identity statements of the form 'The value-range of the function f = q,' where 'q' is any proper name of the formal language that does not have the form 'the value-range of the function g.'

In the literature, it is generally assumed that this solution is implicitly based on the so-called "context principle" that Frege introduced in *Grundlagen* (1884) to solve the parallel problem that the contextual definition 'The number of Fs = the number of Gs iff the Fs and the Gs can be correlated one-to-one' does not determine the reference of the names of a number completely, because it does not determine the truth values of mixed identity statements like 'The number of Jupiter's moons = Julius Caesar.'[1] Understood as a principle concerning reference, the context principle in *Grundlagen* says that, to determine the reference of a term, it is necessary and sufficient to fix the truth values of the sentences in which it occurs. The main evidence for this "contextualist" interpretation of Frege's solution to the problem of referential indeterminacy in *Grundgesetze* is that the "criteria for referentiality" that he lays down in §29 of the first volume seem to incorporate a generalized version of the context principle for reference.

In what follows, I shall argue that the contextualist reading is based on a misunderstanding of the purpose of his criteria of referentiality. His remarks in the second volume of *Grundgesetze* (1903), which are widely ignored, clearly show that their task is not to explain the concept of reference, but the requirement that in logic and mathematics all concepts and relations must be sharply delimited. The crux of the problem of referential indeterminacy is that, in the informal language of the exact sciences, there are (almost) no terms whose reference is completely determined. To fix the reference of a vague predicate or an ambiguous proper name, we depend on predicates and proper names that are themselves vague or ambiguous. Thus, when we define the concept of number in terms of the concept of extension, we do not sharply delimit the concept of number, because the concept of extension is not completely delimited, either. The point of the criteria of referentiality is to define a non-inductive, "holistic" method for the complete determination of reference according to which concepts without sharp boundaries must be sharply delimited step by step in a reciprocal way.

[1]See, for instance, Dummett (1991b, 211), Dummett (1991a, 232), Heck (2012, 129), Linnebo (2004, 76), Linnebo (2019, esp. p. 99), Schirn (2003), Schirn (2018, 237), Weiner (2019, 128, 131–37), and Wright (2019, 415).

The paper is structured as follows. Section 2 briefly explains the context and the nature of the problem of referential indeterminacy. Section 3 is designed to clarify the task of the criteria of referentiality. In Section 4, the holistic method of conceptual determination on which the criteria are implicitly based is explained. Finally, in Section 5, the application of the criteria in Frege's "proof of referentiality" is analyzed. It is argued that this proof must not be read as an inductive, but as a conditional proof.

2. The Context and the Nature of the Problem of Referential Indeterminacy

Frege's main concern in *Grundlagen* is to provide a satisfactory definition of number. In the preface, he complains that the concept of number had not yet been clarified in the mathematics of his time. His definition of number is supposed to transform this vague and ambiguous concept into a scientifically more adequate concept meeting the highest criteria of clarity and precision.[2] However, in contrast to *Grundgesetze*, *Grundlagen* does not contain a theory of definition making explicit what his criteria of adequacy for definitions are. Frege only mentions a single criterion of adequacy explicitly: a definition of number must achieve a "sharp delimitation" of the concept of number. He illustrates this criterion by two heuristic attempts to define the numbers. The first consists in the following inductive definition (see Frege 1884, sec. 55):

> (I) The number 0 belongs to the concept F iff for all x: not F(x). The number 1 belongs to the concept F iff not for all x: not F(x) and for all y and z: if F(y) and F(z), then y=z. The number n+1 belongs to the concept F iff there is an x such that F(x) and the number n belongs to the concept of being a y such that F(y) and y≠x.[3]

On the intended reading, the definienda of (I) are the proper names 'the number 0,' 'the number 1,' etc. In this case, the following problem arises:

[2]See the preface and the first two paragraphs of Frege (1884), especially p. xiv. The Kantian goal of *Grundlagen*, to determine the nature of arithmetical knowledge, is mentioned only in §3.

[3]See Frege (1884, sec. 55)

> [W]e can never—to take a crude example—decide by means of our definitions whether any concept has the number Julius Caesar belonging to it, or whether that same familiar conqueror of Gaul is a number or not. Moreover we cannot by the aid of our suggested definitions prove that, if the number *a* belongs to the concept F and the number *b* belongs to the same concept, then necessarily $a = b$. Thus we should be unable to justify the expression "*the* number that belongs to the concept F", and therefore should find it impossible in general to prove a numerical equality, since we should be quite unable to achieve a determinate number. (Frege 1960, sec. 56)

The problem is that (I) does not completely determine the reference of the definienda. Thus, (I) does not determine the truth value of mixed identity statements like 'The number 5 = Caesar.' Nor does (I) rule out that the identity statements 'The number 5 = Caesar' and 'The number 5 = Cicero' are both true. For this reason, the use of the definite article in statements of the form '*the* number 5 = the number belonging to the concept F' is unjustified.

Frege's second heuristic attempt to arrive at a satisfactory definition of number is to consider "Hume's principle,"

> (C) The number belonging to the concept F = the number belonging to the concept G iff the concepts F and G are equinumerous, i.e., there is a one-to-one function f such that for all x: F(x) if and only if G(f(x)),

as a contextual definition of number whose definiendum is 'the number belonging to the concept F.' Again, the problem is that the definition does not determine the reference of the definiendum completely, because it does not fix the truth values of mixed identity statements like 'The number belonging to the concept *moon of Jupiter* = Caesar.'[4] This problem has become known as "the Caesar problem."[5]

The criterion of adequacy for definitions to which Frege refers in his discussion of the two heuristic definitions is obviously this: a definition of a concept must be *complete*, that is, it must determine unambiguously for every object whether it falls under the concept or not. It implies that a definition of the number belonging to the concept of being a moon of Jupiter is adequate only if it determines for every object x whether x is the number belonging to the concept of being a moon of Jupiter or not.

[4]See Frege (1884, secs. 62–66).

[5]For a detailed explication of the Caesar problem, see Greimann (2025).

Metaphorically speaking, a definition of a concept is adequate only if it "sharply delimits" its extension.

In the literature, it is commonly assumed that the demand for the completeness of definitions was introduced by Frege only in his later work; in *Grundlagen*, it does not play any role.[6] However, in §68 of *Grundlagen*, Frege explicitly says that the two heuristic attempts to define the numbers are unsatisfactory precisely because they do not provide us with a concept of number "with sharp limits to its application":

> Seeing that we cannot by these methods obtain any concept of direction *with sharp limits to its application*, nor therefore, for the same reason, any satisfactory concept of Number either, let us try another way. (Frege 1960, sec. 68, my emphasis)

Moreover, in §74, he even claims that the sharp delimitation of concepts with regard to their extension is the only requirement that concepts must satisfy to be acceptable from the point of view of logic:

> All that can be demanded of a concept from the point of view of logic and with an eye to rigour of proof is only that the limits to its application should be sharp, that it should be determined, with regard to every object whether it falls under that concept or not. (Frege 1960, sec. 74)

To solve the problem of the referential indeterminacy of the names of a number in *Grundlagen*, Frege finally reduces the concept of number to the concept of extension by means of the explicit definition

> (N) The number belonging to the concept F = the extension of the concept of being equinumerous to F.[7]

Obviously, this solution presupposes that the concept of extension itself has sharp boundaries. In *Grundgesetze*, however, Frege surprisingly argues that this condition is not fulfilled. In §3 of the first volume, he first replaces the concept of extension by the more general concept of value-range. It applies to both concepts and functions, where concepts are considered as a special kind of functions, namely, as functions whose value is always a truth value. In §10, he then shows by means of a permutation argument that axiom

[6] See, for instance, Heck (1997, sec. 2) and Heck (2011, 131–32). Exceptions are Kemp (2005, 185), and Blanchette (2012, chap. 3).

[7] See Frege (1884, sec. 68).

(V) The value-range of the function f = the value-range of the function g iff for all x: f(x) = g(x),

does not fix the truth values of mixed identity statements like

(T) The value-range of the function — x = (2+3=5),

and

(F) The value-range of the function $x = (2+3\neq5)$ = (2+3≠5).

Let X be a one-to-one function permuting the objects of the system in such a way that the value of X for a value-range as argument is not (always) identical to this value-range. Since X is one-to-one, X(the value-range of the function f) = X(the value-range of the function g) iff the value-range of the function f = the value-range of the function g. Consequently, (V) does not determine whether the proper name 'the value-range of the function f' denotes the value-range of the function f or X(the value-range of the function f). We can therefore identify the two truth values with any two value-ranges without violating axiom (V).

Frege's strategy to solve the problem of the referential indeterminacy of the names of a value-range is this:

> Now, how is this indeterminacy resolved? By determining for every function, when introducing it, which value it receives for value-ranges as arguments, just as for all other arguments. (Frege 2013a, sec. 10)

To apply this strategy, we must determine which values the identity function $x = y$ receives when x is a value-range and y is a truth value. To this end, Frege identifies the extension of the concept of being the True with the True and the extension of the concept of being the False with the False. These stipulations imply that (T) and (F) are both true (see §10).

To understand this solution, we must first clarify the nature of the problem of referential indeterminacy in *Grundgesetze*. Since the concept of a value-range is primitive, we can presuppose that it is already known which objects the value-ranges are. Consequently, the trivial stipulation that 'The value-range of the function f' denotes the value-range of the function f should already completely determine the reference of these names. We must explain, then, why the referential indeterminacy arises at all. There are two natural answers we must clearly reject. The first is to consider the problem of referential indeterminacy as a model-theoretical problem that arises because the axioms of the system do not rule out unintended interpretations of the proper names. This reading presupposes that Frege aims at an axiomatic fixation of the reference of proper names. But his permutation argument already shows that this is impossible. We can always permute the objects of the system without changing the truth value of any sentence.

According to the second answer, the problem of referential indeterminacy arises because mixed identity statements like (T) and (F) are not logically decidable in the system. Neither (T) nor the negation of (T) is a logical consequence of axiom (V). A partial solution to this proof-theoretical problem is achieved when we add (T) and (F) to the axioms of the system. However, Frege’s stipulation that the value-range of the function — x is identical to the True is neither an axiom nor a definition. Hence, (T) and (F) are not really theorems of his system.

Frege assumes that, even on the intended interpretation of the names of a value-range, according to which the proper name ‘the value-range of the function f’ denotes the value-range of the function f, mixed identity statements like (T) and (F) lack a truth value. This assumption presupposes that the concept of value-range itself does not lay down whether the True and the False are value-ranges. Consequently, the referential indeterminacy of the names of a value-range must derive from the “fuzziness” of the concept of value-range. Like the concept of number, it has no sharp boundaries.

Why does Frege identify the True with the extension of the concept of being the True? The most obvious answer is that the concept of extension does not determine which objects are the extensions of *individual* concepts. When C is a concept under which exactly one object falls, it would be natural to identify the extension of C with this object. Thus, the True may be regarded as the extension of the concept of being the True. However, if C applies to two or more objects, C and the concept of being the extension of C have different extensions. The latter applies only to one object. In this case, the identification of the extension of C with the extension of the concept of being the extension of C would contradict axiom V. Consequently, we cannot generalize the identification of the True with the extension of the concept of being the True and of the False with the extension of the concept of being the False, that is, we cannot stipulate that every object x is identical to the value-range of the concept of being identical to x, as Frege himself notes in the footnote to §10. On the other hand, we are free to identify Caesar with the value-range of the concept of being identical to Caesar. This identification is just as compatible with axiom V as the identification of the True with the value-range of the concept of being the True. These examples already show that the concept of value-range (or extension) does not have sharp boundaries. Otherwise, we could not even stipulate that the True is identical to the value-range of the concept of being the True. The facts would decide whether the True is identical to this value-range or not.

The crux of the problem of referential indeterminacy is consequently this: to transform the concept of number into a clear and precise concept, we need to use concepts that must also be transformed into clear and precise concepts, such as the concept of a value-range and the concept of a truth value. More generally, to determine the reference of a vague or ambiguous term, we need to use terms that are also vague or ambiguous to some degree. Consequently, we cannot use the inductive method to solve the problem of referential indeterminacy. We rather need a holistic method that allows us to determine the reference of vague or ambiguous terms by means of other vague or ambiguous terms. Such a method is implicitly contained in Frege's criteria of referentiality, as we will see later.

3. The Task of the Criteria of Referentiality

We claimed that, in Frege's view, a definition of the form '*a* = the x such that F(x)' is adequate only if the definite description 'the x such that F(x)' picks out a unique object in the universe. This is the standard reading of *Grundlagen*. Weiner has recently argued that Frege did not actually make this requirement.[8] Rather, we must ascribe to him a contextualist conception of reference that can be roughly characterized as follows.[9] In *Grundlagen*, Frege stresses that, to overcome the referential indeterminacy of the names of a number, we must keep in mind that "words have meaning only in the context of a sentence" (Frege 1884,

[8]We have already seen that Frege makes this requirement explicitly in the footnote to §74.

[9]See Wright (2019, especially pp. 121, 134, and 136), and Weiner (2007, 680).

sec. 62). This "context principle" is ambiguous because Frege does not clearly distinguish between sense and reference in *Grundlagen*. Understood as a principle concerning reference, it mainly contains the following claims:

(R_1) The primary bearers of reference are sentences.

(R_2) The reference of a sentence is its truth value.

(R_3) The reference of a term consists in the contribution it makes to the reference of the sentences in which it occurs.

On this reading, the context principle implies that, to fix the reference of a term, it is necessary and sufficient to determine the truth values of all sentences in which it occurs. To eliminate the referential indeterminacy of a proper name, we do not need to construct a definite description picking out the object that this name is supposed to denote, but it is sufficient to close all relevant truth value gaps. This is exactly what Frege does to completely determine the reference of the names of a value-range in *Grundgesetze*.

Prima facie, the contextualist interpretation of Frege's notion of reference is evidenced by the "criteria of referentiality" that he lays down in §29 in the first volume of *Grundgesetze*. The most important are:

(C_1) A proper name *a* has a reference (refers to something, is referential) iff the following conditions are met: (i) every more complex proper name 'f(a)' has a reference, provided that 'f(ξ)' is a name of a function of first order with one argument having a reference; (ii) every one-place name of a function 'g(a, ζ)' has a reference, provided that 'g(ξ, ζ)' is a two-place name of a function of first order with two arguments having a reference; and (iii) every one-place name of a function 'g(ξ, a) has a reference, provided that 'g(ξ, ζ)' is a two-place name of a function of first order with two arguments having a reference.

(C_2) A one-place name of a function of first order 'f(ξ)' has a reference iff every complex proper name 'f(a)' has a reference, provided that 'a' is a proper name having a reference.

(C_3) A two-place name of a function of first order 'g(ξ, ζ)' has a reference iff every complex proper name 'g(a, b)' has a reference, provided that 'a' and 'b' are proper names having a reference.

(C_4) A one-place name of a function of second order has a reference iff for each one-place name of a function of first order that has a reference

> the result of inserting the name of a function of first order in the empty place of the name of a function of second order has a reference.

According to the contextualist reading, the task of these criteria is to explain Frege's notion of reference. They incorporate a "generalised context-principle for reference," as Dummett puts it, according to which the reference of a term is fixed by the references of the more complex terms of which it is a constituent. In contrast to the original context principle in *Grundlagen*, the generalized context principle is not based on the assumption that sentences are the primary bearers of reference. This is a consequence of the "assimilation" of sentences to proper names in *Grundgesetze*, according to which sentences are a species of proper names, namely, proper names of a truth value.[10]

However, in his critical discussion of the generalized context principle, Dummett shows that it suffers from two disastrous defects. First, the generalised principle "is surely incoherent" and it can "scarcely be called a principle, since it embodies no criterion for distinguishing those terms whose reference is to be fixed by fixing that of more complex terms containing them from those whose reference is to be given outright" (see Dummett 1991b: 230, 215). It is obvious that we cannot fix the reference of a term by fixing the reference of the more complex terms of which it is a constituent, because in order to fix the reference of the latter terms, we have to fix the reference of the more complex terms of which they are a constituent, and so on ad infinitum. To make sense, the generalised principle must therefore be restricted to Frege's original context principle in *Grundlagen* according to which the reference of a sentence part consists exclusively in the contribution it makes to the reference of the sentences of which it is a part. According to the latter, the reference of terms is uniquely determined by the truth values of the sentences in which they occur. But the restricted principle is in "flat contradiction" to the permutation argument because the argument "entails that the references of the terms may change though no single sentence undergoes any alteration in truth-value" (see Dummett 1981: 422). The permutation argument rather presupposes the standard conception of reference, according to which the reference of a proper name is an object and the reference of a name of a function is a function.[11] These implications

[10]See Dummett (1991b, 209–13) and Dummett (1991a, 232).

[11]See Dummett (1981, 422–23).

already cast doubt on the contextualist reading of Frege's criteria of referentiality. But there is also an immediate reason why it cannot be sustained: it does not correspond to the actual task that Frege ascribes to these criteria. To see this, we must briefly explain the theory of definition into which they are embedded.

In *Grundgesetze*, Frege sketches both a special and a more general theory of definition. The special theory is presented in the section entitled "Definitions" of the first volume (see Frege 1893, secs. 26–46). Its task is to establish the criteria of adequacy for definitions in the formal language of *Grundgesetze*, the "concept-script." The general theory is outlined in the section "Principles of definition" of the second volume (see Frege 1903, secs. 55–67). It is used by Frege to critically discuss some alternative definitions of number that are not formulated in the formal language of *Grundgesetze*, but in the informal "word languages" [*Wortsprachen*] used by his scientific opponents. The task of the general theory accordingly is to lay down and justify general criteria of adequacy for definitions that are formulated in the latter languages:

> Before examining what foremost mathematicians have taught us about numbers . . ., it will be beneficial to lay dawn and justify in advance some principles of definition that are disregarded by nearly all authors of this area, so that we need not discuss the issue in detail on every occasion. In volume I, we already laid down such principles for concept-script; here, we will mainly be concerned with definitions in ordinary language [*in den Wortsprachen*]. (Frege 2013a, sec. 55)

Unfortunately, Frege does not justify the principles of definition of the special theory in volume I. In particular, he does not explain why the criteria of referentiality are introduced. Their role remains obscure. To clarify this role, we need to relate the principles of definition in volume I to the parallel principles in volume II.

The criteria of referentiality are an integral part of the special theory (see Frege 1893, sec. 29). They are used to spell out the following principle for definitions in concept-script:

> I now lay down the following governing principle for definitions: Correctly formed names must always refer to something. (Frege 2013a, sec. 28)

This principle corresponds to the following "principle of completeness" for the introduction of a word or sign by means of a definition of the general theory:

> In the exact sciences, a word or sign may not be used so long as its reference has not been completely explained or is not otherwise known. . . ." (Frege 2013b, sec. 55)

Since the inductive definition (I) and the contextual definition (C) do not completely determine the reference of their definienda, they are examples of definitions that are ruled out by this general principle.

The criteria of referentiality refer to the principle of the special theory that "correctly formed names must always refer to something." They do not explain Frege's notion of reference, but the requirements that a term must fulfil to be considered as a referential term in the formal language of *Grundgesetze*. For example, Frege holds that logic cannot recognize vague predicates as referential terms. In his "Comments on Sense and *Bedeutung*" (1891/1892), he writes:

> If it is a question of the truth of something—and truth is the goal of logic—we also have to inquire after *Bedeutungen*; we have to throw aside proper names that do not designate or name an object, though they may have a sense; we have to throw aside concept words that do not have a *Bedeutung*. These are not such as, say, contain a contradiction—for there is nothing at all wrong in a concept's being empty—but such as have vague boundaries. It must be determinate for every object whether it falls under a concept or not; a concept word which does not meet this requirement on its *Bedeutung* is *bedeutungslos*. (Frege 1997a: 178)

In *Grundgesetze*, this requirement is incorporated into the criterion (C_2), according to which a predicate 'F(ξ)' has a reference only if the sentence 'F(a)' always has a truth value, where 'a' is any proper name that has a reference. The concept denoted by 'F(ξ)' has sharp boundaries only if 'F(a)' always has a truth value.

At first sight, this criterion implies that most of the sentences of natural language are neither true nor false because they contain predicates that do not denote concepts with sharp boundaries.[12] But this is not exactly what Frege says: he does not say that fuzzy concepts (or partial functions) do not exist, but only that they cannot be recognized in logic.

[12] See, for instance, Kemp (1996).

The reason for this is that fuzzy concepts violate the law of excluded middle. This is made explicit only in the second volume of *Grundgesetze*.

> A definition of a concept (a possible predicate) must be complete; it has to determine unambiguously for every object whether it falls under the concept or not (whether the predicate can be applied to it truly). Thus, there must be no object for which, after the definition, it remains doubtful whether it falls under the concept, even though it may not always be possible, for us humans, with our deficient knowledge, to decide the question. Figuratively, we can also express it like this: a concept must have sharp boundaries. . . . a concept without sharp definition is wrongly called a concept. *Logic cannot recognize such concept-like constructions as concepts; it is impossible to formulate exact laws concerning them. The law of excluded middle is in fact just the requirement, in another form, that concepts have sharp boundaries*. Any object Δ either falls under the concept Φ or it does not fall under it: *tertium non datur*. (Frege 2013a, sec. 56, my emphasis)

The criteria of referentiality must accordingly be understood as an explication of the criteria for having a *logically respectable* reference (a "proper" reference), and not as criteria for having a reference at all. On this reading, the criterion that a name of a function 'f(ξ)' has a proper reference only if 'f(a)' has a reference for any proper name 'a' that has a reference, is to be understood as a reformulation of the requirement that a function f can be recognized in logic only if it is defined for any object a. Again, this is made explicit only in the second volume of *Grundgesetze*:

> Here, once again, we see that the laws of logic presuppose concepts with sharp boundaries and also, therefore, complete explanations of function-names—e.g., of the plus-sign. *In the first volume, we expressed this point thus: every function-name must have a reference*. Accordingly, all conditional definitions, and all piecemeal defining, are to be rejected. Every sign must be completely explained in one stroke, so that, in our terms, it receives a reference. (Frege 2013a, sec. 65, my emphasis)

There can therefore be no doubt that Frege's criteria of referentiality do not tell us what reference is, but what requirements the reference of a term must fulfil if we want to introduce this term into the language of logic. In §30 of the first volume, he explicitly says that they are not to be regarded as explanations of the notion of reference:

> These propositions are not to be regarded as explanations of the expressions 'to have a reference' or 'to refer to something', since their

> application always presupposes that one has already recognised some names as referential; but they can serve to widen the circle of such names gradually. (Frege 2013a, sec. 30)

To explain the notion of having a reference, we need criteria that define this notion inductively. Given the standard compositional notion of reference, we must first explain the conditions under which the primitive expressions have a reference and then explain the conditions under which the complex expressions have a reference.[13] Obviously, Frege's criteria cannot be understood in this way. Nor do they explain the contextualist notion of having a reference, according to which the names of a truth value (sentences) are the primary bearers of reference. To explain this notion, we must first explain the conditions under which the names of a truth value have a reference and then explain under which conditions the names of which they are composed have a reference. But, according to Frege's criteria, any name has a reference only if certain more complex names containing it have a reference.

This feature can be explained by the purpose of the criteria. They are used by Frege primarily to *verify* whether a term has a proper reference.[14] For instance, to verify whether the predicate 'F(ξ)' is referential in the proper sense, we must check whether all sentences of the form 'F(a)' have a truth value, where 'a' is any proper name that has a proper reference. It is in this sense that the reference of a predicate depends on the references of the sentences in which it occurs. But this does not imply that predicates have a reference in the contextualist sense. To fix the reference of 'F(ξ),' it is not sufficient to fix the truth value of all sentences in which 'F(ξ)' occurs, but it must also be determined to which concept it refers. The notion of reference at play here is still the standard one.

The main reason why Frege insists on complete definitions is that the validity of the theorems depends on the completeness of the definitions: "Without final definitions there will be no final theorems" (Frege 1903, sec. 61). Thus, the incomplete definition of a term may lead to theorems that lose their validity when the definition is completed:

> Piecemeal definition also renders the status of theorems uncertain. If, for example, the words 'square root of 9' have been explained restricting

[13] The compositional notion is explained by Frege in Frege (1892).

[14] They are used to this end in his "proof of referentiality," as we will see.

> the domain to the positive whole numbers, then one may, for example, prove the proposition that there is only one square root of 9, which will be immediately overturned when one extends one's considerations to the negative numbers and supplements the definition accordingly. (Frege 2013b, sec. 61)

If we expand the domain of the variables step by step to include more and more kinds of numbers, we change the content of the theorems each time. Therefore, their validity becomes questionable, again:

> When expanding the domain of numbers, e.g., does one in fact always think that ... all general propositions proven hitherto acquire a different thought-content, that also the proofs become obsolete? (Frege 2013b, sec. 65, footnote 2)

Nevertheless, it is widely assumed that Frege gradually expands the domain of his system in the first volume of *Grundgesetze*.[15] At the first stage, the domain contains only the truth values and, at the second stage, the value-ranges are added. However, on this assumption, the axioms and theorems of the system change their content each time we expand the domain. Thus, axiom I, which may be transcribed as

> (I) For all objects x and y: — x → — (— y → — x),[16]

must be read at the first stage as follows:

> (I′) For all truth values x and y: —x → — (— y → — x).

At the second stage, we must read (I) as follows:

> (I″) For all truth values and value-ranges x and y: —x → — (— y → — x).

Since the thought expressed by (I″) is different from the thought expressed by (I′), (I′) and (I″) are different axioms. At later stages, when new objects are introduced, the content of (I) changes again. This would clearly be unacceptable for Frege. In §58 of the second volume, he points out that, if we expand the domain of a system in this way, we must always replace the signs that were used to formulate the old axioms with new signs, in order to avoid ambiguity. Since Frege does not replace the

[15] For an overview over the discussion of the domain of Frege's system, see Bentzen (2019, 10–14).

[16] See Frege (1893, sec. 18).

original signs with new ones when introducing the value-ranges, we must assume that the domain of his system is unrestricted: the variables in (I) refer to all objects without any restriction.

To be sure, there are some passages that strongly suggest that Frege is successively expanding the domain by "introducing" new objects or functions. Thus, in §10 of the first volume, he explicitly says that we have so far only "introduced" the truth values and value-ranges as objects and that we must "introduce" new primitive functions to determine the value-ranges more precisely. However, since, as we here assume, the domain already contains at the current stage all objects and functions that may be "introduced" at later stages, we must not understand the "introduction" of new objects or functions as an expansion of the domain, but as an expansion of the vocabulary of the system. To "introduce" new objects or functions means to introduce new names and to determine their reference in such a way that they refer to the intended objects or functions, respectively. This is exactly what Frege does when he "introduces" a new kind of entities. An expansion of the domain is not involved.

4. The Holistic Method of Conceptual Determination

We have seen that the crux of the problem of referential indeterminacy is that, to determine the reference of a vague or ambiguous term, we need to use terms that are also vague or ambiguous to some degree. To understand Frege's solution to this problem, we must distinguish between a foundationalist (or "inductive") and a holistic theory of the sharp delimitation of concepts. The distinction I have in mind corresponds to the well-known distinction between foundationalist and holistic theories of justification and can be explained as follows. The deductive part of Frege's logical foundation of arithmetic consists of the deduction of the arithmetical laws from the laws of logic. It is implicitly based on a foundationalist theory of justification according to which a system of justified claims (a "theory") has the hierarchical structure of a building in which floor n supports floor n+1. Its main doctrines are:

> (FJ_1) There are claims of a special sort that do not need to be justified because they are self-evident. These claims, which Frege's calls 'axioms,' constitute an appropriate basis for the justification of all other claims.[17]
>
> (FJ_2) To justify a non-basic claim in a satisfactory way, it must be reduced to the special claims by means of suitable chains of logical derivations.
>
> (FJ_3) Being justified is a yes-no question.

According to the alternative holistic approach, a system of justified claims has the holistic structure of a network in which every node supports every other node.[18] It is characterized by the following holistic doctrines:

> (HJ_1) There are no claims constituting an appropriate foundation of justification. Even the axioms of logic are not immune to revision.
>
> (HJ_2) The claims of a theory can be justified only in a reciprocal way. To justify a claim, we have only claims at our disposal that also need to be justified.
>
> (HJ_3) Being justified is a matter of degree. The more the claims of a theory are connected by logical derivations, the better they are justified.

The conceptual part of Frege's logical foundation of arithmetic consists mainly of the definition of the arithmetical concepts in terms of logical concepts. It has a double task. First, it is designed to make the inner structure of arithmetical concepts explicit.[19] Thus, the definition of number reveals that the concept of number implicitly contains the concept of the one-to-one correlation of objects by a function. A definition in this sense is a "conceptual analysis": it "decomposes" the complex sense of the definiendum into the senses of the words that are used to define it. The second task is to transform the vague and ambiguous concepts of arithmetic into clear and precise concepts. A definition of this kind is a definition in the literal sense of "definition": it "delimits" the definiendum. Thus, the identification of numbers with extensions by the definition of number in *Grundlagen* serves to provide the concept of number with sharp boundaries.

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[17] For a detailed reconstruction of Frege's conception of self-evidence, see Jeshion (2001) and Jeshion (2004).

[18] See Quine (1968).

[19] See Frege (1884, xix).

Prima facie, the conceptual part of Frege's foundation of arithmetic is based on a foundationalist theory of conceptual explanation according to which a system of satisfactorily explained concepts has a hierarchical structure in which the concepts of level n provide the concepts of level n+1 with a clear and precise content. Its main doctrines are:

> (FC_1) There are concepts of a special sort that need not be explained because they are clear and precise on their own. These are the logical concepts. They constitute an appropriate basis for the conceptual explanation of all arithmetical concepts.
>
> (FC_2) To explain an arithmetical concept in a satisfactory way, it must be reduced to the basic concepts by appropriate chains of definitions.
>
> (FC_3) Being satisfactorily explained is a yes-no question.

Thus, Kemp assumes that "it is unavoidable, on Frege's conception, that certain concepts—those expressed by the primitive vocabulary of the theory used to express the axioms—be *presupposed* as being sharply delimited" (Kemp 2005, 189). Accordingly, "the worry that nothing in the *Grundgesetze* theory of number has been said to ensure that Julius Caesar is not a value-range must be regarded, and dismissed, as a kind of skeptical worry, not as the kind of question that the theory is itself obliged to answer" in Kemp's view (Kemp 2005, 190).

However, the foundationalist approach cannot be ascribed to Frege, because he clearly rejects (FC_1), at least in his mature system. According to his criteria of referentiality, the informal concepts of propositional logic also lack sharp boundaries because they are not defined for objects that are not truth values. For instance, the functions denoted by the logical connectives 'it is not the case that p' and 'if p, then q' are not defined for objects that are not truth values, neither in ordinary language nor in the informal language of the exact sciences. Sentences like 'It is not the case that 2' and 'If the Moon, then Julius Caesar' are not true or false. Strictly speaking, such sentences are not even well-formed. Similarly, in Frege's first system, the functions denoted by the "negation stroke" (*Verneinungsstrich*) and the "conditional stroke" (*Bedingungsstrich*) are not defined for objects that are not circumstances (see Frege 1879, sec. 5 and 7). This problem is solved in the system of *Grundgesetze* by means of the function — x. It has two special features: first, it is defined for all objects, and second, its value is always a truth value. Since the partial function denoted by 'it is not the case that p' is always defined for the values of — x, the complex function — ¬ — x is also defined for all objects. For example, — ¬ — 2 is the True, because — 2 is the False. Moreover, the complex and the partial function have always the same value for the same truth value as argument. For these reasons, the complex function can be regarded as a concept of negation with sharp boundaries. Frege accordingly construes the negation operator of *Grundgesetze* as a complex expression that is composed of the horizontal stroke and the negation stroke. It does not denote the partial function denoted by 'it is not the case that p' in natural language, but the total function — ¬ — x. The judgement represented by '|— ¬ — 2' which is the infamous "negation of the number 2" made by Frege in §6 of *Grundgesetze*, is actually true, because the number 2 is not the True. The same applies *mutatis mutandis* to the concept of material implication of *Grundgesetze*. According to §12, — the Moon → — Julius Caesar is the True, because the Moon is not the True.

The informal concepts of quantificational logic suffer from exactly the same defect, from Frege's point of view. In ordinary language (broadly construed), the first order quantifier 'for all x: Φ(x)' for instance, is not defined for first order functions Φ that are not concepts. Sentences like 'For all x: (x + 2)' are neither true nor false. This problem is solved in *Grundgesetze* by means of the function — x, again. According to §8, the corresponding quantifier of the formal language is a complex expression that does not denote the partial function 'for all x: Φ(x)' but the complex function — ∀x — Φ(x), which is also defined for functions Φ that are not concepts. Thus, — ∀x (— x + 2) is the False, because the value of the concept — 2 + x is not always the True.

Finally, Frege acknowledges that the logical concept of extension (value-range) is not sharply delimited, either. The definition of number in *Grundlagen* does not really transform the concept of number into a clear and precise concept, because the concept of extension itself is not sufficiently clear. In *Grundlagen*, he already alludes to this problem, and, in *Grundgesetze*, he even explicitly states that the more general concept of value-range is not completely determined.[20] Strictly speaking, the informal concept of being the True and the informal concept of being the False also lack sharp boundaries, because sentences like

[20] See Frege (1884, sec. 107) and Frege (1893, sec. 10).

(T) The value-range of the function — x = (2+3=5)

and

(F) The value-range of the function x = (2+3≠5) = (2+3≠5)

are neither true nor false. To delimit these concepts more sharply, we must stipulate whether the truth values are value-ranges and, if so, which ones. There is only one logical concept which, from Frege's point of view, seems to be sharply delimited from the outset, namely the concept of identity.

We are here confronted with the following dilemma. On the one hand, we cannot use a concept F to sharply delimit a concept G with fuzzy boundaries if F itself has fuzzy boundaries. To avoid a circle or an infinite regress, we need concepts that have sharp boundaries already from the outset. On the other hand, there are (almost) no concepts of this kind. How does Frege resolve this dilemma? The most plausible answer is that he adopts a holistic conception of the complete determination of primitive concepts according to which a system of completely determined concepts has the holistic structure of a network in which the primitive concepts determine each other in a mutual way. Its main doctrines are:

(HC_1) There are (almost) no concepts that are sharply delimited already from the outset. The primitive concepts suffer from the same lack of clarity and precision as all other concepts.

(HC_2) The primitive concepts of a logical system can be determined only in a mutual way. To determine a concept, we have only concepts at our disposal that also need to be determined.

(HC_3) The sharp delimitation of concepts is a matter of degree. The more primitive concepts of a theory are mutually determined, the more sharply they are delimited.

In accordance with (HC_1), Frege does not assume that the primitive concepts of his system have sharp boundaries already when they are introduced. They must be sharply delimited by means of stipulations like the identification of the True with the extension of the concept of being the True. This stipulation must not be regarded as a definition of the True. Frege does not presuppose that we already know which object is the extension of the concept of being the True. But it must also not be regarded as a definition of this extension. He does not presuppose that we already know which object is the True, either. Rather, this stipulation aims to determine both the True and the extension of the concept of being the True more closely. This is strongly suggested by the following comment he makes on the stipulations:

> We have hereby determined the value-ranges *as far as is possible here*. Only when the further issue arises of introducing a function that is not completely reducible to the functions already known will we be able to stipulate what values it should have for value-ranges as arguments; *and this can then be viewed as a determination of the value-ranges as well as of that function*. (Frege 2013a, sec. 10, my emphasis)

For instance, when at a later stage we introduce the primitive function $x = Caesar$, we must stipulate what values it should have for value-ranges as arguments. It would again be natural to identify the value-range of the function $x = Caesar$ with Caesar. This stipulation could then be viewed as a determination of the value-ranges as well as of the function $x = Caesar$, which is the concept of being Caesar. For analogous reasons, the identification of the True with the value-range of the function — x can be regarded both as a determination of the True and as a determination of the value-ranges. In accordance with the holistic doctrine (HC_2), the truth values and the value-ranges are not determined inductively, but mutually. To sharply delimit the concept of a truth value, we must also sharply delimit the concept of a value-range, and vice versa. But this kind of circularity is not vicious; it must be tolerated because there are (almost) no concepts that are sharply delimited already at the outset.

The holistic doctrine (HC_3), that the sharp delimitation of a concept is a matter of degree, is implied by his assumption that the sharp delimitation of concepts must be achieved in stages. He claims that, at the current stage, we have determined the value-ranges as far as possible. At later stages, the problem of their indeterminacy will reoccur. To take this into account, we must relativize the completeness of the sharp delimitation of concepts to stages. Relative to the current stage, the determination of the value-ranges is complete, but relative to later stages, it is incomplete. As soon as we introduce the name 'Julius Caesar,' for instance, we must lay down whether Caesar is the value-range of the function x = Caesar.

The holistic approach implies that the concept-script is not a logically perfect language already at the outset. At the current stage, its terms have a logically respectable, but not a logically perfect reference. Their reference has only been determined as far as possible. They are still vague or ambiguous to a certain degree. According to (HC_3), their logical imperfection can only be eliminated step by step. Every time we introduce a new sort of object, the question arises again as to whether some of the new objects are value-ranges and, if so, which ones. Since, as Frege notes, the general identification of an object x with the value-range of the function y = x contradicts axiom V, we must decide in each individual case whether or not we can identify a sort of object with the corresponding value-ranges.[21] For instance, when at a later stage we introduce physical objects, it would be natural to stipulate that for every physical object x the value-range of the function $y = x$ is identical to x. This step, which amounts to identifying physical objects with their unit sets, is natural because physical objects are "urelements," i.e., they have no elements.[22] But this does not mean that all objects of physics can be treated in the same way. For example, if we identify the space-time regions occupied by physical objects as usual with the sets of the space-time points they contain, we cannot identify these regions also with their unit sets. For the same reason, we must not identify all objects of geometry with their units sets. If, for instance, we identify lines with sets of points, we cannot identify them also with their unit sets, and so on.

It stands to reason that we cannot anticipate the introduction of all other sorts of objects already at the current stage. Therefore, we must determine the value-ranges in stages. The criteria of referentiality only require us to make the concept-script as perfect as possible at every stage. A complete determination of the value-ranges in the absolute sense can only be achieved at the end of the construction of the system, when all relevant sorts of objects have been introduced. To this end, we must introduce not only the objects of arithmetic and logic, but also the objects of all other exact sciences like physics and geometry. To realize the logicist program of reducing arithmetic to logic, this is not necessary, of course. It would be sufficient to introduce all functions and objects that are needed to derive the laws of arithmetic from the laws of logic. But there is no reason to assume that concept-script is designed to be an

[21] See the footnote to §10 in the first volume of *Grundgesetze*.

[22] In Quine's set theory, the urelements are actually identified with their unit sets; see Quine (1963, 31).

ideal language only for logic and arithmetic, and not also for all other exact sciences. To transform it into a logically perfect language, we must extend its vocabulary in such a way that the fulfillment of the criteria of referentiality guarantees the complete determination of all objects and functions in the absolute sense.

Surprisingly, the requirements for having a logically respectable reference of Frege's general theory of definition are more restrictive than the corresponding requirements of his special theory. According to the former, a name of a function can be recognized as a referential term only if it denotes a *total* function, that is, a function defined for all objects as arguments. Its reference must be completely determined in the absolute sense. Since the concept-script is supposed to be a logically perfect language, one would expect the requirements of the special theory to be more restrictive than the requirements of the general theory. Why does Frege require the definitions in the logically imperfect language of the exact sciences to be more complete than the definitions in the conceptual script? To see this, we must apply the criteria of referentiality of the special theory to the informal language of the exact sciences. Obviously, this language is an extension of ordinary language. Its vocabulary includes all expressions of ordinary language like 'Julius Caesar' as well as some technical terms like 'extension of a concept' or 'value-range of a function.' Applied to the informal language of the exact sciences, the criterion (C_4) implies that the name of a function 'the value-range of the function Φ' can be recognized as referential in this language only when we fix the truth value of the sentence

(C) The value-range of the function $x = Caesar$ = Caesar.

To determine the value-ranges in this language, we must actually determine whether Caesar is a value-range. This is a consequence of the fact that the name 'Caesar' is already included in its vocabulary.[23]

[23] For a more detailed discussion of the relationship between the general and the specific theory, see Blanchette (2012, chap. 3), Wehmeier (2015) and Blanchette (2015, 4–7). Note that to completely determine the reference of an expression in the absolute sense, we need a language whose vocabulary allows us to make the necessary stipulations. Let A and B be two different objects that are linguistically "indistinguishable" in the sense that we cannot formulate any open sentence that is satisfied by A but not by B, or vice versa. Let N be proper name with a descriptive content (a "sense") that applies to both A and B. In this case, our vocabulary does not allow us to fix the reference of N completely in the

5. The Application of the Criteria in Frege's Proof of Referentiality

In §§31 and 32 of the first volume of *Grundgesetze*, Frege sketches his so-called "proof of referentiality". It is supposed to show that every well-formed expression of the formal language fulfills the criteria of referentiality. In the literature, it is commonly assumed that Frege is trying here to prove by induction on the complexity of expressions that every well-formed expression has a completely determined reference in the absolute sense.[24] The point of this proof is that it indirectly proves the consistency of the system of *Grundgesetze*. When we show that every proper name refers to a unique object, we can rule that there is a name of a truth value that refers both to the True and the False. Hence, there cannot be any contradiction in the system of *Grundgesetze*.[25]

I see four problems with this reading. First, it ignores the context in which the proof is made. There can be no doubt that the proof is an integral part of the special theory of definition. §§31 and 32 are contained in the section on definitions. But, in this context, it does not make any sense to prove the consistency of the system. Consistency is not a criterion of adequacy for definitions, but for axioms. Second, since Frege explicitly admits that the reference of every well-formed expression is not fully determined at the current stage, his aim cannot be to prove that every well-formed expression has a completely determined reference in the absolute sense. Rather, his aim is only to prove that the reference of all well-formed expressions has been determined as far as possible. For instance, to prove that a name of a truth value is referential, we do not need to show that this name picks out a unique object in the universe which is either the True or the False. It is sufficient to show that all more complex names of a truth value in which it occurs have a truth value. Third, on the inductive reading, the criteria are incomplete.

absolute sense. Thus, the requirement of the general theory to determine the reference of all expressions in the absolute sense amounts in practice to the requirement of the special theory that all expressions of the language must meet the criteria of referentiality.

[24]See, for instance, Heck (2012, 52), Schirn (2018, 226), and Bentzen (2019, 9). An exception is Weiner (2021, 197–98). She argues that the proof must not be considered as an inductive one.

[25]This view goes back to Sluga (1980, 167). It is shared, with some qualifications, by Bentzen (2019, 4), Landini (2012, 169–70), and Heck (2012, 127–28), for example.

Frege explicitly says that “their application always presupposes that one has already recognised some names as referential” (Frege 2013a, sec. 30). For example, to check whether the names of a first order function have a reference, we must presuppose that the proper names have a reference. Conversely, to check whether the proper names have a reference, we must presuppose that the names of a first order function have a reference. Because of this circularity, the criteria can only be used to verify whether, on the condition that certain names have a reference, certain other names also have a reference. When we drop the first clause of the inductive definition of the natural numbers

1. 0 is a natural number.
2. If n is a natural number, n+1 is also a natural number.
3. Nothing else is a natural number,

we can no longer prove that 0+1 is a natural number. We can only prove that, on the condition that 0 is a natural number, 0+1 is also a natural number. To start his proof, Frege accordingly assumes that the names of a truth value are referential.

> We assume that the names of truth-values refer to something, namely either the True or the False. We will then gradually widen the circle of names to be recognised as referential, by demonstrating that the names that are to be added form referential names with those already added, by way of one occupying fitting argument places of the other. (Frege 1893, sec. 31).

Note that Frege does not *prove* that the names of a truth value refer to something. He rather *assumes* this. To turn his proof into a complete inductive proof, we would have to add the criterion

(C_B) If a proper name is a name of a truth value, it has a reference

to his criteria of referentiality and assume that he tacitly derives this criterion from the original context principle for reference. However, there is no textual evidence for this assumption. The original context principle for reference is not mentioned in *Grundgesetze* at all, neither explicitly, nor implicitly.

The fourth and most important problem is that the inductive reading is incompatible with the proper goal of Frege’s proof. This becomes clear from the conclusion that Frege draws from the contradiction discovered by Russell in his system. He does not conclude that there is a name of a truth value that denotes both the True and the False, but that there is a name of a truth value that denotes neither the True nor the False. In the afterword to the second volume of *Grundgesetze*, he argues as follows. Since, in the formal language, the name of the class of all classes that do not belong to themselves is a well-formed expression, it must have a reference. But the contradiction discovered by Russell implies that the object denoted by this name does not fulfill the law of excluded middle. Let K be the class of all classes that do not belong to themselves. Both the assumption that K has the property of belonging to K and the assumption that K does not have this property lead to a contradiction. Hence, K has neither of the two properties. Frege claims:

> What attitude should we take to this [the contradiction]? Should we assume the law of excluded middle fails for classes? Or should we assume that there are cases where to an incontestable concept no class corresponds as its extension? In the first case we would find ourselves compelled to deny the full objecthood of classes. For if classes were proper objects, then the law of excluded middle would have to hold of them. (Frege 2013b, 254)

According to the conclusions drawn by Frege, the contradiction establishes that there is a truth-value gap in his system. The sentence 'K belongs to K' is neither true nor false, because both the assumption that it is true and the assumption that it is false lead to a contradiction. From this he concludes that the proper name 'K' either does not denote any object or it denotes an "improper object" to which the law of excluded middle does not apply because it has neither the property of belonging to itself nor the property of not belonging to itself. He rejects the recognition of "improper objects" because "they would not be admissible arguments for all first-level functions" (Frege 2013b, 254). Consequently, 'K' does not denote any object at all. This is the reason why he admits in his letter to Russell dated June 22, 1902, that his explanations in §31 "do not suffice to secure a *Bedeutung*" for every well-formed expression" (Frege 1997b, 254). Consequently, the real point of the "proof of referentiality" is not to prove that the system is consistent, but to prove that it fulfills the principle of bivalence. In this case, the proof can actually be regarded as an integral part of the special theory of definition. As we have seen, Frege regards the fulfilment of the law of excluded middle as a fundamental criterion of adequacy for definitions.

According to our holistic reading, (C_B) is not as an implicit premise of Frege's proof, but its conclusion. He is not trying to prove by induction that every proper name refers to a single object, but he is trying to prove that every time we have introduced a new kind of proper names, we have sharply delimited all concepts and all relations of the system with respect to the objects denoted by these names, and every time we have introduced a new name of a first-order function, we have defined this function for all objects that are denoted by the proper names that have already been introduced, and so on. In this way, we establish that there are no truth-value gaps. To prove that the system fulfills the law of excluded middle, it is sufficient to show that there is no truth-value name that refers neither to the True nor to the False.

The first names that are introduced in *Grundgesetze* are the primitive names of a function like '— ξ,' '— ¬ — ξ,' '— ξ → — ζ,' 'ξ=ζ' and '— ∀x — Φ(x)' (in §§5–8). To show that the names of a function of first order '— ¬ — ξ,' '— ξ → — ζ' and 'ξ=ζ' fulfill the criteria of referentiality, we must show that, under the assumption that '*a*' and '*b*' have a reference, the more complex names '— *a*,' '— ¬ — *a*,' '— *a* → — *b*' and '*a*=*b*' also have a reference, where '*a*' and '*b*' are any proper names that have been introduced so far. At this stage (in §§5–8), the proper names that have been introduced are all names of a truth value. They are not primitive names, but complex names like '—∀x— x=x' and '—∀x — — ¬ — x' which are composed of the eight primitive names of a function.

Consequently, to apply the criteria to the first names we have introduced, we must assume that the names of a truth value have a reference. This is the reason why we are entitled to recognize the latter names as referential from the outset. We must make this assumption just because the criteria do not provide a basis for their inductive application. To prove, for instance, that 'ξ=ζ' fulfills (C_3), we must assume that the names of a truth value have a reference and show that, under this assumption, '*a*=*b*' always has a reference, where '*a*' and '*b*' are any names of a truth value that have already been introduced. But this does not mean that we must consider (C_B) as an implicit premise of the proof. The point is rather that the criteria only allow us to prove the *conditional* statement that, if '*a*' and '*b*' are proper names of a truth value that have a reference, '*a*=*b*' also has a reference. This is guaranteed by the stipulation in §7 that '*a*=*b*' denotes the True iff *a* is identical to *b*. Consequently, we may add '$\xi = \zeta$' to the circle of names that are recognized as referential. Its reference has been completely determined at the current stage. At a later stage (in §9), the names of a value-range are introduced. To show that '$\xi = \zeta$' has a reference, we must then show that, on the assumption that 'a' and 'b' are any names of a truth value or a value-range that have a reference, '$a = b$' also has a reference. Conversely, to show that a name of a value-range 'a' has reference, we must assume that '$\xi = \zeta$' has a reference and show that '$a = b$' also has a reference, provided that 'b' is any name of a truth value or a value-range that has a reference. In both cases, the referentiality of the names in question is guaranteed by axiom V and the identification of the truth values with two special value-ranges. In this way, Frege proves in §31 that the eight primitive names of a function have a reference. From this he finally concludes in §32 that also the complex names that are built from them (like '$\mid$— ∀x—x=x') have a reference.

Obviously, the order we have chosen to prove that every well-formed expression is referential is not essential. To apply the criteria, we could just as well have assumed that the primitive names of a first-order function '— ξ,' '— ¬ — ξ,' '— ξ → — ζ,' and '$\xi = \zeta$' are referential. This assumption allows us to apply (C_1) to the proper names of a truth value. To show that the latter names are referential, we must again show that the more complex proper names '— a,' '— ¬ — a,' '— a → — b' and '$a = b$' are referential, where 'a' and 'b' are two names of a truth-value. Since, for instance, the function — x has been defined for the truth values as arguments, the complex name '— a' has a reference for every name of a truth value 'a'. In this way, we show that, on the condition that the names of a first-order function have a reference, all other names also have a reference.

Consequently, we must not read Frege's proof as an inductive, but as a conditional proof. It is designed to show that, on the condition that the names of a truth value have a reference, all other names also have a reference. This is already sufficient to show that every well-formed expression fulfills the criteria of referentiality. We do not claim that we have determinately identified two objects, the True and the False, and shown that all truth-value names of his system name one and only of these two objects. We still do not know exactly which objects the truth values are supposed to be. We only claim that we have determined the reference of all truth-value names as far as possible, by fixing the truth values of all equations like (T) and (F). From this we finally derive the conclusion that the criterion (C_B) is fulfilled.

The flaw in Frege's proof is that the fulfilment of the criteria of referentiality does not actually guarantee that the system fulfills the principle of bivalence. Since we have not delimited the concept of a value-range in relation to objects that are not truth values, the sentence 'There is an object that is not a value-range,' for example, is neither true nor false. To correct the proof, we must replace (C_B) with the weaker conclusion that we have closed all truth-value gaps as far as this is possible at the current stage.

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